

# A METRIC DISCREPANCY RESULT WITH GIVEN SPEED

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ABSTRACT. It is known that the discrepancy  $D_N\{kx\}$  of the sequence  $\{kx\}$  satisfies  $ND_N\{kx\} = O((\log N)(\log \log N)^{1+\varepsilon})$  a.e. for all  $\varepsilon > 0$ , but not for  $\varepsilon = 0$ . For  $n_k = \theta^k$ ,  $\theta > 1$  we have  $ND_N\{n_k x\} \leq (\Sigma_\theta + \varepsilon)(2N \log \log N)^{1/2}$  a.e. for some  $0 < \Sigma_\theta < \infty$  and  $N \geq N_0$  if  $\varepsilon > 0$ , but not for  $\varepsilon < 0$ . In this paper we prove, extending results of Aistleitner-Larcher [6], that for any sufficiently smooth intermediate speed  $\Psi(N)$  between  $(\log N)(\log \log N)^{1+\varepsilon}$  and  $(N \log \log N)^{1/2}$  and for any  $\Sigma > 0$ , there exists a sequence  $\{n_k\}$  of positive integers such that  $ND_N\{n_k x\} \leq (\Sigma + \varepsilon)\Psi(N)$  eventually holds a.e. for  $\varepsilon > 0$ , but not for  $\varepsilon < 0$ . We also consider a similar problem on the growth of trigonometric sums.

## 1. INTRODUCTION

A sequence  $\{x_k\}$  of real numbers is said to be uniformly distributed modulo 1 if

$$\frac{1}{N} \#\{k \leq N : \langle x_k \rangle \in [a, b)\} \rightarrow b - a, \quad (N \rightarrow \infty),$$

for all  $0 \leq a < b \leq 1$ , where  $\langle x \rangle$  denotes the fractional part  $x - [x]$  of a real number  $x$ . The discrepancy  $D_N\{x_k\}$ , also denoted by  $D_N(x_1, \dots, x_N)$ , is used to measure the speed of convergence:

$$D_N\{x_k\} = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \#\{k \leq N : \langle x_k \rangle \in [a, b)\} - (b - a) \right|.$$

For arithmetic progressions  $\{kx\}$  with  $x \notin \mathbf{Q}$ , Bohl [10], Sierpiński [24], and Weyl [26] independently proved that they are uniformly distributed modulo 1. A metric result of Khintchine [20] implies

$$ND_N\{kx\} = O((\log N)(\log \log N)^{1+\varepsilon}) \quad \text{a.e. for any } \varepsilon > 0 \quad (1)$$

and this fails for  $\varepsilon \leq 0$ . The discrepancy of exponentially growing sequences has also been investigated extensively. By assuming the

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Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \dots), \quad (2)$$

Philipp [23] proved, using Takahashi's method [25], that

$$\frac{1}{4\sqrt{2}} \leq \overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} \leq \frac{1}{\sqrt{2}} \left( 166 + \frac{664}{q^{1/2} - 1} \right) \quad \text{a.e.} \quad (3)$$

For improvements of (3), see [3] for the lower bound, and [18] for the upper bound. In case of geometric progressions, an exact law of the iterated logarithm holds: for any  $\theta \notin [-1, 1]$  there exists a constant  $\Sigma_\theta \geq 1/2$  with

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N \log \log N}} = \Sigma_\theta \quad \text{a.e.}$$

If  $\theta^j \notin \mathbf{Q}$  for any  $j \in \mathbf{N}$ , then  $\Sigma_\theta = \frac{1}{2}$ , otherwise  $\Sigma_\theta > \frac{1}{2}$ . For a  $\theta$  which is a power root of an integer, of a large rational number, or of a ratio of odd integers, the concrete value of  $\Sigma_\theta$  is evaluated. See [12, 14, 15, 16, 17]. For conditions to have an exact law of the iterated logarithm in (3), see [1, 5].

Since there is a big difference between (1) and (3), it is natural to ask if for intermediate speeds  $\Psi(N)$  between  $(\log N)(\log \log N)^{1+\varepsilon}$  and  $(N \log \log N)^{1/2}$  one can find a sequence  $\{n_k\}$  of integers such that the growth speed of  $D_N\{n_k x\}$  is  $\Psi(N)$  in the above sense. For all  $\gamma \in (0, 1/2]$ , Aistleitner and Larcher [6] constructed an increasing sequence  $\{n_k\}$  of integers such that  $ND_N\{n_k x\} = O(N^\gamma)$  and  $ND_N\{n_k x\} = \Omega(N^{\gamma-\varepsilon})$  a.e. for all  $\varepsilon > 0$ . They also constructed (see [7]) a sequence  $\{n_k\}$  with polynomial growth such that  $ND_N\{n_k x\} = O((\log N)^{2+\varepsilon})$  a.e. for all  $\varepsilon > 0$ .

The main result of the present paper is the following

**Theorem 1.** *Let  $\{\Psi(N)\}$  be a sequence of real numbers. Assume that there exists a constant  $N_0$  such that*

$$0 < \Psi(N) \leq \Psi(N+1) \quad \text{for all } N \geq N_0, \quad (4)$$

$$\Psi(N) \geq (\log N)(\log \log N)^{1+\varepsilon} \quad \text{for some } \varepsilon > 0 \text{ and } N \geq N_0, \quad (5)$$

$$\Psi^2(N+1) - \Psi^2(N) = o(\log \log \Psi^2(N)). \quad (6)$$

*Then for any  $\Sigma > 0$ , there exists a sequence  $\{n_k\}$  of positive integers satisfying  $1 \leq n_{k+1} - n_k \leq 2$  and*

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\Psi(N)} = \Sigma \quad \text{a.e.} \quad (7)$$

Note that for the function  $\Psi^2(N) = N \log \log N$  we have

$$\Psi^2(N+1) - \Psi^2(N) \sim \log \log \Psi^2(N)$$

and thus condition (6) means that the jumps of  $\Psi^2(N)$  are of smaller order of magnitude than those of  $N \log \log N$ . Naturally, this implies that  $\Psi^2(N) = o(N \log \log N)$  and thus the conditions of Theorem 1 bound the function  $\Psi^2(N)$  between  $(\log N)(\log \log N)^{1+\varepsilon}$  and  $N \log \log N$  and require a certain smoothness of growth. Typical examples are  $\Psi(N) = N^\alpha (\log N)^\beta (\log \log N)^\gamma$  where the parameters  $\alpha, \beta, \gamma$  are chosen so that the order of growth of  $\Psi^2(N)$  is between the previous bounds. Note that the theorem does not cover  $\Psi(N) = (N \log \log N)^{1/2}$ ; the existence of  $\{n_k\}$  with (7) is already proved in [4] for  $0 < \Sigma < \infty$ , and in [2] for  $\Sigma = \infty$ . See also [9, 14].

As a related problem, we can ask if there exists a sequence  $\{n_k\}$  such that  $\sum_{k=1}^N \cos 2\pi n_k x$  grows with a given speed  $\Psi(N)$ . The law of the iterated logarithm by Erdős-Gál [11] states

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N \cos 2\pi n_k x = 1 \quad \text{a.e.} \quad (8)$$

for  $\{n_k\}$  satisfying the Hadamard gap condition (2). As we will see in Section 4, for any  $D > 0$  there exists an increasing  $\{n_k\}$  such that (8) holds with the norming factor replaced by  $c\sqrt{N}(\log \log N)^D$ . The following theorem shows that any growth speed  $O(\sqrt{N}(\log \log N)^D)$  with small jumps is possible for  $\sum_{k=1}^N \cos 2\pi n_k x$ .

**Theorem 2.** *Let  $\{\Psi(N)\}$  be an sequence of real numbers. Assume that there exists a constant  $N_0$  and  $D > 0$  such that (4),*

$$\Psi(N) \rightarrow \infty, \quad \text{and} \quad \Psi^2(N+1) - \Psi^2(N) = o((\log \log \Psi^2(N))^D).$$

*Then there exists a strictly increasing sequence  $\{n_k\}$  of positive integers such that*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\Psi(N)} \sum_{k=1}^N \cos 2\pi n_k x = 1 \quad \text{a.e.} \quad (9)$$

In conclusion, we mention a number of open problems related to our results. Let  $\mathcal{G}$  denote the class of functions  $\Psi(N)$ ,  $N = 1, 2, \dots$  such that for some increasing sequence  $\{n_k\}$  relation (7) holds for some constant  $0 < \Sigma < \infty$ . From Theorem 1 it follows that  $\mathcal{G}$  contains all smoothly increasing functions  $\Psi(N)$  with speed between  $(\log N)(\log \log N)^{1+\varepsilon}$  for some  $\varepsilon > 0$  and  $(N \log \log N)^{1/2}$ . By a classical result of W. Schmidt (see e.g. Kuipers and Niederreiter [22], p. 109) for any infinite sequence  $\{x_k\}$  we have  $ND_N\{x_k\} \geq c \log N$  for

infinitely many  $N$  with an absolute constant  $c$  and thus  $\mathcal{G}$  contains no functions  $\Psi(N) = o(\log N)$ . Hence assumption (5) in Theorem 1 is nearly optimal; whether  $\Psi(N) = (\log N)(\log \log N)^\alpha$ ,  $0 \leq \alpha \leq 1$  belongs to  $\mathcal{G}$  remains open. Concerning upper bounds for functions in  $\mathcal{G}$ , the results of Baker [8] and Berkes and Philipp [9] imply that

$$ND_N\{n_k x\} \leq \text{const} \cdot N^{1/2}(\log N)^\gamma \quad \text{a.e.}$$

holds for all  $\{n_k\}$  if  $\gamma > 3/2$  but not if  $\gamma \leq 1/2$ . This implies that for  $\gamma > 3/2$  we have  $N^{1/2}(\log N)^\gamma \notin \mathcal{G}$  and makes it plausible (but does not prove) that  $(N \log N)^{1/2} \in \mathcal{G}$ . If this is true, condition (6) in Theorem 1 can be replaced by

$$\Psi^2(N+1) - \Psi^2(N) = o(\log \Psi^2(N))$$

allowing all smoothly growing functions  $\Psi(N) = O(N \log N)^{1/2}$ , an essentially optimal result. Similar remarks hold for Theorem 2.

## 2. KEY PROPOSITION

We begin with proving a weaker version of Theorem 1.

**Proposition 3.** *For any sequence  $\{\psi(N)\}$  satisfying*

$$\psi(0) = 0, \quad \psi(N) \leq \psi(N+1), \quad (10)$$

$$(\log N)(\log \log N)^{1+\varepsilon} = o(\psi(N)) \quad \text{for some } \varepsilon > 0, \quad (11)$$

$$\psi^2(N+1) - \psi^2(N) \leq \frac{1}{2}(4 \vee \log \log \psi^2(N)), \quad (12)$$

there exists a sequence  $\{n_k\}$  of positive integers satisfying  $1 \leq n_{k+1} - n_k \leq 2$  and

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\psi(N)} = \frac{\sqrt{2}}{4} \quad \text{a.e.} \quad (13)$$

Set  $G(x) = x/(4 \vee \log \log x)$ , where  $\log \log x$  is meant as  $-\infty$  for  $x \leq 1$ . Note that  $G(x)$  is increasing. By (12), we can derive

$$G(\psi^2(N+1)) - G(\psi^2(N)) \leq \frac{\psi^2(N+1) - \psi^2(N)}{4 \vee \log \log \psi^2(N)} \leq \frac{1}{2}. \quad (14)$$

Let  $\nu_i$  be the smallest  $\nu$  satisfying  $2i^3 \leq G(\psi^2(i^3 + \nu))$ . Note that  $\nu_0 = 0$ . By (14), we have

$$G(\psi^2(i^3 + \nu_i)) = 2i^3 + e_i \quad \text{for some } 0 \leq e_i < 1/2. \quad (15)$$

Set  $\Delta_i = \mathbf{N} \cap (2(i-1)^3, 2i^3]$  and  $\eta_i = 2i^3 - 2(i-1)^3$ .

By using (14), we have

$$\begin{aligned} \eta_i - \frac{1}{2} &\leq 2i^3 - 2(i-1)^3 + e_i - e_{i-1} \\ &= G(\psi^2(i^3 + \nu_i)) - G(\psi^2((i-1)^3 + \nu_{i-1})) \leq \frac{1}{2} \left( \frac{1}{2} \eta_i + \nu_i - \nu_{i-1} \right). \end{aligned}$$

By  $\eta_i \geq 2$ , we have

$$\nu_i - \nu_{i-1} \geq (3/2)\eta_i - 1 \geq \eta_i \quad \text{and} \quad \nu_i \geq 2i^3. \quad (16)$$

Set  $\mu_k = 2\nu_i + 2(k - 2i^3)$  for  $k \in \Delta_i$ . By  $\mu_{2i^3+1} = 2\nu_{i+1} - 2\eta_{i+1} + 2 \geq 2\nu_i + 2 > \mu_{2i^3}$ , we see that  $\{\mu_k\}$  is strictly increasing.

We now introduce some notation. Denote by  $\mathbf{1}_{[a,b]}$  the indicator function of  $[a, b)$ , and put  $\tilde{\mathbf{1}}_{[a,b]}(x) = \mathbf{1}_{[a,b]}(\langle x \rangle) - (b - a)$ . Then we have

$$ND_N\{x_k\} = ND_N(x_1, \dots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]}(x_k) \right|.$$

Put  $S = \{2^{-l}i : l \in \mathbf{N}, i = 0, 1, \dots, 2^l\}$ ,  $S^{2<} = \{(a, b) : a, b \in S, a < b\}$ ,  $\phi_C(t) = \sqrt{Ct(1 \vee \log \log t)}$ , and  $\sigma_{a,b} = \sqrt{(b-a)(1 - (b-a))}$ . Let  $\{X_k\}$  be a sequence of independent random variables satisfying  $P(X_k = 1) = P(X_k = -1) = 1/2$ .

**Lemma 4.** *We have*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]}(\mu_k x) X_k \right| = \sigma_{a,b} \quad (17)$$

for all  $(a, b) \in S^{2<}$ , a.e., a.s.

*Proof.* Since  $\mu_k$  is a strictly increasing sequence of integers, by Weyl's theorem [27],  $\{\mu_k x\}$  is uniformly distributed modulo 1 a.e. Hence,

$$B_N := \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]}^2(\mu_k x) \sim N \int_0^1 \tilde{\mathbf{1}}_{[a,b]}^2(y) dy = N\sigma_{a,b}^2 \rightarrow \infty \quad \text{a.e.}$$

if  $b - a \neq 0, 1$ . By Kolmogorov's law of the iterated logarithm [21]

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_2(B_N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]}(\mu_k x) X_k \right| = 1 \quad \text{a.s., a.e.,}$$

we see that (17) holds a.s., a.e. if  $0 < b - a < 1$ . Clearly (17) holds if  $b - a = 0, 1$ . Since  $S^{2<}$  is countable, we see that (17) holds for all  $(a, b) \in S^{2<}$ , a.s., a.e. By Fubini's theorem, we have the conclusion.  $\square$

**Lemma 5.** *Suppose that  $l \in \mathbf{N}$  and  $0 \leq i < 2^l$ , we have*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_2(N)} \sup_{0 < c < 2^{-l}} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[2^{-l}i, 2^{-l}i+c]}(\mu_k x) X_k \right| \leq 4 \cdot 2^{-l/2} \quad \text{a.e., a.s.}$$

*Proof.* Denote  $\mathbf{1}_{[a,b]}(\langle x \rangle)$  simply by  $\mathbf{1}_{[a,b]}x$ . By noting

$$b_N = \sum_{k=1}^N \mathbf{1}_{[2^{-l}i, 2^{-l}(i+1)]}(\mu_k x) \sim N \int_0^1 \mathbf{1}_{[2^{-l}i, 2^{-l}(i+1)]}(y) dy = N2^{-l} \quad \text{a.e.}$$

and by following the proof of Lemma 4 of [13], we can prove

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_2(N)} \sup_{0 < c < 2^{-l}} \left| \sum_{k=1}^N \mathbf{1}_{[2^{-l}i, 2^{-l}(i+c)]}(\mu_k x) X_k \right| \leq \sqrt{10 \cdot 2^{-l}} \quad \text{a.e., a.s.}$$

Thus together with the law of the iterated logarithm

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 < c < 2^{-l}} \frac{c}{\phi_2(N)} \left| \sum_{k=1}^N X_k \right| = \overline{\lim}_{N \rightarrow \infty} \frac{2^{-l}}{\phi_2(N)} \left| \sum_{k=1}^N X_k \right| \leq 2^{-l} \quad \text{a.s.,}$$

we have the conclusion.  $\square$

For  $0 \leq a < b \leq 1$ , take  $l$  with  $b - a > 2^{-l}$  and take the largest  $i$  and  $j$  such that  $2^{-l}i \leq a < 2^{-l}j \leq b$ . Then we have  $\mathbf{1}_{[a,b]} = \mathbf{1}_{[2^{-l}i, 2^{-l}j]} - \mathbf{1}_{[2^{-l}i, a]} + \mathbf{1}_{[2^{-l}j, b]}$  and  $\tilde{\mathbf{1}}_{[a,b]} = \tilde{\mathbf{1}}_{[2^{-l}i, 2^{-l}j]} - \tilde{\mathbf{1}}_{[2^{-l}i, a]} + \tilde{\mathbf{1}}_{[2^{-l}j, b]}$ , which implies

$$\begin{aligned} & \max_{0 \leq i < j \leq 2^l} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[2^{-l}i, 2^{-l}j]}(\mu_k x) X_k \right| \\ & \leq \overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]}(\mu_k x) X_k \right| \\ & \leq \max_{0 \leq i < j \leq 2^l} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[2^{-l}i, 2^{-l}j]}(\mu_k x) X_k \right| \\ & \quad + 2 \max_{0 \leq i \leq 2^l} \overline{\lim}_{N \rightarrow \infty} \sup_{0 < a \leq 2^{-l}} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[2^{-l}i, 2^{-l}(i+a)]}(\mu_k x) X_k \right|. \end{aligned}$$

By applying two lemmas above, we have

$$\frac{1}{2} \leq \overline{\lim}_{N \rightarrow \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]}(\mu_k x) X_k \right| \leq \frac{1}{2} + 8 \cdot 2^{-l/2} \quad \text{a.e., a.s.}$$

which implies

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]}(\mu_k x) X_k \right| = \frac{1}{2} \quad \text{a.e., a.s.} \quad (18)$$

By the relation  $ND_N\{x_k + y\} = ND_N\{x_k\}$  and (1), we have

$$\eta_i D_{\eta_i}(\mu_{2(i-1)^3+1}x, \mu_{2(i-1)^3+2}x, \dots, \mu_{2i^3}x) = \eta_i D_{\eta_i}\{2kx\} = O((\log \eta_i)^2).$$

Noting  $ND_N\{\mu_k x\} \leq \sum_{i=1}^j \eta_i D_{\eta_i}(\mu_{2(i-1)^3+1}x, \mu_{2(i-1)^3+2}x, \dots, \mu_{2i^3}x)$  for  $N \in \Delta_j$ , we have

$$ND_N\{\mu_k x\} = O\left(\sum_{i=1}^j (\log \eta_i)^2\right) = O(N^{1/3}(\log N)^2) = o(\sqrt{N}) \quad \text{a.e.}$$

by  $j - 1 < (N/2)^{1/3}$ . This together with (18) implies

$$\lim_{N \rightarrow \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]} \langle \mu_k x \rangle \frac{X_k + 1}{2} \right| = \frac{1}{4} \quad \text{a.e., a.s.} \quad (19)$$

Note that  $\{\mu_k\}$  and  $\{2k - 1\}$  are mutually disjoint. Let  $\{\lambda_k\}$  be an arrangement in increasing order of  $\{\mu_k\} \cup \{2k - 1\}$ . By  $\mu_{2i^3} = 2\nu_i$ , we have  $\#\{k : \mu_k \leq 2\nu_i\} = 2i^3$  and  $\#\{k : 2k - 1 \leq 2\nu_i\} = \nu_i$ , and thereby we have  $\#\{k : \lambda_k \leq 2\nu_i\} = 2i^3 + \nu_i$  and  $\lambda_{2i^3+\nu_i} = 2\nu_i$ . We set

$$Y_k = \begin{cases} 1 & \lambda_k \notin 2\mathbf{N}, \\ (X_k + 1)/2 & \lambda_k \in 2\mathbf{N}, \end{cases}$$

$I_N = \#\{k \leq N : \lambda_k \notin 2\mathbf{N}\}$ ,  $J_N = \#\{k \leq N : Y_k = 1, \lambda_k \in 2\mathbf{N}\}$ , and  $H_N = \#\{k \leq N : Y_k = 1\} = I_N + J_N$ . We have  $I_{2i^3+\nu_i} = \#\{k \leq 2i^3+\nu_i : \lambda_k \notin 2\mathbf{N}\} = \#\{k : 2k-1 \leq 2\nu_i\} = \nu_i$  and  $H_{2i^3+\nu_i} = J_{2i^3+\nu_i} + \nu_i$ . By the law of large numbers we have  $J_{2i^3+\nu_i} \sim \frac{1}{2} \#\{k : \mu_k \leq 2\nu_i\} = i^3$  a.s. By (14), we have

$$|G(\psi^2(H_{2i^3+\nu_i})) - G(\psi^2(i^3 + \nu_i))| \leq \frac{1}{2} |H_{2i^3+\nu_i} - (i^3 + \nu_i)| = \frac{1}{2} |J_{2i^3+\nu_i} - i^3|.$$

Dividing by  $G(\psi^2(i^3 + \nu_i)) = 2i^2 + e_i$ , we have

$$\left| \frac{G(\psi^2(H_{2i^3+\nu_i}))}{2i^3 + e_i} - 1 \right| \leq \frac{1}{2} \left| \frac{J_{2i^3+\nu_i}}{2i^3 + e_i} - \frac{i^3}{2i^3 + e_i} \right| \rightarrow 0 \quad \text{a.s.}$$

Therefore we have  $G(\psi^2(H_{2i^3+\nu_i})) \sim 2i^3 + e_i \sim 2i^3 \sim 2J_{2i^3+\nu_i}$  a.s. Since  $J_N$  and  $H_N$  are increasing, for  $N \in [(i-1)^3 + \nu_{i-1}, i^3 + \nu_i]$  we have

$$1 \sim \frac{G(\psi^2(H_{2(i-1)^3+\nu_{i-1}}))}{2J_{2i^3+\nu_i}} \leq \frac{G(\psi^2(H_N))}{2J_N} \leq \frac{G(\psi^2(H_{2i^3+\nu_i}))}{2J_{2(i-1)^3+\nu_{i-1}}} \sim 1,$$

and thereby,

$$2J_N \sim G(\psi^2(H_N)) \quad \text{a.s.} \quad (20)$$

By (1), we see  $ND_N\{(2k-1)x\} = O((\log N)(\log \log N)^{1+\varepsilon/2})$ , which implies  $ND_N\{(2k-1)x\} = o((\log N)(\log \log N)^{1+\varepsilon})$  or

$$\lim_{N \rightarrow \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{A_N} \left| \sum_{k \leq N: \lambda_k \notin 2\mathbf{N}} \tilde{\mathbf{1}}_{[a,b]} \langle \lambda_k x \rangle Y_k \right| = 0 \quad \text{a.e., a.s.} \quad (21)$$

for  $A_N = (\log I_N)(\log \log I_N)^\varepsilon$ . Since  $H_N \geq I_N$ , it is valid for  $A_N = (\log H_N)(\log \log H_N)^\varepsilon$ . Because of (11), we see that (21) holds for  $A_N = \sqrt{2} \psi(H_N)$ .

By (19), we have

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{A_N} \left| \sum_{k \leq N: \lambda_k \in 2\mathbf{N}} \tilde{\mathbf{1}}_{[a,b]} \langle \lambda_k x \rangle Y_k \right| = \frac{1}{4} \quad \text{a.e., a.s.} \quad (22)$$

for  $A_N = \phi_2(\#\{k \leq N : \lambda_k \in 2\mathbf{N}\})$ . By  $J_N \sim \frac{1}{2} \#\{k \leq N : \lambda_k \in 2\mathbf{N}\}$  a.s., we see that (22) is valid for  $A_N = \sqrt{2} \phi_2(J_N) \sim \phi_2(2J_N)$ . (20) and  $\phi_2^2(G(\psi^2(N))) \sim 2\psi^2(N)$  imply  $\phi_2^2(J_N) \sim \phi_2^2(G(\psi^2(H_N)))/2 \sim \psi^2(H_N)$  a.s. Hence (22) holds for  $A_N = \sqrt{2} \psi(H_N)$ . Combining these, we have

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{\sqrt{2} \psi(H_N)} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b]} \langle \lambda_k x \rangle Y_k \right| = \frac{1}{4} \quad \text{a.e., a.s.}$$

Denoting by  $\{n_k\}$  the subsequence  $\{\lambda_k : Y_k = 1\}$ , we have (13) a.s.

### 3. PROOF OF THEOREM 1

By (6), we have  $\Psi^2(N) = o(N \log \log \Psi^2(N))$  and  $G(\Psi^2(N)) = o(N)$ . For any  $C > 0$ , we see  $G(\phi_C^2(N)) \sim CN$  and hence  $G(\Psi^2(N)) \leq G(\phi_C^2(N))$  or  $\Psi^2(N) \leq \phi_C^2(N)$  for large  $N$ . Since it holds for any  $C > 0$ , we see that  $\Psi^2(N) = o(\phi_C^2(N))$ .

By (6), we can take  $N_1 > N_0$  such that for all  $N \geq N_1$ ,

$$(2\sqrt{2} \Sigma \Psi(N+1))^2 - (2\sqrt{2} \Sigma \Psi(N))^2 \leq \frac{1}{2} \log \log (2\sqrt{2} \Sigma \Psi(N))^2. \quad (23)$$

Take  $c \in (0, \frac{1}{4})$  such that  $\phi_c^2(N_1) < (2\sqrt{2} \Sigma \Psi(N_1))^2$  holds. We have  $(2\sqrt{2} \Sigma \Psi(N))^2 < \phi_c^2(N)$  for large  $N \geq N_1$ . Denote  $N_2$  the minimum of such  $N$ . Putting

$$\psi(N) = \begin{cases} \phi_c(N) & N < N_2, \\ 2\sqrt{2} \Sigma \Psi(N) & N \geq N_2, \end{cases}$$

it is clear that  $\psi(N)$  satisfies (10) and (11). As to the condition (12), we first prove it for  $\phi_c^2(N)$ .

In the case  $\log \log(N+1) \geq 1$ , i.e.  $N \geq 15$ , we see  $(N+1)(\log \log(N+1) - \log \log N) \leq ((N+1)/N)/\log N \leq 2/\log 15 < \log \log 15 \leq$

$\log \log N$  and  $(N + 1) \log \log(N + 1) - N \log \log N < 2 \log \log N$ . If  $c \log \log N \leq 1$ , then  $2c \log \log N \leq 2 \leq \frac{1}{2}(4 \vee \log \log \phi_c^2(N))$ . If  $c \log \log N \geq 1$ , then  $2c \log \log N \leq \frac{1}{2} \log \log N \leq \frac{1}{2} \log \log(cN \log \log N) \leq \frac{1}{2}(4 \vee \log \log \phi_c^2(N))$ . Therefore, when  $\log \log(N + 1) \geq 1$ , we have  $\phi_c^2(N + 1) - \phi_c^2(N) \leq 2c \log \log N \leq \frac{1}{2}(4 \vee \log \log \phi_c^2(N))$ . When  $\log \log(N + 1) \leq 1$ , clearly we have  $\phi_c^2(N + 1) - \phi_c^2(N) \leq c \leq \frac{1}{4} \leq \frac{1}{2}(4 \vee \log \log \phi_c^2(N))$ .

By  $\psi^2(N_2) - \psi^2(N_2 - 1) \leq (2\sqrt{2} \Sigma \Psi(N_2))^2 - \phi_c^2(N_2 - 1) \leq \phi_c^2(N_2) - \phi_c^2(N_2 - 1)$  together with (23), we conclude that  $\psi(N)$  satisfies (12).

Hence we can apply Proposition 3 to have the conclusion.

#### 4. PROOF OF THEOREM 2

Take an integer  $d \geq D \vee 2$  to satisfy

$$\Psi^2(N + 1) - \Psi^2(N) = o((\log \log \Psi^2(N))^d). \quad (24)$$

Put  $M_k = 2^{d-1} \binom{k}{d}$ ,  $L_k = \min\{n \mid \Psi^2(n) \geq (2^{d-1}/d!)M_k(\log \log M_k)^d\}$ , and  $L_k^+ = L_k + M_{k+1} - M_k$ .

There exists  $K_-$  such that  $\max_{N \leq N_0} \Psi(N) < (2^{d-1}/d!)M_k(\log \log M_k)^d$  for all  $k \geq K_-$ . From now on, we consider only for  $k \geq K_-$ , for which we have  $L_k > N_0$ .

By (24) and  $\Psi^2(L_k - 1) < (2^{d-1}/d!)M_k(\log \log M_k)^d$ , we have

$$\begin{aligned} (2^{d-1}/d!)M_k(\log \log M_k)^d &\leq \Psi^2(L_k) \\ &= o((\log \log \Psi^2(L_k - 1))^d) + \Psi^2(L_k - 1) \\ &\leq o((\log \log (M_k(\log \log M_k)^d))^d) + (2^{d-1}/d!)M_k(\log \log M_k)^d, \end{aligned}$$

$\Psi^2(L_k)/(2^{d-1}/d!)M_k(\log \log M_k)^d \rightarrow 1$ ,  $\log \log \Psi^2(L_k) - \log \log M_k \rightarrow 0$  and  $\log \log \Psi^2(L_k) \sim \log \log M_k$  in turn. Combining

$$\begin{aligned} \Psi^2(L_{k+1}) - \Psi^2(L_k - 1) &\geq (2^{d-1}/d!)(M_{k+1}(\log \log M_{k+1})^d - M_k(\log \log M_k)^d) \\ &\geq (2^{d-1}/d!)(M_{k+1} - M_k)(\log \log M_{k+1})^d \end{aligned}$$

and  $\Psi^2(L_{k+1}) - \Psi^2(L_k - 1) = (L_{k+1} - L_k + 1)o((\log \log \Psi^2(L_{k+1}))^d)$ , we have

$$\frac{M_{k+1} - M_k}{L_{k+1} - L_k + 1} \leq \frac{o((\log \log \Psi^2(L_{k+1}))^d)}{(2^{d-1}/d!)(\log \log M_{k+1})^d} = o(1).$$

Hence we see that there exists a  $K_0$  such that

$$L_{k+1} - L_k > M_{k+1} - M_k \quad \text{i.e.,} \quad L_{k+1} > L_k^+ \quad (k \geq K_0). \quad (25)$$

By (24) we have  $\Psi^2(N) \leq o(N(\log \log \Psi^2(N))^d)$ , Thereby  $\log \Psi^2(N) < \log N + d \log \log \log \Psi^2(N)$ , and  $\log \Psi^2(N) \leq 2 \log N$  or  $\Psi^2(N) \leq N^2$  for large  $N$ . Hence  $\Psi^2(N) = o(N(\log \log N)^d)$ . Hence we see  $\Psi^2(M_k) = o(M_k(\log \log M_k)^d) = o(\Psi^2(L_k))$ . It implies  $M_k < L_k$  for large  $k$ . Take such  $k \geq K_0$  and denote by  $k_0$ . We see  $M_{k_0} < L_{k_0}$ .

We define an non-decreasing sequence  $\{a_k\}$  of positive integers as below. Put  $a_1 = \dots = a_{k_0} = 3$ , take  $a_{k_0+1}$  large enough to satisfy  $a_{k_0+1} \geq a_{k_0}$  and

$$\gamma_{k_0+1}^+ := \frac{1}{2}a_{k_0+1}^{k_0+1} \geq \frac{3}{2}a_{k_0}^{k_0} + (L_{k_0} - 1 - M_{k_0}) =: \gamma_{k_0+1}^- \quad (26)$$

For  $k \geq k_0$ , inductively take  $a_{k+2}$  large enough to satisfy  $a_{k+2} \geq a_{k+1}$  and

$$\gamma_{k+2}^+ := \frac{1}{2}a_{k+2}^{k+2} \geq \frac{3}{2}a_{k+1}^{k+1} + (L_{k+1} - L_k^+) =: \gamma_{k+2}^- \quad (27)$$

Put  $\rho_j = a_j^j$ . Since  $\rho_j$  satisfies the Hadamard gap condition  $\rho_{j+1}/\rho_j \geq a_{j+1} \geq 3$ , by the law of the iterated logarithm we have

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_1(N)} \sum_{j=1}^N \cos 2\pi \rho_j x = \overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_1(N)} \left| \sum_{j=1}^N \cos 2\pi \rho_j x \right| = 1 \quad \text{a.e.} \quad (28)$$

From this, we drive

$$\overline{\lim}_{N \rightarrow \infty} \frac{d!}{\phi_1(N)^d} \sum_{1 \leq m_1 < \dots < m_d \leq N} \prod_{j=1}^d \cos 2\pi \rho_{m_j} x = 1 \quad \text{a.e.} \quad (29)$$

For a function  $f(m_1, \dots, m_d)$  on  $\{1, \dots, N\}^d$ , define a signed measure  $\nu$  on  $\{1, \dots, N\}^d$  by

$$\nu(A) = \sum_{(m_1, \dots, m_d) \in A} f(m_1, \dots, m_d) \quad (A \subset \{1, \dots, N\}^d).$$

Let  $J = \{(j, k) \mid 1 \leq j, k \leq N, j \neq k\}$ . For  $(j, k) \in J$ , put  $A_{(j,k)} = \{(m_1, \dots, m_d) \in \{1, \dots, N\}^d \mid m_j = m_k\}$ .

Putting

$$f(m_1, \dots, m_d) = \prod_{j=1}^d \cos 2\pi \rho_{m_j} x$$

and by applying the inclusion-exclusion principle

$$\begin{aligned} \nu\left(\{1, \dots, N\}^d \setminus \bigcup_{j \in J} A_j\right) &= \nu(\{1, \dots, N\}^d) - \sum_{j \in J} \nu(A_j) \\ &\quad + \sum_{j_1, j_2 \in J: j_1 \neq j_2} \nu(A_{j_1} \cap A_{j_2}) - \dots + \nu\left(\bigcap_{j \in J} A_j\right), \end{aligned}$$

we see that

$$\left| \sum_{m_1, \dots, m_d \leq N: m_j \neq m_k ((j, k) \in J)} \prod_{j=1}^d \cos 2\pi \rho_{m_j} x - \left( \sum_{k=1}^N \cos 2\pi \rho_k x \right)^d \right|$$

can be bounded by a linear combination of

$$\left| \prod_{j=1}^{\beta} \sum_{k=1}^N \cos^{\alpha_j} 2\pi \rho_k x \right| \quad (\alpha_1 + \dots + \alpha_{\beta} = d, \max_{j=1}^{\beta} \alpha_j \geq 2).$$

Note that we can verify

$$\begin{aligned} 0 &\leq \overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_1(N)^d} \left| \prod_{j=1}^{\beta} \sum_{k=1}^N \cos^{\alpha_j} 2\pi \rho_k x \right| \\ &\leq \prod_{j=1}^{\beta} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_1(N)^{\alpha_j}} \left| \sum_{k=1}^N \cos^{\alpha_j} 2\pi \rho_k x \right| = 0 \quad \text{a.e.} \end{aligned}$$

because

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_1(N)^{\alpha}} \left| \sum_{k=1}^N \cos^{\alpha} 2\pi \rho_k x \right| \leq \overline{\lim}_{N \rightarrow \infty} \frac{N}{\phi_1(N)^{\alpha}} = 0$$

holds for  $\alpha \geq 2$ . Hence by (28) we have

$$\begin{aligned} &\overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_1(N)^d} \sum_{m_1, \dots, m_d \leq N: m_j \neq m_k ((j, k) \in J)} \prod_{j=1}^d \cos 2\pi \rho_{m_j} x \\ &= \overline{\lim}_{N \rightarrow \infty} \frac{1}{\phi_1(N)^d} \left( \sum_{k=1}^N \cos 2\pi \rho_k x \right)^d = 1 \quad \text{a.e.} \end{aligned}$$

and thereby we see (29).

Let  $\mathcal{S}_0$  be a collection of  $(b_1, b_2, \dots) \in \{-1, 0, 1\}^{\mathbf{N}}$  such that  $b_i = 0$  for all large  $i$ .

**Lemma 6.** *The mapping  $\mathcal{S}_0 \ni (b_1, b_2, \dots) \mapsto \sum_{i=1}^{\infty} b_i a_i \in \mathbf{Z}$  is injective.*

*Proof.* Because of  $|\sum_{i=1}^{I-1} b_i a_i^i| \leq \sum_{i=1}^{I-1} a_{I-1}^i < \frac{1}{2} a_I^I$ , we have

$$\sum_{i=1}^I b_i a_i^i \in \left( (b_I - \frac{1}{2}) a_I^I, (b_I + \frac{1}{2}) a_I^I \right),$$

and if  $b_I \neq 0$ , then

$$\sum_{i=1}^I b_i a_i^i \in \left( -\frac{3}{2} a_I^I, -\frac{1}{2} a_I^I \right) \cup \left( \frac{1}{2} a_I^I, \frac{3}{2} a_I^I \right) =: C_I. \quad (30)$$

Take  $(b_1, b_2, \dots) \in \mathcal{S}_0$  and  $(b'_1, b'_2, \dots) \in \mathcal{S}_0$  and assume  $\sum_{i=1}^{\infty} b_i a_i^i = \sum_{i=1}^{\infty} b'_i a_i^i$ . By putting  $I = \max\{i \mid b_i \neq 0\}$  and  $I' = \max\{i \mid b'_i \neq 0\}$ , then we see that  $\sum_{i=1}^{\infty} b_i a_i^i \in C_I$  and  $\sum_{i=1}^{\infty} b'_i a_i^i \in C_{I'}$ . By  $\frac{3}{2} a_I^I \leq \frac{1}{2} a_{I+1}^{I+1}$ , we see that  $C_I$  ( $I = 1, 2, \dots$ ) are mutually disjoint and  $\max\{i \mid b_i \neq 0\} = \max\{i \mid b'_i \neq 0\}$ . Because  $\left( (b - \frac{1}{2}) a_I^I, (b + \frac{1}{2}) a_I^I \right)$  ( $b \in \mathbf{Z}$ ) are mutually disjoint, we see  $b_I = b'_I$ . Hence we have  $\sum_{i=1}^{I-1} b_i a_i^i = \sum_{i=1}^{I-1} b'_i a_i^i$ . In the same way, we can verify  $b_i = b'_i$  for all  $i < I$ , and see that the mapping is injective.  $\square$

By this lemma, we see that

$$\rho_{m_d} + \varepsilon_{d-1} \rho_{m_{d-1}} + \dots + \varepsilon_1 \rho_{m_1} \quad (31)$$

with  $m_1 < m_2 < \dots < m_d$  and  $\varepsilon_1, \dots, \varepsilon_d = \pm 1$  are all distinct. Denote by  $\{l_i\}$  the arrangement in increasing order of this family.

Note that  $M_k$  equals to the number of the sum of the type (31) with  $m_1 < m_2 < \dots < m_d \leq k$  and  $\varepsilon_1, \dots, \varepsilon_d = \pm 1$ . By (30),

$$l_i \in \left( \frac{1}{2} a_N^N, \frac{3}{2} a_N^N \right), \quad (M_{N-1} < i \leq M_N). \quad (32)$$

Clearly

$$\prod_{j=1}^d \cos 2\pi \rho_{m_j} x = \frac{1}{2^{d-1}} \cos 2\pi (\rho_{m_d} + \varepsilon_{d-1} \rho_{m_{d-1}} + \dots + \varepsilon_1 \rho_{m_1}) x,$$

and

$$\sum_{1 \leq m_1 < \dots < m_d \leq N} \prod_{j=1}^d \cos 2\pi \rho_{m_j} x = \frac{1}{2^{d-1}} \sum_{k=1}^{M_N} \cos 2\pi l_k x.$$

Hence by (29), we have

$$\overline{\lim}_{N \rightarrow \infty} \frac{d!}{2^{d-1} \phi_1(N)^d} \sum_{k=1}^{M_N} \cos 2\pi l_k x = 1 \quad \text{a.e.} \quad (33)$$

Put

$$B_N(x) = \max_{M_N+1 \leq Q \leq M_{N+1}} \left| \sum_{k=M_N+1}^Q \cos 2\pi l_k x \right|.$$

By the Carleson-Hunt inequality [19] we have

$$\int_0^1 B_N^4(x) dx \leq C \int_0^1 \left( \sum_{k=M_N+1}^{M_{N+1}} \cos 2\pi l_k x \right)^4 dx$$

where  $C$  is an absolute constant. Put

$$C_N(x) = \sum_{m_1, \dots, m_{d-1} \leq N-1; m_i \neq m_j (i \neq j)} \prod_{j=1}^{d-1} \cos 2\pi \rho_{m_j} x.$$

By

$$\begin{aligned} \sum_{k=M_N+1}^{M_{N+1}} \cos 2\pi l_k x &= 2^{d-1} \sum_{m_1 < \dots < m_{d-1} < m_d = N} \prod_{j=1}^d \cos 2\pi \rho_{m_j} x \\ &= \frac{2^{d-1}}{d!} C_N(x) \cos 2\pi N x \end{aligned}$$

we have

$$\int_0^1 B_N^4(x) dx \leq C \left( \frac{2^{d-1}}{d!} \right)^4 \int_0^1 C_N^4(x) dx$$

As before, by the inclusion-exclusion principle, we see that  $|C_N(x)|$  can be bounded from above by a linear combination of

$$\left| \prod_{j=1}^{\beta} \sum_{k=1}^{N-1} \cos^{\alpha_j} 2\pi \rho_k x \right| \quad (\alpha_1 + \dots + \alpha_{\beta} = d-1, \alpha_j \geq 1).$$

Put  $S = \sum_{j=1}^{\beta} \alpha_j \mathbf{1}(\alpha_j > 1)$  and  $T = \sum_{j=1}^{\beta} \mathbf{1}(\alpha_j = 1)$ .  $S + T = d-1$  is clear. For  $\alpha \geq 2$ , we bound  $|\sum_{k=1}^{N-1} \cos^{\alpha_j} 2\pi \rho_k x| \leq N \leq N^{\alpha/2}$  to have

$$\left| \prod_{j=1}^{\beta} \sum_{k=1}^{N-1} \cos^{\alpha_j} 2\pi \rho_k x \right| \leq N^{S/2} \left| \sum_{k=1}^{N-1} \cos 2\pi \rho_k x \right|^T.$$

By applying Theorem 8.20 of Zygmund [28], we have

$$\int_0^1 \left( \prod_{j=1}^{\beta} \sum_{k=1}^N \cos^{\alpha_j} 2\pi \rho_k x \right)^4 dx = O(N^{2S} N^{2T}) = O(N^{2(d-1)}).$$

Therefore we have

$$\int_0^1 B_N^4(x) dx = O(N^{2(d-1)}) \quad \text{and} \quad \sum_{N=1}^{\infty} \int_0^1 \left( \frac{B_N(x)}{N^{d/2}} \right)^4 dx < \infty.$$

By applying the Beppo-Levi Theorem we have  $B_N = o(N^{d/2})$  a.e. By noting  $M_N \sim N^d 2^{d-1}/d!$  and combining with (33), we have

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{(2^{d-1}/d!)N(\log \log N)^d}} \sum_{i=1}^N \cos 2\pi l_i x = 1 \quad \text{a.e.} \quad (34)$$

Put

$$n_i = \begin{cases} l_i & \text{if } i \leq M_{k_0}, \\ l_{M_{k_0}} + (i - M_{k_0}) & \text{if } M_{k_0} < i < L_{k_0}, \\ l_{M_k+i+1-L_k} & \text{if } L_k \leq i < L_k^+, \\ n_{L_k^+-1} + (i+1-L_k^+) & \text{if } L_k^+ \leq i < L_{k+1} \quad (k \geq k_0), \end{cases}$$

We can verify that  $\{n_k\}$  is strictly increasing. Actually by (32) and (26), we see

$$n_{L_{k_0}} = l_{M_{k_0+1}} > \gamma_{k_0+1}^+ \geq \gamma_{k_0+1}^- > l_{M_{k_0}} + (L_{k_0} - 1 - M_{k_0}) = n_{L_{k_0}-1},$$

and by (27) we see for  $k \geq k_0$ ,

$$n_{L_{k+1}} = l_{M_{k+1}+1} > \gamma_{k+2}^+ \geq \gamma_{k+2}^- > l_{M_{k+1}} + (L_{k+1} - L_k^+) = n_{L_{k+1}-1}.$$

Put  $E = [1, M_{k_0}] \cup \bigcup_{k=k_0}^{\infty} [L_k, L_k^+)$ ,  $F = \mathbf{N} \setminus E$ ,  $E_N = E \cap [1, N]$ ,  $F_N = F \cap [1, N]$ , and  $\eta_N = \#E_N$ . By  $\eta_{L_k} = M_k + 1$ , we have  $\Psi^2(L_k) \sim (2^{d-1}/d!) \eta_{L_k} (\log \log \eta_{L_k})^d$ . By  $\Psi^2(L_{k+1}) \sim \Psi^2(L_k)$ , we have

$$\Psi^2(N) \sim (2^{d-1}/d!) \eta_N (\log \log \eta_N)^d \quad (35)$$

By (34), we see that

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{A_N} \sum_{i \in E_N} \cos 2\pi n_i x = 1 \quad \text{a.e.}$$

holds for  $A_N = \sqrt{(2^{d-1}/d!) \eta_N \log \log \eta_N}$ , and by (35) we see that it holds for  $A_N = \Psi(N)$ .

If  $N \in [L_{k-1}^+, L_k)$ , we have  $|\sum_{i=L_{k-1}^+}^N \cos 2\pi n_i x| \leq 2/|\sin \pi x|$ . By  $\Psi^2(N) \sim \eta_{L_k} \log \log \eta_{L_k} \sim (2^{d-1}/d!) k^d \log \log k$ , we can see that

$$\max_{N \in [L_{k-1}^+, L_k)} \left| \sum_{i \in F_N} \cos 2\pi n_i x \right| \leq \frac{2k}{|\sin \pi x|} = o(\Psi(N)) \quad \text{a.e.}$$

Hence we can verify (9).

## REFERENCES

- [1] C. Aistleitner, *On the law of the iterated logarithm for the discrepancy of lacunary sequences*, Trans. Amer. Math. Soc., **362** (2010), 5967-5982.
- [2] C. Aistleitner & K. Fukuyama, *On the law of the iterated logarithm for trigonometric series with bounded gaps*, Probab. Theory Related Fields, **154** (2012) 607–620.
- [3] C. Aistleitner & K. Fukuyama, *Extremal discrepancy behaviour of lacunary sequences*, Monatsh. Math., **177** (2015) 167–184.
- [4] C. Aistleitner & K. Fukuyama, *On the law of the iterated logarithm for trigonometric series with bounded gaps II*, J. Théor. des Nombres Bordeaux, (to appear).
- [5] C. Aistleitner, K. Fukuyama & Y. Furuya, *Optimal bound for the discrepancies of lacunary sequences*, Acta Arith., **158** (2013) 229–243.
- [6] C. Aistleitner & G. Larcher, *On sequences with prescribed metric discrepancy behavior*, Monatsh. Math., (to appear), DOI: 10.1007/s00605-015-0860-2
- [7] C. Aistleitner & G. Larcher, *Metric results on the discrepancy of sequence  $\{a_n\}_{n \geq 1}$  modulo one for integer sequence  $\{a_n\}$  of polynomial growth*, Mathematika, **62** (2016) 478–491.
- [8] R. Baker, *Metric number theory and the large sieve*, J. London Math. Soc. 24 (1981), 34–40.
- [9] I. Berkes & W. Philipp, *The size of trigonometric and Walsh series and uniform distribution mod 1*, J. London Math. Soc. (2), **50** (1994) 454–464.
- [10] P. Bohl, *Über ein in der Theorie der säkulären Störungen vorkommendes Problem*, J. Reine Angew. Math., **135** (1909), 189-283.
- [11] P. Erdős & S. Gál I, *On the law of the iterated logarithm I, II*, Nederl. Akad. Wetensch. Proc. Ser. A 58 (Indag. Math. 17) (1955) 65–76, 77–84.
- [12] K. Fukuyama, *The law of the iterated logarithm for discrepancies of  $\{\theta^n x\}$* . Acta Math. Hungar., **118** (2008), 155-170.
- [13] K. Fukuyama, *A law of the iterated logarithm for discrepancies: non-constant limsup*, Monatsh. Math., **160** (2010), 143-149.
- [14] K. Fukuyama, *A central limit theorem and a metric discrepancy result for sequence with bounded gaps*, Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp, Eds. I. Berkes, R. Bradley, H. Dehling, M. Peligrad, and R. Tichy, Kendrick press, 2010, pp. 233-246.
- [15] K. Fukuyama, *Metric discrepancy results for alternating geometric progressions*, Monatsh. Math., **171** (2013) 33–63.
- [16] K. Fukuyama, *A metric discrepancy result for the sequence of powers of minus two*, Indag. Math. (NS), **25** (2014) 487-504.
- [17] K. Fukuyama & M. Yamashita, *Metric discrepancy results for geometric progressions with large ratios*, Monatsh. Math., (to appear), DOI: 10.1007/s00605-015-0791-y.
- [18] K. Fukuyama, Y. Mitsuhata, *Bounded law of the iterated logarithm for discrepancies of permutations of lacunary sequences*, Summer School on the Theory of Uniform Distribution, RIMS Kôkyûroku Bessatsu, **B29** (2012) 65–88.
- [19] R. A. Hunt, *On the convergence of Fourier series*, Orthogonal Expansions and their Continuous Analogues, (Proc. Conf. Edwardsville, Ill.,1967) 235-255. Southern Illinois Univ. Press, Carbondale, Ill. 1968.

- [20] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. **92**, (1924) 115-125
- [21] A. Kolmogoroff, *Über das Gesetz des iterierten Logarithmus*, Math. Ann. **101**, (1929) 126-135
- [22] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*. Wiley, 1974.
- [23] W. Philipp, *Limit theorems for lacunary series and uniform distribution mod 1*, Acta Arith. **26**, (1975) 241-251
- [24] W. Sierpiński, *Sur la valeur asymptotique d'une certaine somme*, Bull. Int. Acad. Polon. Sci. (Cracovie) A, **1910** (1910), 9-11.
- [25] S. Takahashi, *An asymptotic property of a gap sequence*, Proc. Japan Acad., **38** (1962), 101-104.
- [26] H. Weyl, *Über die Gibbssche Erscheinung und verwandte Konvergenzphänomene*, Rend. Circ. Mat. Palermo, **30** (1910) 377-407.
- [27] H. Weyl, *Über die Gleichverteilung von Zahlen modulo Eins*, Math. Ann., **77** (1916), 313-352.
- [28] A. Zygmund, *Trigonometric series, Vol I.*, Cambridge University Press, 1959.

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