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A topology on the class of unequivocal rings

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The fundamental notions used in this note can be found in [3] and [6]. All rings considered here will be associative. By a radical of a ring we mean a radical in the sense of AMITSUR and KUROSH. As it is well known, a nonzero ring A is called unequivocal, if for any radical $\boldsymbol{R}$, A is either $\boldsymbol{R}$-radical or $\boldsymbol{R}$-semisimple. The class $\boldsymbol{X}$ of unequivocal rings was studied by N. DIVINSKY [2]; it contains properly the class of nonzero simple rings.

The aim of this note is to show that the KUROSH lower radical construction as well as the semisimple class corresponding to the upper radical construction define topological closure operators on sets of unequivocal rings. The class of all unequivocal rings can be represented by the union of these topological spaces.

Let $\boldsymbol{L}$ be a nonempty class of rings. $\boldsymbol{L}_1$ denotes the homomorphic closure of $\boldsymbol{L}$;

$\boldsymbol{L}_\nu = \{A|$ every nonzero homomorphic image of A has a nonzero $\boldsymbol{L}_\varkappa$-ideal for some $\varkappa < \nu\}$

for $\nu = 2, 3, \ldots, \omega_0$. Occasionally we write $L_\nu = L_\nu(\boldsymbol{L})$. Then $L(\boldsymbol{L}) = \boldsymbol{L}_{\omega_0}$ is the lower radical class defined by $\boldsymbol{L}$ and this is the smallest radical class containing the class $\boldsymbol{L}$ (see [6]).

We shall use the following result of LEAVITT [4]:

Proposition 1. *Let $\{\boldsymbol{R}_i\}$ $(i \in I)$ be a set of radical classes. Then the relation*

$$S\Big(L\Big(\bigcup_{i\in I} \boldsymbol{R}_i\Big)\Big) = \bigcap_{i\in I} S(\boldsymbol{R}_i)$$

holds, where $S(\boldsymbol{R}_i)$ denotes the class of all $\boldsymbol{R}_i$-semisimple rings, $i \in I$.

Let $\boldsymbol{X}$ be the class of all unequivocal rings and let $\boldsymbol{X}'$ be a representative subclass of $\boldsymbol{X}$, i.e. $\forall A \in \boldsymbol{X}\ \exists\ !A' \in \boldsymbol{X}'$, $A \cong A'$. For every cardinal number α we put

$\boldsymbol{X}_\alpha = \{A \in \boldsymbol{X}' \mid$ the cardinal number of A is not larger than $\alpha\}$.

It is clear that $\boldsymbol{X}_\alpha$ is a set.

Let us define the operator P acting on the set of all subsets of $\boldsymbol{X}_\alpha$ as follows. Put

$$P(\mathbf{S}) = \begin{cases} \Phi & \text{if} \quad \mathbf{S} = \Phi \\ \boldsymbol{X}_\alpha \cap L(\mathbf{S}) & \text{if} \quad \mathbf{S} \neq \Phi \end{cases}$$

for every $\mathbf{S} \subseteq \boldsymbol{X}_\alpha$ (Φ denotes the empty set).

Theorem 1. *P is a topological closure operator on $\boldsymbol{X}_\alpha$, which preserves even arbitrary unions. That is, P satisfies the following properties:*

1) $$P(\Phi) = \Phi$$

2) $$P(\boldsymbol{S}) \supseteqq \boldsymbol{S}$$

3) $$P\big(P(\boldsymbol{S})\big) = P(\boldsymbol{S})$$

4) $$P\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) = \bigcup_{i\in I} P(\boldsymbol{S}_i) \qquad (I \text{ is any index set}).$$

Proof. The properties 1) and 2) are obvious, thus we first establish 3).

$$P\big(P(\boldsymbol{S})\big) = \boldsymbol{X}_\alpha \cap L\big(P(\boldsymbol{S})\big) = \boldsymbol{X}_\alpha \cap L\big(\boldsymbol{X}_\alpha \cap L(\boldsymbol{S})\big).$$

Since $L\big(\boldsymbol{X}_\alpha \cap L(\boldsymbol{S})\big) \subseteqq L\big(L(\boldsymbol{S})\big) = L(\boldsymbol{S})$, we have

$$P\big(P(\boldsymbol{S})\big) \subseteqq \boldsymbol{X}_\alpha \cap L(\boldsymbol{S}) = P(\boldsymbol{S}).$$

By 2) we have $P\big(P(\boldsymbol{S})\big) \supseteqq P(\boldsymbol{S})$. Therefore $P\big(P(\boldsymbol{S})\big) = P(\boldsymbol{S})$ holds.
To establish 4) first of all we prove the following equality:

(1) $$L\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) = L\Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big).$$

Since $\bigcup_{i\in I} \boldsymbol{S}_i \subseteqq \bigcup_{i\in I} L(\boldsymbol{S}_i) \subseteqq L\Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big)$, by the property of the lower radical we have

$$L\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) \subseteqq L\Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big).$$

On the other hand

$$\bigcup_{i\in I} \boldsymbol{S}_i \supseteqq \boldsymbol{S}_j, \qquad j \in I;$$

so

$$L\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) \supseteqq L(\boldsymbol{S}_j), \qquad j \in I.$$

Therefore we have $L\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) \supseteqq \bigcup_{i\in I} L(\boldsymbol{S}_i)$. Again, by the property of the lower radical, the inclusion $L\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) \supseteqq L\Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big)$ holds, and the equality (1) is proved.

By the definition of the operator P we have

$$\bigcup_{i\in I} P(\boldsymbol{S}) = \bigcup_{i\in I} \big(\boldsymbol{X}_\alpha \cap L(\boldsymbol{S}_i)\big) = \boldsymbol{X}_\alpha \cap \Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big).$$

On the other hand we have

$$\boldsymbol{X}_\alpha \cap \Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big) \subseteqq \boldsymbol{X}_\alpha \cap L\Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big) = \boldsymbol{X}_\alpha \cap L\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) = P\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big).$$

Therefore the inclusion $\bigcup_{i\in I} P(\boldsymbol{S}_i) \subseteqq P\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big)$ holds. By Proposition 1 and equality (1) we have

$$SL\Big(\bigcup_{i\in I} \boldsymbol{S}_i\Big) = SL\Big(\bigcup_{i\in I} L(\boldsymbol{S}_i)\Big) = \bigcap_{i\in I} SL(\boldsymbol{S}_i).$$

Hence if $A \in \boldsymbol{X}_\alpha$ and $A \notin \bigcup_{i \in I} L(\boldsymbol{S}_i)$, then $A \in SL(\boldsymbol{S}_i)$ $(i \in I)$ since A is an unequivocal ring. We have $A \in \bigcap_{i \in I} SL(\boldsymbol{S}_i) = SL\left(\bigcup_{i \in I} \boldsymbol{S}_i\right)$ so $A \notin L\left(\bigcup_{i \in I} \boldsymbol{S}_i\right)$. From this follows

$$\bigcup_{i \in I} P(\boldsymbol{S}_i) = \boldsymbol{X}_\alpha \cap \left(\bigcup_{i \in I} L(\boldsymbol{S}_i)\right) \supseteq \boldsymbol{X}_\alpha \cap L\left(\bigcup_{i \in I} \boldsymbol{S}_i\right) = P\left(\bigcup_{i \in I} \boldsymbol{S}_i\right).$$

Therefore $P\left(\bigcup_{i \in I} \boldsymbol{S}_i\right) = \bigcup_{i \in I} P(\boldsymbol{S}_i)$. The theorem is proved.

Remark. The above theorem is also valid for any set $\boldsymbol{Y}$ of unequivocal rings, instead of $\boldsymbol{X}_\alpha$.

The operator P defines a topology on the set $\boldsymbol{X}_\alpha$. We shall denote this topology by G_α.

Proposition 2. *A set $\boldsymbol{S} \subset \boldsymbol{X}_\alpha$ is G_α-closed if and only if $\boldsymbol{S} = \boldsymbol{X}_\alpha \cap \boldsymbol{R}$, where $\boldsymbol{R}$ is some radical class.*

Proof. Let $\boldsymbol{S} \subset \boldsymbol{X}_\alpha$ be a G_α-closed set, then $\boldsymbol{S} = P(\boldsymbol{S}) = \boldsymbol{X}_\alpha \cap L(\boldsymbol{S})$. We take $\boldsymbol{R} = L(\boldsymbol{S})$.

Conversely let $\boldsymbol{S} = \boldsymbol{X}_\alpha \cap \boldsymbol{R}$. Since $\boldsymbol{X}_\alpha \cap \boldsymbol{R} \subseteq \boldsymbol{R}$ and R is a radical class, we have $L(\boldsymbol{X}_\alpha \cap \boldsymbol{R}) \subseteq \boldsymbol{R}$ and

$$P(\boldsymbol{S}) = \boldsymbol{X}_\alpha \cap L(\boldsymbol{X}_\alpha \cap \boldsymbol{R}) \subseteq \boldsymbol{X}_\alpha \cap \boldsymbol{R} = \boldsymbol{S}.$$

Thus $P(\boldsymbol{S}) = \boldsymbol{S}$ follows and $\boldsymbol{S}$ is a G_α-closed set.

Proposition 3. *A set $\boldsymbol{G} \subseteq \boldsymbol{X}_\alpha$ is G_α-open if and only if $\boldsymbol{G} = \boldsymbol{X}_\alpha \cap S\boldsymbol{R}$, where $S\boldsymbol{R}$ is the $\boldsymbol{R}$-semisimple class of some radical $\boldsymbol{R}$.*

Proof. Let $\boldsymbol{G} \subseteq \boldsymbol{X}_\alpha$ be a G_α-open set, then $\boldsymbol{S} = \boldsymbol{X}_\alpha \setminus \boldsymbol{G}$ is a G_α-closed set ($\boldsymbol{Y} \setminus \boldsymbol{Z}$ is the set-theoretical difference). By Proposition 2 we have $\boldsymbol{X}_\alpha \setminus \boldsymbol{G} = \boldsymbol{X}_\alpha \cap \boldsymbol{R}$ where $\boldsymbol{R}$ is some radical class.

Given a ring $A \in \boldsymbol{G} \subseteq \boldsymbol{X}_\alpha$, then $A \notin \boldsymbol{X}_\alpha \setminus \boldsymbol{G} = \boldsymbol{X}_\alpha \cap \boldsymbol{R}$, so $A \notin \boldsymbol{R}$. Since A is an unequivocal ring, so A is $\boldsymbol{R}$-semisimple, $A \in S\boldsymbol{R}$. Hence we have $A \in \boldsymbol{X}_\alpha \cap S\boldsymbol{R}$, so $\boldsymbol{G} \subseteq \boldsymbol{X}_\alpha \cap S\boldsymbol{R}$. Conversely, given a ring $A \in \boldsymbol{X}_\alpha \cap S\boldsymbol{R}$ then A is an $\boldsymbol{R}$-semisimple ring. Since $A \neq \{0\}$, so $A \notin \boldsymbol{R}$. Hence we have $A \notin \boldsymbol{X}_\alpha \cap \boldsymbol{R} = \boldsymbol{X}_\alpha \setminus \boldsymbol{G}$, i.e. $A \in \boldsymbol{G}$ and $\boldsymbol{X}_\alpha \cap S\boldsymbol{R} \subseteq \boldsymbol{G}$. Thus $\boldsymbol{G} = \boldsymbol{X}_\alpha \cap S\boldsymbol{R}$ holds.

Now let $\boldsymbol{G} = \boldsymbol{X}_\alpha \cap S\boldsymbol{R}$ for some radical $\boldsymbol{R}$. We have

$$\boldsymbol{X}_\alpha \setminus \boldsymbol{G} = \boldsymbol{X}_\alpha \setminus (\boldsymbol{X}_\alpha \cap S\boldsymbol{R}) = \boldsymbol{X}_\alpha \setminus S\boldsymbol{R} = \boldsymbol{X}_\alpha \cap \boldsymbol{R},$$

since $\boldsymbol{X}_\alpha$ consists of unequivocal rings only. By Proposition 2, $\boldsymbol{X}_\alpha \setminus \boldsymbol{G} = \boldsymbol{X}_\alpha \cap \boldsymbol{R}$ is G_α-closed, so $\boldsymbol{G}$ is G_α-open. The proposition is proved.

Proposition 4. *For every infinite α the space $(\boldsymbol{X}_\alpha, G_\alpha)$ is not compact. More exactly $(\boldsymbol{X}_{\aleph_0}, G_{\aleph_0})$ contains a sequence having no point of accumulation in any $(\boldsymbol{X}_\alpha, G_\alpha)$, $\alpha \geqq \aleph_0$.*

Proof. Consider the sequence of the simple rings $Z_p = Z/(p)$, $p \geqq 2$, where p is a prime number. We show it has no point of accumulation. In fact, let $A \in \boldsymbol{X}_\alpha$ be a ring, which is a point of accumulation of this sequence, i.e. $A \in \bigcap_{q=2}^{\infty} P\{Z_p, p \geqq q\}$. Consider the following class $\boldsymbol{R}_q$ of rings: $\boldsymbol{R}_q = \{A \mid \forall\, a, 0 \neq a \in A\ O(a) \neq 0$ and all prime factors of $O(a)$ are $\geqq q\}$. Analogously to ARMENDARIZ-LEAVITT [1] one can show this is a radical class. Evidently $\{Z_p, p \geqq q\} \subseteq \boldsymbol{R}_q$, $A \in \bigcap_{q=2}^{\infty} P\{Z_p, p \geqq q\} \subseteq \bigcap_{q=2}^{\infty} \boldsymbol{R}_q$. This implies by the definition of $\boldsymbol{R}_q$ that $A = \{0\}$, which however does not belong to $\boldsymbol{X}$, a contradiction.

Proposition 5. *For every infinite α the space $(\boldsymbol{X}_\alpha, G_\alpha)$ is not $T_{0'}$. More exactly for any $A \in \boldsymbol{X}_\alpha$ of cardinality $< \alpha$ there is an $A' \in \boldsymbol{X}_\alpha$, so that any neighbourhood of A contains A' and conversely.*

Proof. Let A' be the discrete direct sum of α copies of A. Then A' has cardinality α, so A and A' are not isomorphic, however by DIVINSKY [2] $P(\{A\}) = P(\{A'\})$, which is equivalent to our statement.

Proposition 6. *For every infinite α the space $(\boldsymbol{X}_\alpha, G_\alpha)$ contains A_1, A_2 so that $A_1 \in P(\{A_2\})$, but $A_2 \notin P(\{A_1\})$.*

Proof. Let A_1 = zeroring on $C(p^\infty)$, A_2 = zeroring on $C(p)$ (p prime). Then by DIVINSKY [2] $A_1, A_2 \in \boldsymbol{X}_\alpha$, $A_1 \in P(\{A_2\})$, since A_2 is an ideal of A_1. However evidently $A_2 \notin P(\{A_1\})$, since $A_2 \notin L_1(\{A_1\})$, and for $\nu \geqq 2$ $A_2 \in L_\nu(\{A_1\}) \Leftrightarrow A_2 \in L_\varkappa(\{A_1\})$ for some $\varkappa < \nu$.

Proposition 7. For every infinite α the simple rings in $\boldsymbol{X}_\alpha$ are not dense in $(\boldsymbol{X}_\alpha, G_\alpha)$. More exactly $(\boldsymbol{X}_{\aleph_0}, G_{\aleph_0})$ contains a ring, which is not in the closure of the set of simple rings in any $(\boldsymbol{X}_\alpha, G_\alpha)$, $\alpha \geqq \aleph_0$.

Proof. Let A_1 = zeroring on the infinite cyclic group, $\boldsymbol{S} = \{A \in \boldsymbol{X}_\alpha \mid A \text{ is simple}\}$. Then by DIVINSKY [2] $A_1 \in \boldsymbol{X}_\alpha$, but $A_1 \notin P(\boldsymbol{S})$, since $A_1 \notin L_1(\boldsymbol{S})$ and for $\nu \geqq 2$ $A_1 \in L_\nu(\boldsymbol{S}) \Leftrightarrow A_1 \in L_\varkappa(\boldsymbol{S})$ for some $\varkappa < \nu$.

Proposition 8. *For every infinite α the space $(\boldsymbol{X}_\alpha, G_\alpha)$ is not connected. More exactly it contains infinitely many subsets which are both closed and open.*

Proof. Let $\mathscr{F}$ be any finite set of finite fields, containing with any $F \in \mathscr{F}$ all its subfields too. Then by STEWART [5] $\mathscr{B}(\mathscr{F}) = \{A, A$ is a subdirect sum of some copies of fields in $\mathscr{F}\}$ is a class of rings, which is both a radical class and a semisimple class (and in fact all non-trivial such classes are of this form). Hence $\mathscr{B}(\mathscr{F}) \cap \boldsymbol{X}_\alpha$ is both closed and open in X_α. Further all $\big(\mathscr{B}(\mathscr{F}) \cap \boldsymbol{X}_{\aleph_0}\big)$-s are distinct, since the only finite fields contained in $\mathscr{B}(\mathscr{F})$, and hence in $\mathscr{B}(\mathscr{F}) \cap \boldsymbol{X}_{\aleph_0}$, are the ones belonging to $\mathscr{F}$.

Theorem 2. *The closed sets of $(\boldsymbol{X}_\alpha, G_\alpha)$ constitute the open sets of a topological space $(\boldsymbol{X}_\alpha, H_\alpha)$, where analogously the closure of a set $\boldsymbol{T} \neq \Phi$ is the intersection of $\boldsymbol{X}_\alpha$ with the semisimple class of the upper radical determined by $\boldsymbol{T}$.*

Proof. By Theorem 1, 4) in $(\boldsymbol{X}_\alpha, G_\alpha)$ any union of closed sets is closed. This implies that the closed sets of $(\boldsymbol{X}_\alpha, G_\alpha)$ satisfy the axioms for the open sets in the topology $(\boldsymbol{X}_\alpha, H_\alpha)$. The rest is obvious by Proposition 3.

Remark. Our Theorem 1, the remark following it, the analogues of Propositions 2, 3, further Propositions 5, 6, 8 are valid for $(\boldsymbol{X}_\alpha, H_\alpha)$ too (only change the role of A_1, A_2).

Proposition 9. *For every infinite α the simple rings in $\boldsymbol{X}_\alpha$ are not dense in $(\boldsymbol{X}_\alpha, H_\alpha)$. More exactly $(\boldsymbol{X}_{\aleph_0}, H_{\aleph_0})$ contains a ring which is not in the closure of the simple rings in any $(\boldsymbol{X}_\alpha, H_\alpha)$, $\alpha \geqq \aleph_0$.*

Proof. By the construction of the upper radical (SZÁSZ [6], § 3) one sees similarly to Proposition 7 that the zeroring on $C(p^\infty)$ (which belongs to $\boldsymbol{X}_\alpha$ by DIVINSKY [2]) does not belong to the semisimple class of the upper radical determined by the simple rings in $\boldsymbol{X}_\alpha$.

Remark. We do not know if the simple rings in $(\boldsymbol{X}_\alpha, G_\alpha)$, resp. $(\boldsymbol{X}_\alpha, H_\alpha)$ are nowhere dense. Further we conjecture that $(\boldsymbol{X}_\alpha, H_\alpha)$ is not compact. It would be interesting to know all classes $\boldsymbol{Y} \subset \boldsymbol{X}$, for which $\boldsymbol{Y} \cap \boldsymbol{X}_\alpha$ is both closed and open, for each α.

Proposition 10. *If $\alpha \leqq \beta$ then $(\boldsymbol{X}_\alpha, G_\alpha)$ resp. $(\boldsymbol{X}_\alpha, H_\alpha)$ is a subspace of the topological space $(\boldsymbol{X}_\beta, G_\beta)$, resp. $(\boldsymbol{X}_\beta, H_\beta)$.*

Proof. We have $\boldsymbol{X}_\alpha \subseteqq \boldsymbol{X}_\beta$. We may consider only the case of $(\boldsymbol{X}_\alpha, G_\alpha)$. Let $\boldsymbol{G} \subseteqq \boldsymbol{X}_\alpha$ be an open set in $\boldsymbol{X}_\alpha$, then by Proposition 3, $\boldsymbol{G} = \boldsymbol{X}_\alpha \cap S\boldsymbol{R}$, where $\boldsymbol{R}$ is some radical class. Since $\boldsymbol{X}_\alpha \subseteqq \boldsymbol{X}_\beta$, we have $\boldsymbol{G} = \boldsymbol{X}_\alpha \cap S\boldsymbol{R} = \boldsymbol{X}_\alpha \cap (\boldsymbol{X}_\beta \cap S\boldsymbol{R})$. Similarly the intersection with $\boldsymbol{X}_\alpha$ of any open set in $\boldsymbol{X}_\beta$ will be open in $\boldsymbol{X}_\alpha$.

We note yet that in a larger universe it is legitimate to introduce the above topologies on the whole of $\boldsymbol{X}$, for which the analogues of the above statements hold.

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