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In this paper connections between certain radicals of particular rings are considered. In this context a less known radical, namely the Fuchsian zeroid pseudo-radical $F(A)$ and its generalization studied by the second author will play an important role. For its importance, and even for the sake of completeness, we present in the first half of the paper known results mostly done by the second author. New results are given concerning MHR-rings (rings not necessarily having 1 and satisfying the minimum condition for principal right ideals). Important examples are given with the aim of differentiating the Fuchsian zeroid pseudo-radical with other radicals.

DEFINITION 1. Generalization of the Fuchs zeroid radical (after F. Szász [11]): Let L be a complete lattice with minimal element 0 and maximal element 1. Moreover let S be a *semifilter* or *upper subset* of L, defined by:

(1) if $x\in S$ and $x\leqq y$, then also $y\in S$.

DEFINITION 2. An upper subset $S\subseteq L$ is called of *finite character*, shortly an *f-upper subset*, if

(2) $\bigvee_{\gamma\in\Gamma} X_\gamma\in S$ implies the existence of a finite subset Γ^* of Γ such that already $\bigvee_{\gamma\in\Gamma^*} X_\gamma\in S$ holds.

PROPOSITION 3. *The union of an arbitrary family of f-upper subsets is again an f-upper subset of L.*

PROPOSITION 4. *The intersection of a finite number of f-upper subsets is again an f-upper subset of L.*

DEFINITION 5. Let $x\in L$, but $x\notin S$, for an f-upper subset S of L. Then $y\in L$ is called an *(S, x)-element,* if one has the relation $x\vee y\notin S$.

DEFINITION 6. An element $z\in L$ with $z\notin S$ is said to be a *strong* (S, x)-*element,* if $z\vee y\notin S$ always holds, provided that y is an arbitrary (S, x)-element.

PROPOSITION 7. *Every strong (S, x)-element is also an (S, x)-element.*

THEOREM 8. *The complete union $Z_s(x)$ of all strong (S, x)-elements is again a strong (S, x)-element $Z_s(x)$ of L, which must coincide with the intersection of all maximal (S, x)-elements.*

PROOF. Follows from F. Szász's book [14] (see Theorem 45.1).

REMARKS (9.1) It is trivial that the union of a finite number of strong (S, x) -elements is again a strong (S, x)-element.

(9.2) In a complete lattice $x \leq y$ iff there exists an x^* such that $x \vee x^* = y$ holds, and $\leq$ is a partial ordering. The word "maximal" in Theorem 8 must be understood in the sense of this ordering$\leq$.

DEFINITION 10. If $\mathscr{M}$ is an arbitrary family of f-upper subsets of L, then $z_{\mathscr{M}}(x) = \bigwedge_{S_\alpha \in \mathscr{M}} z_{S_\alpha}(x)$ is called the *$\mathscr{M}$-zeroid pseudo-radical* of $x \in L$ with $x \notin S_\alpha$ for every $S_\alpha \in \mathscr{M}$.

DEFINITION 11. If $x \in S_\alpha$ for every $S_\alpha \in \mathscr{M}$, put $z_{\mathscr{M}}(x) = 1$.

DEFINITION 12. Let L be a *lattice ordered groupoid* satisfying for every triple a, b, c of elements of L:

(a) $a . b \leq a \wedge b$

(b) $(a \vee b)c \leq ac \vee bc$, furthermore

(c) $a(b \vee c) \leq ab \vee ac$.

REMARK 13. Our system of axioms (a), (b) and (c) is a little different from the usual one for lattice ordered groupoids (see G. Birkhoff [2]), but for our purpose the axioms (a), (b) and (c) are the most useful and suitable.

DEFINITION 14. A subset S of L is said to be an *f-subgroupoid* of the lattice ordered groupoid, if the following properties hold:

(d) S is an f-upper subset of $(L, \leq)$.

(e) S is a subgroupoid of $(L, .)$.

DEFINITION 15. The empty set ϕ is always considered as an f-subgroupoid of $(L, \leq, .)$.

PROPOSITION 16. *Every f-subgroupoid S is a filter of L.* (see Kowalsky [8]).

PROPOSITION 17. *The set theoretical union of an arbitrary ascending chain of f-subgroupoids of $(L, \leq, .)$ is again an f-subgroupoid of L.*

PROPOSITION 18. *The intersection of a finite number of f-subgroupoids of L is again an f-subgroupoid of* $L(\leq, .)$.

DEFINITION 19. Let $\mathscr{N}$ be an arbitrary family of f-subgroupoids S_α of L. Put

$$Z_{\mathscr{N}}(x) = \bigwedge_{S_\alpha \in \mathscr{N}} Z_{S_\alpha}(x). \tag{20}$$

In (20) the element $Z_{\mathscr{N}}(x)$ is called the *algebraic* $\mathscr{N}$*-zeroid pseudo-radical* of $x \in L$ with $x \notin S_\alpha$ for every $S_\alpha \in \mathscr{N}$.

THEOREM 21. *The element* $Z_{\mathscr{N}}(x)$ *is always a complete intersection of prime elements.*

PROOF. Follows from Szász's book [14] (Theorem 45. 2).

REMARK 22. As well known, the element p of L is called *prime*, if $p \leq ab$ always implies $p \leq a$ or $p \leq b$ (or eventually also $p \leq a \vee b$ holds; in some cases of $a, b \in L$).

THEOREM 23. *Let L be the complete lattice of all two-sided ideals of a not necessarily associative ring A. Then for* $\mathscr{T} \in L$ *we have* $\mathscr{T} = Z_{\mathscr{N}}(0)$ *if and only if the factoring* $A/\mathscr{T}$ *is semiprime.* (see Mary Gray [6]).

REMARK 24. Our ring A does not generally have a unity element 1.

PROOF. (of Theorem 23) follows from Theorem 45. 3 of [14].

DEFINITION 25. By a *weak block* of a (not necessarily associative) ring A we understand a subset B such that $\mathscr{T}_1 \lhd A$, $\mathscr{T}_2 \lhd A$, $\mathscr{T}_1 \cap B \neq \phi$ and $\mathscr{T}_2 \cap B \neq \phi$ always imply $\mathscr{T}_1 . \mathscr{T}_2 \cap B \neq \phi \neq \mathscr{T}_2 . \mathscr{T}_1 \cap B$.

PROPOSITION 26. *Let L be the lattice of all two-sided ideals of the ring A and B be a weak block of A. Then the ideals* $\mathscr{T}_\alpha \lhd A$ *satisfying* $\mathscr{T}_\alpha \cap B \neq \phi$ *form an f-subgroupoid of L.*

PROOF. It is almost trivial.

DEFINITION 27. An arbitrary subset B of the ring A is called a *block* of A, if

(g) B is a weak block of A,

(h) $x \in B$ implies $x^n \in B$ for every positive exponent $n \in Z$.

PROPOSITION 28. *The complementary subset* $C(P)$ *of every prime ideal* P *of the ring A always is a weak block A.*

PROOF. It is trivial by the definition.

PROPOSITION 29. *If B is a weak block of the ring A such that its complementary subset $C(B)$ is an ideal of A, then $C(B)$ is a prime ideal of A.*

PROPOSITION 30. *Every multiplicative subgroupoid M satisfying $0\notin M$ of the multiplicative groupoid of the ring A is a block of A.*

REMARK 31. If F is a commutative and associative field, and x and y are noncommutative indeterminates over F, then one can find in the skew polynomial ring $F[x, y]$ some blocks, which are not subsemigroups M with relation $0\notin M$.

REMARK 32. Let D be a division ring, $n\in Z$, $n\geqq 2$, and $D_n=A$ be the total matrix ring over D of type $n\times n$. Then $A=D_n$ contains the subset B, which is a weak block, but B is not a block of A.

THEOREM 33. *The upper Baer nil radical* [3] *is always contained in $Z_{\mathscr{N}}(0)$, provided $S_\alpha\in\mathscr{N}$, and $\mathscr{T}\in S_\alpha$ with property $\mathscr{T}_\alpha\cap B_\gamma\neq\phi$ for some blocks B_γ of the ring A.*

PROOF. It follows from Theorem 45.4 of [14].

EXAMPLE 34.1. Let A be an associative ring, generally without unity element 1, M be a right A-module, H_1 and H_2 be subsets of M and N be a non-zero submodule of M. Then the subset $S_1=[x;\ x\in A,\ N_x=N]$ is a multiplicative subsemigroup of (A_j) with $0\notin S_1$.

EXAMPLE 34.2. With the above notation, $S_2=[x;\ x\in A,\ nx=n$ for every $n\in N]$ is again a multiplicative subsemigroup with $0\notin S_2$.

EXAMPLE 34.3. Put $S_3=[x:\ x\in A,\ H_2x\subseteq H_1,\ H_1\subsetneqq H_2]$. Then $x, y\in S_3$ imply $xy\in S_3$ with $0\notin S_3$.

EXAMPLE 34.4. Put $S_4=[x:\ x\in A,\ H_2x\subsetneqq H_1,\ H_1\subsetneqq H_2]$. Then $0\notin S_4$ and S_4 is a multiplicative subsemigroup of (A_j).

EXAMPLE 35. The left units (i.e. elements, which are left divisors of every element of the associative ring A) form a multiplicative subsemigroup S_5 with $0\notin S_5$.

EXAMPLE 36. The left-right dual of S_5 is S_6.

EXAMPLE 37. The left unity elements 1_l of A form a multiplicative subsemigroup S_7 with property $0 \notin S_7$.

EXAMPLE 38. The left-right dual of S_7 is S_8.

EXAMPLE 39. Every non-zero idempotent $e=e^2 \in A$ (with $e \neq 0$) forms alone a semigroup $S_9=[e]$ with $0 \notin S_9$.

EXAMPLE 40. The subset of all non left divisors of zero forms a subsemigroup $S_{10}=S_l$ with condition $0 \notin S_{10}$.

EXAMPLE 41. The left-right dual of S_{10} is S_{11}.

DEFINITION 42. Let A be an associative ring; put $K_r=[\mathscr{T};\ \mathscr{T} \lhd A,\ \mathscr{T} \cap S_{10} \neq \phi]$, $K_l=[\mathscr{T}:\ \mathscr{T} \lhd A,\ \mathscr{T} \lhd S_{11} \neq \phi]$, and $\mathscr{N}=[K_r, K_l]$. Then the ideal: (43) $Z_{\mathscr{N}}(0)=Z(A)$ is called the *zeroid pseudo-radical* of the ring A in sense of L. Fuchs [5].

DEFINITION 44. For $\mathscr{T} \lhd A$, the ideal $Z_{\mathscr{N}}(\mathscr{T})$ is called the *radical* of $\mathscr{T}$ in the sense of van Leuwen [9].

THEOREM 45. *If A has no non-zero divisors of zero, then $Z(A)=0$ holds.*

PROOF. It follows from Definition 44.

THEOREM 46. *The Fuchsian zeroid pseudo-radical $F(A)$ of an arbitrary associative ring is always an intersection of some prime ideals.*

PROOF. It follows from 42, 43 and Theorem 23.

COROLLARY 47. *The factor ring $A/F(A)$ is semiprime.*

PROOF. It follows immediately from Theorem 4.

We have yet the sharper.

THEOREM 48. *In the factor-ring $A/F(A)$ the ideal $\overline{0}=F(A)/F(A)$ is the only nil ideal.*

PROOF. Let $C/F(A)$ be a nil ideal of $A/F(A)$ and $B \lhd A$. Then we obviously have for every $c \in C$ the relation $c^k \in F(A)$ for some $k \geqq 1$, $k \in Z$. Thus with every element $b \in B$ holds:

$$(b+c)^k \in F(A)+B, \tag{49}$$

by the non-commutative analogue of Newton's binomial expansion for $(b+c)^k$. By our definition, every element of $F(A)+B$ is a left divisor of zero, consequently $b+c$ always also is a left divisor of zero, therefore $C\sqsubseteq F(A)$, since we can similarly verify that every element of $C+D$ is also a right divisor of zero, provided that every element of $D\lhd A$ is a right divisor of zero in the ring A.

THEOREM 50. *If A is an arbitrary MHR-ring with a right unity element $1_r=e$, then $F(A)=G(A)$ holds, where $G(A)$ is the Brown-McCoy radical of A.* (By its importance we repeat the proof.)

PROOF. If $C\lhd A$ contains a non-left-divisor of zero $c\in C$, then let $(c_0)_r$ be a minimal right ideal among these principal right ideals $(c)_r$. Then by definition:

$$(c_0)_r=(c_0^2)_r \tag{51}$$

holds. Now (51) implies, by $e=1_r\in A$, an equation $c_0=c_0^2 \cdot d$ with some $d\in A$. Hence $c_0 \cdot (c_0 \cdot d-1_r)=0$ implies by our assumption on c_0 evidently $c_0 \cdot d_r=1_r\in C$; consequently $x=x.1_r\in C$ for every $x\in A$. Therefore $C=A$. In other words, if $C\neq A$, then every element of C is a left divisor of zero. Thus the intersection of all proper two-sided ideals coincides with the ideal $Z_l(0)=Z_{\mathscr{N}}(0)=G(A)$, where $G(A)$ is the Brown-McCoy radical, $\mathscr{N}_1=[S_B]$, the set having only the element S_B, and $S_B=[\mathscr{T}:\ \mathscr{T}\lhd A,\ \mathscr{T}\cap B\neq\phi]=S_l=S_{11}$, the semigroup defined in Example 40. Put $Z_r(0)=Z_{\mathscr{N}_2}(0)$, where $\mathscr{N}_2=[S_C]$, $S_C=[\mathscr{T}:\ \mathscr{T}\lhd A, \mathscr{T}\cap C\neq\phi]$ $=S_r=S_{10}$, the semigroup defined in Example 41, and here B and C are blocks in sense of Definition 27. Thus we have, by Theorem 33, $U(A)\sqsubseteq Z_r(0)\ Z_l(0)$ $=F_r(A)\ F_l(A)=F(A)$. Moreover, by F. Szász's paper [21], part II, on MHR-rings, one has $U(A)=G(A)$, since $l_r\in A$, and hence

$$G(A)=U(A)\sqsubseteq F(A)\sqsubseteq Z_l(0)=G(A). \tag{52}$$

THEOREM 53. *If A is a von Neumann regular ring with two-sided unity element 1, then $F(A)=G(A)$ holds.*

PROOF. If $\mathscr{T}\lhd A$ and $\mathscr{T}\neq A$, $i\in\mathscr{T}$, then by $i\in iAi$ there exists an element $j\in A$ such that $i=iji$. Now $\mathscr{T}\neq A$ implies $1\notin\mathscr{T}$, and thus $ji-1\notin\mathscr{T}$ and $ij-1\notin\mathscr{T}$. Moreover we have

$$i(ji-1)=iji-i=0=(ij-1)i, \tag{54}$$

which completes the proof.

REMARK 55. By Example 57 to follow, there exists a ring A such that $F(A)$ has the cardinality $\aleph_\alpha$, where $\aleph_\alpha$ is an arbitrary cardinal number and A is at the same time also a von Neumann regular ring with two-sided unity element.

REMARK 56. The existence of a ring A with properties pointed out in Remark 48, disproves a false proposition of article 6 of a paper of L. Fuchs [5]. Thus our ring also shows that our generalized zeroid pseudo-radicals are generally different from $Z(0)=F(A)$ and from the van Leeuwen's zeroid pseudo-radical $Z(\mathscr{T})$ of an ideal $\mathscr{T}\lhd A$ of the ring A [9].

EXAMPLE 57. Let D be a division ring of arbitrary cardinality, V be a left D-vector space of dimension $\aleph_\alpha$, where $\aleph_\alpha$ is fixed, but arbitrarily given. Moreover, let A be the ring of all linear transformations of V i.e., A is the ring of all D-endomorphisms of the left D-module V, which can be also considered as a right A-module. Thus V is a (D, A)-bimodule. By a generalization of F. Szász [20], our ring A must be von Neumann regular. Furthermore, we evidently have $1\in A$ and almost trivially $|F(A)|=\aleph_\alpha$, where $|X|$ denotes the cardinality of a set X.

Moreover, put

$$\mathscr{T}_\nu=[a;\ a\in A,\ \dim_D(V, a)<\aleph_\nu] \tag{58}$$

This yields a chain $(0)\subsetneqq\mathscr{T}_1\subsetneqq\mathscr{T}_2\subsetneqq\cdots\subsetneqq A$ with $\mathscr{T}_\nu\lhd A$ for arbitrary ordinal number ν. Now we consider the following f-upper subsemigroups of the semigroup $S^*=[(0),\ \mathscr{T}_0,\ \mathscr{T}_1,\ \cdots,\ \mathscr{T}_\alpha,\ A]=K_{-1}$:

$$\begin{aligned}
K_0&=[\mathscr{T}_0,\ \mathscr{T}_1,\ \cdots,\ \mathscr{T}_\alpha,\ A]\\
K_1&=\quad\ [\mathscr{T}_1,\ \cdots,\ \mathscr{T}_\alpha,\ A]\\
&\ \vdots\\
K_\alpha&=[\mathscr{T}_\alpha,\ A]\\
K_{\alpha+1}&=[A]\\
K_{\alpha+2}&=\phi.
\end{aligned}$$

Then $S^{**}=[K_{-1},\ K_0,\ K_1,\ K_{\alpha+2}]$ is an upper f-semigroup, and the corresponding zeroid pseudo-radical satisfies obviously $0\neq F(A)=G(A)=Z_{K_{\alpha+1}}(0)=\aleph_\alpha\neq Z_{S^{**}}(0)$, and $|F(A)|=\aleph_\nu$, too. This completes the proof.

DEFINITION 59. Let $a_1,\ a_2,\ \cdots,\ a_m$ be left divisors of zero of the ring A i.e. there exist elements $0\neq b_i\in A$ such that

$$a_1b_1=a_2b_2=\cdots=a_mb_n=0 \tag{60}$$

holds. If there exists at least one element $0\neq b\in A$ such that already

$$a_1b=a_2b=\cdots=a_mb=0 \tag{61}$$

holds, then A is called a ***right uniform ring.***

DEFINITION 62. Similarly, by left-right duality, also the left uniform rings can be introduced. If A is both left and right uniform, then A is said to be a *uniform ring.*

PROPOSITION 63. *Let for $n\geqq 2$, $n\in Z$, the ring A_n be the set of all matrices of type $n\times n$ over A. Put $\mathscr{T}\lhd A_n$, where A is a uniform ring. Then every element x of $\mathscr{T}$ is left divisor of zero in A_n if and only if there exists an ideal $K\subset A$ such that $A_n\subseteq K_n$ holds, where every element of K is a left divisor of zero in A.*

PROOF. It follows from 45.9 of [14].

THEOREM 64. *If the ring A is uniform, then we have*

$$F(A_n)=(F(A))_n \text{ for every } n\geqq 1,\ n\in Z. \tag{65}$$

PROOF. It follows from Proposition 63 and from Theorem 45.10 of [14].

EXAMPLE 65. Every ring with a non-zero left-annihilator is a left uniform ring.

REMARK 66. Every homomorphic image of the ring A has no non-zero left annihilator if and only if $a\in aA+AaA$ holds for every element $a\in A$ (see [16], [17], [18], [19]). A is an E_5 of the sense of Szász.

REMARK 67. Every ring having non non-zero divisors of zero is both left uniform and right uniform.

PROBLEM 68. Construct a ring A such that $F(A/F(A))\neq 0$ holds.

PROBLEM 69. Construct a left uniform ring A which is not right uniform.

PROBLEM 70. Does $F(A_n)=(F(A))_n$ hold for every left uniform ring A?

DEFINITION 71. The *generalized nil radical* N_g [3] is the special (upper) radical, determined by the class of all rings without non-zero divisors of zero.

DEFINITION 72. The *Thierrin corpoidal radical* T is the special (upper) radical determined by the class of all division rings.

PROPOSITION 73. *$T(A)\supseteq N_g$ always holds for any ring A.*

PROOF. It follows from Definitions 63 and 64.

DEFINITION 74. Let S be the upper radical determined by the class of all simple MHR-rings without divisors of zero.

PROPOSITION 75. *If for every maximal completely prime ideal $C_\alpha \lhd A$, (see [11]) of a ring A, A/C_α has a non-zero idempotent $\bar{e}=e+C$, $e\notin C$ and $e^2-e\in C_\alpha$, C being the heart of A, then $T(A)\supseteq S(A)$ and $S(A)\supsetneqq \bigcap_\alpha C_\alpha$.*

PROOF. $e^2\equiv e \pmod{C}$ implies $e^2x\equiv ex \pmod{C}$, $e(ex-x)\in C$, $e\notin C$. Hence, $ex\equiv x \pmod{C}$ for every $x\in A$. But then we have $yex\equiv yx \pmod{C}$ for every $y\in A$. Thus $(ye-y)x\in C$, for all $x, y\in A$. Consequently $ye\equiv y \pmod{C}$. Therefore $\bar{e}=1\in A/C$, and therefore, $T(A)\supseteq S(A)\supsetneqq \bigcap(C_\alpha)=V(A)$.

PROPOSITION 76. *$V(A)$ coincides with the special (upper) radical determined by the class of all rings with 1, but without non-zero divisors.*

PROOF. Clear (see [3]).

DEFINITION 77. A ring A is called *reduced*, if A has no non-zero nilpotent elements.

REMARKS 78.1. A non-zero nilpotent ring is not reduced, but every ring without non-zero divisors of zero is reduced.

REMARK 78.2. By a result, independently obtained by V.A. Andrunakievich —Yu.M. Rjabuhin [1] and P.N. Stewart [13], A is reduced if and only if $0=\bigcap_\alpha C_\alpha$ where $C_\alpha \lhd A$ are completely prime ideals.

THEOREM 79. [1]. *For a ring A the following are equivalent:*

(1) *Every homomorphic image B of A is reduced.*

(2) *$(a)=(a^2)$ holds for every principal two-sided ideal (a) of A.*

(3) *$(a)\cdot(b)=(ab)=(a)\cap(b)$ hold for every pair of elements $a, b\in A$.*

(4) *$a\in a^2A+Aa^2+Aa^2A$, for every $a\in A$.*

(5) *Every ideal is an intersection of some completely prime ideals.*

PROOF. Apply Zorn's Lemma [1].

THEOREM 80. *Let A be an MHR-ring such that every subdirectly irreducible homomorphic image B of A is without non-zero divisors of zero, then $G(A)=T(A)=Be(A)$, where $Be(A)$ denotes the Behrens radical of the ring A.*

PROOF. By the above $1\in C$, where C is the heart (i.e. the non-zero intersection of all non-zero two-sided ideals) of B. But then, as well known, $B=C\oplus D$ holds by a two-sided Pierce decomposition of B with the central idempotent element 1 of B. Now, B being subdirectly irreducible, $C=B$ holds; whence $G(A)=T(A)=Be(A)$, by F. Szász's paper [21] (MHR-rings, II) namely every simple MHR-ring without non-zero divisors of zero is a division ring.

The following new results can also be easily proved.

PROPOSITION 81. *For every ring A satisfying $A\neq F(A)$ and*

(82) $A=\sum_{m=1}^{n}\oplus A_n$*(where the rings A_i are arbitrary, but the number of direct summands is finite) holds $F(A)=\Sigma\oplus F(A_n)$.*

PROPOSITION 83. *If $A=\sum_{\gamma\in\Gamma}\oplus A_\gamma$ for arbitrary infinite set Γ of indices γ, then one has $F(A)=A$.*

PROPOSITION 84. *If $A=B\oplus\sum_{\gamma\in\Gamma}\oplus C_\gamma$ satisfying $F(B)=B$, and C_γ are arbitrary rings, then $F(A)=A$.*

THEOREM 85. *The homomorphic closure $\overline{C}$ of the class C of all zeroid pseudo-radical rings $A=F(A)$ consists of all associative rings.*

PROOF. Put an arbitrary ring $A\in\overline{C}$, and an arbitrary ring B such that $B=F(B)\in C$. Then, by Proposition 75, we obtain $D=A\oplus B\in C$, and $A\cong D/B\in\overline{C}$ *holds, which completes the proof.*

THEOREM 86. *The Amitsur-Kurosh lower radical class $L(C)$ determined by the class C of all Fuchsian zeroid pseudo-radical rings, consists of all associative rings.*

PROOF. $L(C)\supseteq\overline{C}$, and Theorem 85 can be applied to C.

PROBLEM 87. *Let us find a one-sided analogue of our Theorem 72.*

REMARK 88. In a strong connection with Remark 57, let us finally yet point out, that by F. Szász [19] a ring A such that every homomorphic image has no non-zero two-sided annihilators satisfies $a\in aA+Aa+AaA$ for every $a\in A$, and conversely. These rings form the class of all E_6-rings, which is bigger than that of E_5-rings. By [18] and [19] the class of all E_5-rings, and the class of all E_6-rings, are Amitsur-Kurosh radical classes, satisfying

$$C(E_5)\neq C(E_6)$$

EXAMPLE 89. Let A be the finite commutative ring $A=\{a\}$, generated by a single element $a\in A$ such that $a+a(=2a)=a^3-a^2=a^3+a^2=0$ are valid. Then one has $A=\{a^2\}\oplus\{a^2-a\}$ is a ring theoretical direct sum (applying a Pierce decomposition of $\{a\}$ by $(a^2)^2=a^3=a^2$). Thus, we have, by Proposition 75, evidently $F(A)=A$ with $|A|=4$, but $(A)=\{a^2-a\}$, $((A))^2=0$, $|(A)|=2$, whence $F(A)=A\neq(A)\neq 0$ even in case $|A|=4$.

EXAMPLE 90. Let Q be the set of all rational numbers r/s with $r,\ s\in Z,\ s\neq 0$. Let A be the subset of all rational numbers $a=\frac{2k}{2l+1}$, where $k,\ l\in Z$. Then A is, with the usual addition and multiplication a ring, which is a Jacobson radical ring, since $A=\mathscr{T}(A)$ holds. On the other side A does not contain non-zero divisors of zero, since it can be embedded into the field Q. But therefore $F(A)=0$, consequently even in case $|A|=\aleph_0$ can occur also the situation

$$0=F(A)\neq A=J(A)=L(C(A)),$$

where C is the class of all Fuchsian zeroid pseudo-radical rings.

PROBLEM 91. Does $G(A)=F(A)$ always hold for every MHR-ring A, and for the Brown-McCoy radical $G(A)$ of the ring A?

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