Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 35 (3—4), (1980), 399—402.

ON THE RADICAL CLASSES AND THE TRANSFREE-IMAGES OF RINGS

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1. The purpose of this note is to consider the relation between the transfree-images and the radical classes in category of associative rings. This stays in connection to the solution of problem 8 of [2].

The notion of transfree-images is dual to that of subdirect embedding (cf. [3]). An object A of the category $\mathscr{C}$ is said to be a transfree-image of the free product $\coprod_{i\in I} A_i(\varphi_i)$ if there exists an epimorphism $\gamma\colon \coprod_{i\in I} A_i(\varphi_i)\to A$ such that all maps $\gamma\colon\varrho_i\colon A_i\to A$, $i\in I$ are normal monomorphisms.

Instead of a transfree-image of the free product $\coprod_{i\in I} A_i(\varrho_i)$ we speak of a transfree-image of the objects A_i, $i\in I$.

A class M of rings is said to be a radical class in sense of Amitsur and Kurosh if the following conditions are satisfied:

(i) M is homomorphically closed.

(ii) The sum of all M-ideals of a ring A is an M-ideal.

(iii) M is closed under extensions, that is if B and $A/B\in M$ then also $A\in M$.

The lower radical class defined by the class M is the smallest radical class containing M.

2. Assume that the ring A is a transfree-image of rings A_i, $i\in I$, by an epimorphism γ. Following the definition all maps $\gamma\colon\varrho_i\colon A_i\to A$, $i\in I$ are normal monomorphisms, so they are embeddings and their images are ideals of the ring A.

Let M be an arbitrary abstract class of rings. We put

$$T_r(M)=\{A|A \text{ is a transfree-image of some } M\text{-rings}\}.$$

LEMMA 1. *Every nonzero $T_r(M)$-ring has a non-zero M-ideal.*

This statement follows immediately from the above remark.

DEFINITION. The class M is said to be closed under transfree-images if $T_r(M)=M$.

THEOREM 1. *For every class M of rings there always exists the smallest class $\overline{M}$ satisfying: $\overline{M}\supseteq M$ and $\overline{M}$ is closed under transfree-images.*

PROOF. The class of all rings is closed under transfree-images and it contains M. Since the intersection of classes being closed under transfree-images, is again such a class, we have $\overline{M}=\bigcap(M_\gamma|M_\gamma\supseteq M$ and M_γ is closed under transfree-images).

THEOREM 2. *If M is a homomorphically closed class of rings, then also the class $T_r(M)$ is homomorphically closed.*

Proof. Let A be a transfree-image of M-rings A_i, $i\in I$, by an epimorphism γ, and $\bar{A}$ a homomorphic image of A by a homomorphism f. Then the images $\bar{A}_i$ of each A_i, $i\in I$ are M-ideals in $\bar{A}$. Further by the universal property of the free product there exist uniquely determined mappings

$$\bar{\gamma}\colon \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) \to \bar{A} \quad \text{and} \quad \varphi\colon \coprod_{i\in I} A_i(\varrho_i) \to \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i)$$

such that the following diagrams are commutative

$$\begin{array}{ccc} \bar{A}_i & \xrightarrow{\bar{\varrho}_i} & \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) \\ & \searrow^{k_i} & \downarrow \bar{\gamma} \\ & & \bar{A} \end{array} , \qquad \begin{array}{ccc} A_i & \xrightarrow{\varrho_i} & \coprod_{i\in I} A_i(\varrho_i) \\ \downarrow f_i & & \downarrow \varphi \\ \bar{A}_i & \xrightarrow{\bar{\varrho}_i} & \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) \end{array}$$

where $f_i=f\cdot\varrho_i$ and k_i is an embedding. Clearly, the mappings $\bar{\gamma}\colon\bar{\varrho}_i$, $i\in I$ are normal monomorphisms. We can easily show that the square

$$\begin{array}{ccc} \coprod_{i\in I} A_i(\varrho_i) & \xrightarrow{\gamma} & A \\ \downarrow \varphi & & \downarrow f \\ \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) & \xrightarrow{\bar{\gamma}} & \bar{A} \end{array}$$

is commutative. Hence we have $\bar{\gamma}\varphi=f\cdot\gamma$. Since $f\cdot\gamma$ is an epimorphism, so is $\bar{\gamma}$. Thus $\bar{A}\in T_r(M)$ holds. The theorem is proved.

Theorem 3. *Every radical class is closed under transfree-images.*

Proof. Let M be a radical class and $A\in T_r(M)$. $M(A)$ denotes the sum of all M-ideals of A. Suppose $M(A)\neq A$. By Theorem 2 $\dfrac{A}{M(A)}$ is a nonzero $T_r(M)$-ring, and hence by Lemma 1 it contains a non-zero M-ideal $\dfrac{A'}{M(A)}$. By the condition (iii), A' is an M-ideal of A, so $A'\leqq M(A)$, a contradiction. Thus $A\in M$ holds. This completes the proof.

Corollary. *For every class M of rings the inclusion $T_r(M)\subseteqq\mathscr{L}(M)$ holds, where $\mathscr{L}(M)$ is the lower radical class defined by M.*

Let us consider the subclass of $T_r(M)$ defined as

$$T_{r_0}(M)=\left\{A \,\middle|\, \begin{array}{l} A \text{ is a transfree-image of some } M\text{-rings} \\ \text{by an epimorphism which is a surjection} \end{array}\right\}.$$

Lemma 2. *Assume that M is an abstract class of rings. The ring A belongs to $T_{r_0}(M)$ if and only if in A there exist M-ideals B_i, $i\in I$ such that $\sum_{i\in 1} B_i=A$.*

The proof is trivial.

THEOREM 4. *The class M of rings is a radical class if and only if the following conditions are satisfied:*
(A) *M is homomorphically closed.*
(B) *M is closed under transfree-images.*
(C) *M is closed under extensions.*

PROOF. Theorem 3 and the definition of radical yield the necessity.

For the sufficiency we only must show that condition (B) implies condition (ii). If condition (B) is valid, then $M \subseteqq T_{r_0}(M) \subseteqq T_r(M) = M$. By Lemma 2 it is clear that condition (ii) is satisfied.

Next, with the help of transfree-images we shall get a new construction which does give the lower radical. In order to do this we consider the following operator W acting on classes of rings by

$$W(M) = \left\{ A \,\middle|\, \frac{A}{B} \in M \text{ for some } M\text{-ideal } B \text{ of } A \right\}.$$

LEMMA 3. *If M is a homomorphically closed class then $W(M)$ is homomorphically closed, too.*

PROOF. Let A be in $W(M)$ and B any proper ideal of A. By the definition of $W(M)$ there exists an ideal C of A such that C and A/C both belong to M. Since the class M is homomorphically closed so we have

$$\frac{B+C}{B} \cong \frac{C}{B \cap C} \in M, \quad \frac{A}{B} \Big/ \frac{B+C}{B} \cong \frac{A}{B+C} \in M.$$

Thus A/B belongs to $W(M)$ and so $W(M)$ is homomorphically closed. The lemma is proved.

Now, let M be any class of rings. Define $K_1(M)$ to be the homomorphic closure of M. For every ordinal $\alpha > 1$, put

$$K_\alpha(M) = \begin{cases} T_r(K_{\alpha-1}(M)) & \text{if } \alpha \text{ is not a limit ordinal} \\ W(\bigcup\limits_{\beta<\alpha} K_\beta(M)) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Finally define $K(M) = \bigcup K_\alpha(M)$, where the union is taken over all ordinals α. Clearly, if α and β are ordinals with $\beta \leqq \alpha$ then $K_\alpha(M) \subseteqq K_\alpha(M)$.

LEMMA 4. *For every ordinal $\alpha \geqq 1$, $K(M)$ is homomorphically closed. Hence $K(M)$ is a homomorphically closed class.*

PROOF. $K_1(M)$ is homomorphically closed. Let $\alpha > 1$ be an ordinal and suppose $K_\beta(M)$ is homomorphically closed for all $\beta < \alpha$.

If α is not a limit ordinal, then by Theorem 2 and the induction hypothesis, $K_\alpha(M) = T_r(K_{\alpha-1}(M))$ is homomorphically closed.

Let α be a limit ordinal. Clearly, by the induction hypothesis, the class $\bigcup\limits_{\beta<\alpha} K_\beta(M)$ is homomorphically closed. So by Lemma 3, the class $K_\beta(M) = W(\bigcup\limits_{\beta<\alpha} K_\beta(M))$ is also homomorphically closed. Thus by transfinite induction $K_\alpha(M)$ is homomorphically closed for all ordinals α. It follows immediately that $K(M)$ is homomorphically closed.

THEOREM 5. $K(M)=\mathscr{L}(M)$.

PROOF. We use Theorem 4 to show that $K(M)$ is a radical class. By Lemma 4, the class $K(M)$ satisfies condition (A). Suppose that a ring A is a transfree-image of $K(M)$-rings A_i, $i\in I$. Then for $i\in I$ there exists an ordinal α_i such that $A_i\in K_{\alpha_i}(M)$. Let α be an ordinal greater than all α_i-s, $i\in I$. Hence every ring A_i, $i\in I$ belongs to $K_\alpha(M)$. So we have

$$A\in T_r(K_\alpha(M)) = K_{\alpha+1}(M) \subseteq K(M).$$

Thus, condition (B) is satisfied.

Now, let A have an ideal B such that both B and A/B are in $K(M)$. Then, there exist ordinals α_1 and α_2 such that $B\in K_{\alpha_1}(M)$, $A/B\in K_{\alpha_2}(M)$. We take a limit ordinal α greater than the ordinals α_i, $i=1, 2$. Then both B and A/B belong to the class $\bigcup\limits_{\beta<\alpha} K_\beta(M)$. So we have

$$A\in W\Big(\bigcup_{\beta<\alpha} K_\beta(M)\Big) = K_\alpha(M) \subseteq K(M).$$

Hence condition (C) is valid. Thus $K(M)$ is a radical class.

By the minimality of $\mathscr{L}$ among radical classes containing M, it is enough to show $K(M)\subseteq\mathscr{L}(M)$. This is accomplished by proving $K_\alpha(M)\subseteq\mathscr{L}(M)$ for every ordinal.

Clearly, $K_1(M)\subseteq\mathscr{L}(M)$. Let α be an ordinal exceeding one, and assume $K_\beta(M)\subseteq\mathscr{L}(M)$ for all ordinals $\beta<\alpha$. Suppose $A\in K_\alpha(M)$.

If α is not a limit ordinal, then we have

$$A\in K_\alpha(M) = T_r(K_{\alpha-1}(M)) \subseteq T_r(\mathscr{L}(M)) = \mathscr{L}(M).$$

Let α be a limit ordinal, then by the definition we have

$$A\in K_\alpha(M) = W\Big(\bigcup_{\beta<\alpha} K_\beta(M)\Big).$$

Therefore there exists an ideal B in A such that both B and A/B belong to $\bigcup\limits_{\beta<\alpha} K_\beta(M)$. By the induction hypothesis, it is clear that $\bigcup\limits_{\beta<\alpha} K_\beta(M)\subseteq\mathscr{L}(M)$. From this $A\in\mathscr{L}(M)$ follows by condition (C) of Theorem 4, and the theorem is proved.

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(Received October 19, 1978)

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