THE JOIN-REPRESENTATION OF SOME INTERSECTIONS IN COMPLETE LATTICES

Dedicated to Professor Tatsujiro Shimizu

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For the lattice-theoretical notions, used here, we refer to G. Birkhoff [1].

Let L be a complete lattice with the minimal element 0 and maximal element 1. A subset S of L will be called a lower subset of L, if $x \leqq y$ and $y \in S$ always imply $x \in S$. Furthermore a lower subset S is said to be f-lower subset of L, if $\bigwedge_{\gamma \in \Gamma} x_\gamma \in S$ implies the existence of a finite subset Δ of Γ such that $\bigcap_{\delta \in \Delta} x_\delta \in S$ holds. Obviously, one has $1 \in S$ for a lower subset S of L, if and only if $S=L$ holds. The empty subset ϕ let us consider, as a lower subset, as well as an f-lower subset. The intersection of an arbitrary set-theoretical union of f-lower subsets of L, as well as the set-theoretical intersection of finite number of f-lower subsets of L are again f-lower subsets of L.

Let $x \in L$ be an element such that $x \notin S$ for an f-lower subset of L. Then an element $y \in L$ is called an (x,S)-element, if $x \cap y \notin S$ holds. Furthermore, an element $z \in L$ is said to be a strong (x,S)-element, if $z \cap y$ is an (x,S)-element for every (x,S)-element y of L.

It is evident, that every strong (x,S)-element is also an (x,S)-element, and that x itself is a strong (x,S)-element for every f-lower subset S such that $x \notin S$ holds.

Theorem. The intersection $z(x,S)$ of all strong (x,S)-element satisfies $z(x,S) \leqq x$, and it is again a strong (x,S)-element, which coincides with the join of all minimal (x,S)-elements of L.

Proof. Obviously, $z(x,S) \leqq x$ holds, since x is a strong (x,S)-element. Let $z_1, z_2, \cdots, z_n$ be arbitrary strong (x,S)-elements and y and arbitrary (x,S)-element of L. Then one has

$$(*) \quad (z_1 \cap z_2 \cap \cdots \cap z_n) \cap y = z_1 \cap (z_2 \cap (\cdots \cap (z_n \cap y)))$$

whence this intersection $(*)$ is an (x,S)-element, and thus $z_1 \cap z_2 \cap \cdots \cap z_n$ is a strong (x,S)-element. Put $z(x,S) = \bigwedge_{\gamma \in \Gamma} z_\gamma$, where z_γ runs all strong(x,S)-

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elements of L. For the arbitrary (x,S)-element y, let us define

$$w_\gamma = z_\gamma \cap y \cap x$$

if $z(x_1S)$ would be not a strong (x,S)-element, then y could be chosen such that

$$z(x,S) \cap y \cap x \in S$$

holds, which implies

$$(**) \quad \bigwedge_{\gamma\in\Gamma} w_\gamma \in S$$

Therefore, by virtue of the definition of the f-lower subsets $(**)$ yields at once with a finite set Δ of indices:

$$\bigcup_{\delta\in\Delta} w_\delta = {}_{\delta\in\Delta} z_\delta \cap y \cap x \in S$$

which is, by the above assertion, a contradiction. Thus $z(x,S)$ is a strong (x,S)-element, indeed.

The intersection of any descending chain of (x,S)-elements being again an (x,S)-element of L, by virtue of Zorn's lemma, there exist minimal (x,S)-elements in L, which generally fail tobe minimal elements in L; they are only minimal element in $\geqq$ as a suitable minimal (x,S)-element, which follows from this application of Zorn's lemma. Let $j=j(x,S)$ be the join (i.e. complete union) of all minimal (x,S)-element, y an arbitary (x,S)-element and m a minimal (x,S)-element with $m \leqq y$, of L. Then one obviously has, for $j=j(x,S)$ the relation

$$(***) \quad j \cap y \cap x \geqq m \cap m \cap x = m \cap x \notin S$$

and thus $(***)$ implies, by the definition of the lower subsets $j \cap y \cap x \notin S$. Therefore $j \cap y$ is an (x,S)-element, consequently j is a strong (x,S)-element of L, which implies $j \geqq z(x,S)$.

Conversely, $j \leqq z(x,S)$ can be shown, as follows. $z(x,S) \cap m$ is an (x,S)-element for every minimal (x,S)-element m of L, since (x,S) is strong (x,S)-element. By the minimality of m among all (x,S)-elements we have $m=z(x,S) \cap m \leqq z(x,S)$, whence also

$$z(x,S) \geqq \bigvee_{w\in\Omega} m_w = j = j(x,S)$$

where m_w runs all minimal (x,S)-elements of L.

Thus the desired equality $z(x,S)=j(x,S)$ holds, indeed, which completes the proof.

Problem. Let the complete lattice L be, at the same time, also a groupoid, written multiplicatively, generally without the requirements $(a \cup b)c=$

$ac \cup bc$ and $a(b \cup c) = ab \cup ac$, and let us call an element $p \in L$ prime, if $f \cdot g \leqq p$ always implies $f \leqq p$ or $g \leqq p$, where $f, g \in L$. Then, what is a nontrivial necessary and sufficient condition every strong (x, S)-element to be prime? (Characterize in this groupoid also the minimal (x, S)-elements!)

Remark. The result of this note dualizes some generalizations [6] of some results of [2], [3], [4] and [5].

References

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