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A Sufficient Covering Condition for a Left Duo Ring to be Nilpotent

Dedicated to Professor Kiyoshi Iséki on his 60th birthday

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The purpose of this note is to give a partial solution, for the case of left duo rings, of Problem 93 of the book [8] of the second author.

We simultaneously give a sufficient condition, formulated with the help of set-theoretical coverings (i.e. inclusions in the set-theoretical unions) by right ideals and of set-theoretical intersections, a left duo ring to be nilpotent. Assuming the same covering condition, original Problem 93 of [8] asks of arbitrary associative ring the ring to be only Jacobson radical ring, and thus this Problem reamins still open.

It may be remarked that coverings of groups by subgroups were already discussed by P. G. Kontorovich [4], whose eleven papers are mentioned in Kurosh's book [6]. Coverings of a ring by minimal left ideals were studied by L. G. Kovács and J. Szép [5].

All rings, considered in this note, are assumed to be associative. For the general notions, used here, we refer to Jacobson. Furthermore, a ring is called a *left* (*right*) *duo ring*, if every its right (left) ideal is also

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a twosided ideal. Following Feller [2], a ring is said to be a *duo ring*, if it is a left duo ring and at the same time a right duo ring, i.e. every one-sided ideal is twosided (see also Thierrin [9] and the new book of Steinfeld [7], page 93).

Theorem. *Let A be a left duo ring, and let $n \geq 2$ be a natural number. Assume that $R_1, R_2, \cdots, R_n$ are right ideals of A satisfying the following three requirements*:

(1) $\bigcap_{i=1}^{n} R_i = 0$,

(2) $\bigcup_{i=1}^{n} R_i = A$, *but*

(3) $A_j = \bigcup_{\substack{i \neq j \\ i=1}}^{n} R_i \neq A$.

Then $A^{2^{n-1}} = 0$ holds, i.e. A is nilpotent with nilpotence degree at most 2^{n-1}.

Proof. Since A is a left duo ring, every right ideal is twosided. Thus R_i's are twosided ideals. Let D_k be, for $1 \leq k \leq n$ the sum of all ideals of the form $R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$. Then by (2) we have $D_1 = \sum R_i = A$. We verify $A^{2^{k-1}} \subset D_k$, $k = 1, 2, \cdots, n$ by induction on k. If $k = 1$, then $A = D_1$ which is trivially satisfied. Assume now that the proposition is valid for k, and we prove then that it is true also for $k+1$. If $A^{2^{k-1}} \subset D_k$ then $A^{2^k} = A^{2^{k-1}} \cdot A^{2^{k-1}} \subset D_k^2$, moreover by definition of D_k it is sufficient to show that if

$$x \in R_{i_1} \cap \cdots \cap R_{i_k} \quad \text{and} \quad y \in R_{j_1} \cap R_{j_2} \cap \cdots \cap R_{j_k}$$

then $x \cdot y \in D_{k+1}$. Now we distinguish two cases

(α) If there exists an index j_m such that for $\ell = 1, 2, \cdots, k$, the relation $j_m \neq i_\ell$ holds, then

$$x \cdot y \in R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k} \cap R_{j_m} \subset D_{k+1},$$

A being a left duo ring.

(β) If every j_m equals to an i_ℓ then we may assume e.g. $i_1 = j_1$, $i_2 = j_2$ and $i_k = j_k$. Since $k \leq n-1$, there exists an index k' such that $i_{k'}$ is different from all $i_1, i_2, \cdots, i_k$. Furthermore, by condition (2) and (3) there exists an element $u \in A$ such that $u \in R_{i_{k'}}$ but $u \notin R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_k}$. Put $v = y - u$. Then obviously $v \notin R_{i_1} \cup \cdots \cup R_{i_k}$, since $u \notin R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_k}$. By condition (2) there exists an index k'' such that $v \in R_{i_{k''}}$ with $i_{k''} \neq i_m$, $m = 1, \cdots, k$. But then

$$xu \in R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k} \cap R_{i_{k'}} \subset D_{k+1},$$

and

$$xv \in R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k} \cap R_{i_{k''}} \subset D_{k+1}$$

Moreover $x \cdot y = x(u + v) \in D_{k+1}$ implies $A^{2^k} \subset D_{k+1}$. Now, by (1) $D_n = R_1 \cap R_2 \cap \cdots \cap R_n = 0$ also holds, which implies $A^{2^{n-1}} = 0$. This completes the proof of the theorem.

Corollary. *If A is subcommutative (i.e. $xA = Ax$ holds for every $x \in A$; see D. Barbilian* [1]*), then A is nilpotent under the covering conditions* (1), (2) *and* (3) *of the above theorem.*

Proof. From $(x)_r = Zx + xA = Zx + Ax = (x)_\ell$ (where Z denotes the ring of rational integers) it follows that A is a duo ring, and the theorem can be applied to this case.

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