MATHEMATICS SEMINAR NOTES
Vol. 7 (1979)

On the Homomorphic Closure of Some Semisimple Classes

By Tran Trong HUE and Ferenc Andor SZÁSZ

All rings considered in this note are associative. As is well-known, a non-empty class R of rings is a radical class in the sense of Kurosh-Amitsur if the following conditions (i)-(iii) are satisfied:

(i) R is homomorphically closed.

(ii) Every ring A has an R-ideal $R(A)$ containing all R-ideals of A ($R(A)$ is called the *R-radical of* A).

(iii) $R(A/R(A)) = 0$.

A class S of rings is semisimple if the following conditions are satisfied:

(iv) Every nonzero ideal of an S-ring has a nonzero homomorphic image in S.

(v) If every nonzero ideal of a ring A has a nonzero homomorphic image in S, then A is an S-ring.

In [5] Kurosh has shown that the rings having zero R-radical form a semisimple class, and the semisimple classes are closed under subdirect sums. For related concepts and results we refer to [4] and [7].

In [2] Andrunakievic has treated R-semisimple rings whose homomorphic

Received April 10, 1979.

images are again S-semisimple. Such S-semisimple rings were called *strongly S-semisimple rings*. In [8] F. Szász has shown that the classes

$$K_n = \{ A \mid a^n = a \quad \text{for each} \quad a \in A \}$$

are strongly semisimple classes, i.e. homomorphically closed semisimple classes. Later in [9] R. Wiegandt has shown that these classes are all homomorphically closed semisimple classes.

A natural question to ask is: What are the homomorphic closures of other semisimple classes ? This note gives an answer in the case of some concrete semisimple classes. It is interesting that we get the assertion: every well-known radical ring is a homomorphic image of some corresponding semisimple ring.

We shall use the following result of Amitsur (Theorem 4 of [1]):

Theorem 1. *Every ring is a homomorphic image of a subdirect sum of full matrix rings of finite order over the ring of all integers.*

The following is our main result.

Theorem 2. *If a radical R coincides with Koethe radical on the left artinian rings, then the homomorphic closure of the R-semisimple class consists of all rings.*

Proof. Assume that the radical class R coincides with Koethe radical on the left artinian rings. Then by Theorem 13 in the book [4], the inclusion $R \subset F$ holds, where F is upper radical class determined by the class of all matrix rings over division rings. Hence the R-semisimple class contains the F-semisimple class. So we only have to show the homomorphic closure of the F-semisimple class consists of all rings.

Let $M_n[A]$ denote the full $n \times n$ matrix ring over the ring A, and (n) the ideal of Z, the ring of integers, generated by n. Since $\bigcap_p M_n[(p)] = 0$ where p runs over all prime numbers, it follows that $M_n[Z]$ is a subdirect sum of the rings $M_n[Z]/M_n[(p)]$, $p = 2, 3, 5, \cdots$. Moreover

$$M_n[Z]/M_n[(p)] \cong M_n[Z/(p)] = M_n[Z_p].$$

The rings $M_n[Z_p]$ are F-semisimple. Since the semisimple classes are closed under subdirect sums, the ring $M_n[Z]$ is F-semisimple. From this, by Theorem 1, it follows immediately that the homomorphic closure of F-semisimple class is the class of all rings. This completes the proof.

Corollary. *Every Baer, Levitzki, Koethe, Jacobson and Brown-McCoy radical ring is a homomorphic image of some Baer, Levitzki, Koethe, Jacobson and Brown-McCoy semisimple ring, respectively.*

Proof. By Theorem 28.3 in [10] the above radicals coincide with Koethe radical on left artinian rings. Thus the assertion is an immediate consequence of Theorem 2.

Remark 1. Theorem 33.11 in [10] assert that there exists a Baer, Levitzki, Koethe, Jacobson and Brown-McCoy semisimple ring A, such that a nonzero homomorphic image B of A is Baer, Levitzki, Koethe, Jacobson and Brown-McCoy radical ring, respectively. The corollary is a sharpening of this result.

Remark 2. A ring A is said to be *regular* in the sense of von Neumann [6] if $a \in aAa$ for all $a \in A$. In [3] B. Brown and N. H. McCoy have proved that the class Q of all Neumann regular rings is a radical class, and the Q-radical satisfies the matrix equality $Q(M_n[A]) = M_n[Q(A)]$ for every ring

A. From the proof of Theorem 2, it is easy to see that the above assertion is valid for the Neumann regular rings, too.

References

[1] S. A. Amitsur: The identities of PI-rings, Proc. Amer. Math. Soc., 4 (1953), 27-34.

[2] V. A. Andrunakievic: Radicals in associative rings, Mat. Sb., 44 (1958), 179-212.

[3] B. Brown and N. H. McCoy: The maximal regular ideal of a ring, Proc. Amer. Math. Soc., 1 (1950), 165-171.

[4] N. Divinsky: Rings and radicals, Univ. Toronto Press, 1965.

[5] A. G. Kurosh: Radicals of rings and algebras, Mat. Sb., 33 (1953), 13-26.

[6] J. von Neumann: On regular rings, Proc. Nat. Acad. Sci. U. S. A., 22 (1936), 707-713.

[7] F. Szász: Radikale der Ringe, Akadémiai Kiadó, Budapest, 1975.

[8] F. Szász: Beiträge zur Radikaltheorie der Ringe, Publ. Math. Debrecen, 17 (1970), 267-271.

[9] R. Wiegandt: Homomorphically closed semisimple classes, Studia Univ. Babeş-Bolyai Cluj, Ser. Math.-Mech., 12 (1972), 17-20.

[10] R. Wiegandt: Radical and semisimple classes of rings, Queen's Papers in Pure and Appl. Math., Kingston, 1974.

Mathematical Institute of Hungarian Academy of Science
Budapest V. Realtanoda u. 13-15
Hungary.