# SECOND-ORDER SOBOLEV INEQUALITIES ON A CLASS OF RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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Abstract. Let $(M, g)$ be an $n-$dimensional complete open Riemannian manifold with nonnegative Ricci curvature verifying $\rho\Delta_g\rho \geq n-5 \geq 0$, where $\Delta_g$ is the Laplace-Beltrami operator on $(M, g)$ and $\rho$ is the distance function from a given point. If $(M, g)$ supports a second-order Sobolev inequality with a constant $C > 0$ close to the optimal constant $K_0$ in the second-order Sobolev inequality in $\mathbb{R}^n$, we show that a global volume non-collapsing property holds on $(M, g)$. The latter property together with a Perelman-type construction established by Munn (J. Geom. Anal., 2010) provide several rigidity results in terms of the higher-order homotopy groups of $(M, g)$. Furthermore, it turns out that $(M, g)$ supports the second-order Sobolev inequality with the constant $C = K_0$ if and only if $(M, g)$ is isometric to the Euclidean space $\mathbb{R}^n$.

## 1. Introduction

It is well known that the validity of first-order Sobolev inequalities on Riemannian manifolds strongly depend on the curvature; this is a rough conclusion of the famous AB-program initiated by Th. Aubin in the seventies, see the monograph of Hebey [13] for a systematic presentation. To be more precise, let $(M, g)$ be an $n-$dimensional complete Riemannian manifold, $n \geq 3$, and consider for some $C > 0$ the first-order Sobolev inequality

$$\left(\int_M |u|^{2^*} \mathrm{d}v_g\right)^{\frac{2}{2^*}} \leq C \int_M |\nabla_g u|^2 \mathrm{d}v_g,\ \forall u \in C_0^\infty(M), \qquad (\mathbf{FSI})_C$$

where $2^* = \frac{2n}{n-2}$ is the first-order critical Sobolev exponent, and $\mathrm{d}v_g$ and $\nabla_g$ denote the canonical volume form and gradient on $(M, g)$, respectively. On one hand, inequality $(\mathbf{FSI})_C$ holds on any $n-$dimensional Cartan-Hadamard manifold $(M, g)$ (i.e., simply connected, complete Riemannian manifold with nonpositive sectional curvature) with the optimal Euclidean constant $C = c_0 = [\pi n(n-2)]^{-1} \left(\Gamma(n)/\Gamma(\frac{n}{2})\right)^{2/n}$ whenever the Cartan-Hadamard conjecture holds on $(M, g)$, e.g., $n \in \{3, 4\}$. On the other hand, due to Ledoux [17], if $(M, g)$ has nonnegative Ricci curvature, inequality $(\mathbf{FSI})_{c_0}$ holds if and only if $(M, g)$ is isometric to the Euclidean space $\mathbb{R}^n$. Further first-order Sobolev-type inequalities on Riemannian/Finsler manifolds can be found in Bakry, Concordet and Ledoux [2], Druet, Hebey and Vaugon [8], do Carmo and Xia [6], Xia [24]-[26], Kristály [15]; moreover, similar Sobolev inequalities are also considered on 'nonnegatively' curved metric measure spaces formulated in terms of the Lott-Sturm-Villani-type curvature-dimension condition or the Bishop-Gromov-type doubling measure condition, see Kristály [14] and Kristály and Ohta [16].

With respect to first-order Sobolev inequalities, much less is know about higher-order Sobolev inequalities on curved spaces. The first studies concern the AB-program for Paneitz-type operators on compact Riemannian manifolds, see Djadli, Hebey and Ledoux [7], Hebey [12] and Biezuner and Montenegro [3]. Recently, Gursky and Malchiodi [11] studied strong maximum principles for Paneitz-type operators on complete Riemannian manifolds with semi-positive $Q-$curvature and nonnegative scalar curvature.

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The aim of the present paper is to establish rigidity results on Riemannian manifolds with nonnegative Ricci curvature supporting second-order Sobolev inequalities. In order to present our results, let $(M,g)$ be an $n$-dimensional complete open Riemannian manifold, $n \geq 5$, $B(x,r)$ be the geodesic ball with center $x \in M$ and radius $r > 0$, and $\mathrm{vol}_g[B(x,r)]$ be the volume of $B(x,r)$. We say that $(M,g)$ supports the *second-order Sobolev inequality for* $C > 0$ if

$$\left(\int_M |u|^{2^\sharp} \mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \leq C \int_M (\Delta_g u)^2 \,\mathrm{d}v_g, \quad \forall u \in C_0^\infty(M), \qquad (\mathbf{SSI})_C$$

where $2^\sharp = \frac{2n}{n-4}$ is the second-order critical Sobolev exponent, and $\Delta_g$ is the Laplace-Beltrami operator on $(M,g)$. Note that the Euclidean space $\mathbb{R}^n$ supports $(\mathbf{SSI})_{K_0}$ for

$$K_0 = \left[\pi^2 n(n-4)(n^2-4)\right]^{-1} \left(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right)^{4/n}. \qquad (1)$$

Moreover, $K_0$ is optimal, see Edmunds, Fortunato and Janelli [9], Lieb [19] and Lions [20], and the unique class of extremal functions is

$$u_{\lambda,x_0}(x) = \left(\lambda + |x-x_0|^2\right)^{\frac{4-n}{2}}, \ x \in \mathbb{R}^n,$$

where $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ are arbitrarily fixed.

To state our results, we need a technical assumption on the manifold $(M,g)$; namely, if $\rho$ is the distance function on $M$ from a given point $x_0 \in M$, we say that $(M,g)$ satisfies the *distance Laplacian growth condition* if

$$\rho \Delta_g \rho \geq n-5\,.$$

Now, our main result reads as follows.

**Theorem 1.1.** *Let $n \geq 5$ and $(M,g)$ be an $n$-dimensional complete open Riemannian manifold with nonnegative Ricci curvature which satisfies the distance Laplacian growth condition. Assume that $(M,g)$ supports the second-order Sobolev inequality $(\mathbf{SSI})_C$ for some $C > 0$. Then the following properties hold:*

(i) $C \geq K_0$;
(ii) *if in addition $C \leq \frac{n+2}{n-2}K_0$, then we have the global volume non-collapsing property*

$$\mathrm{vol}_g[B(x,r)] \geq (C^{-1}K_0)^{\frac{n}{4}} \omega_n r^n \quad \textit{for all } r > 0, x \in M\,,$$

*where $\omega_n$ is the volume of the $n-$dimensional Euclidean unit ball.*

**Remark 1.1.** The distance Laplacian growth condition on $(M,g)$ is indispensable in our argument which shows the genuine second-order character of the studied problem. We notice that the counterpart of this condition in the first-order Sobolev inequality $(\mathbf{FSI})_C$ is the validity of an eikonal inequality $|\nabla_g \rho| \leq 1$ a.e. on $M$, which trivially holds on any complete Riemannian manifold (and any metric measure space with a suitable derivative notion). Further comments on this condition will be given in Section 3.

Having the global volume non-collapsing property of geodesic balls of $(M,g)$ in Theorem 1.1 (ii), we shall prove that once $C > 0$ in $(\mathbf{SSI})_C$ is closer and closer to the optimal Euclidean constant $K_0$, the Riemannian manifold $(M,g)$ approaches topologically more and more to the Euclidean space $\mathbb{R}^n$. To describe quantitatively this phenomenon, we recall the construction of Munn [22] based on the double induction argument of Perelman [23]. In fact, Munn determined explicit lower bounds for the volume growth of the geodesic balls in terms of certain constants which guarantee the triviality of the $k$-th homotopy group $\pi_k(M)$ of $(M,g)$. More precisely, let $n \geq 5$ and for $k \in \{1,...,n\}$, let us denote by $\delta_{k,n} > 0$ the smallest positive solution to the equation

$$10^{k+2} C_{k,n}(k) s \left(1 + \frac{s}{2k}\right)^k = 1$$

in the variable $s > 0$, where

$$C_{k,n}(i) = \begin{cases} 1 & \text{if} \quad i = 0, \\ 3 + 10C_{k,n}(i-1) + (16k)^{n-1}(1 + 10C_{k,n}(i-1))^n & \text{if} \quad i \in \{1, ..., k\}. \end{cases}$$

We now consider the smooth, bijective and increasing function $h_{k,n} : (0, \delta_{k,n}) \to (1, \infty)$ defined by

$$h_{k,n}(s) = \left[1 - 10^{k+2} C_{k,n}(k) s \left(1 + \frac{s}{2k}\right)^k\right]^{-1}.$$

For every $k \in \{1, ..., n\}$, let

$$\alpha_{MP}(k,n) = \begin{cases} 1 - \left[1 + \frac{2}{h_{1,n}^{-1}(2)}\right]^{-1} & \text{if} \quad k = 1, \\ 1 - \left[1 + \left(\frac{1 + \ldots + \frac{h_{k-1,n}^{-1}(1 + \frac{\delta_{k,n}}{2k})}{2(k-1)}}{h_{1,n}^{-1}\left(1 + \ldots + \frac{h_{k-1,n}^{-1}(1 + \frac{\delta_{k,n}}{2k})}{2(k-1)}\right)}\right)^n\right]^{-1} & \text{if} \quad k \in \{2, ..., n\}, \end{cases}$$

be the so-called *Munn-Perelman constant*, see Munn [22, Tables 4 and 5, p. 749-750].

Following the idea from Kristály [14], our quantitative result reads as follows:

**Theorem 1.2.** *Under the same assumptions as in Theorem 1.1, we have*

(i) *if $C \leq \frac{n+2}{n-2} K_0$, the order of the fundamental group $\pi_1(M)$ is bounded above by $\left(\frac{C}{K_0}\right)^{\frac{n}{4}}$ (in particular, if $C < 2^{\frac{4}{n}} K_0$, then $M$ is simply connected*);

(ii) *if $C < \alpha_{MP}(k_0, n)^{-\frac{4}{n}} K_0$ for some $k_0 \in \{1, ..., n\}$ then $\pi_1(M) = ... = \pi_{k_0}(M) = 0$;*

(iii) *if $C < \alpha_{MP}(n, n)^{-\frac{4}{n}} K_0$ then $M$ is contractible*;

(iv) *$C = K_0$ if and only if $(M, g)$ is isometric to the Euclidean space $\mathbb{R}^n$.*

## 2. Proof of Theorems 1.1&1.2

Throughout this section, we assume the hypotheses of Theorem 1.1 are verified, i.e., $(M, g)$ is an $n$-dimensional complete open Riemannian manifold with nonnegative Ricci curvature which satisfies the distance Laplacian growth condition and supports the second-order Sobolev inequality $(\mathbf{SSI})_C$ for $C > 0$.

(i) The inequality $C \geq K_0$ follows in a similar way as in Djadli, Hebey and Ledoux [7, Lemmas 1.1&1.2] by using a geodesic, normal coordinate system at a given point $x_0 \in M$.

(ii) Before starting the proof explicitly, we notice that one can assume that $C > K_0$; otherwise, if $C = K_0$ then we can assume that $(\mathbf{SSI})_C$ holds with $C = K_0 + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small, and then letting $\varepsilon \to 0$. Now, we split the proof into five steps.

**Step 1.** *ODE via the Euclidean optimizer.* We consider the function $G : (0, \infty) \to \mathbb{R}$ defined by

$$G(\lambda) = \int_{\mathbb{R}^n} \frac{\mathrm{d}x}{(\lambda + |x|^2)^{n-2}}.$$

The layer cake representation shows that for every $\lambda > 0$,

$$G(\lambda) = 2(n-2)\omega_n \int_0^\infty \frac{t^{n+1}}{(\lambda + t^2)^{n-1}} \mathrm{d}t = \frac{2^{4-n}\pi^{\frac{n+1}{2}}}{(n-4)\Gamma(\frac{n-1}{2})} \lambda^{\frac{4-n}{2}}. \tag{2}$$

Clearly, $G$ is smooth on $(0, \infty)$.

We recall now by (1) that

$$\left(\int_{\mathbb{R}^n} |u_\lambda|^{2^\sharp} \mathrm{d}x\right)^{\frac{2}{2^\sharp}} = K_0 \int_{\mathbb{R}^n} (\Delta u_\lambda)^2 \, \mathrm{d}x\,,$$

where

$$u_\lambda(x)=(\lambda+|x|^2)^{\frac{4-n}{2}},\ x\in\mathbb{R}^n,$$

and $\lambda>0$ is arbitrarily fixed. In terms of the function $G$, the above equality can be rewritten as

$$\left(\frac{G''(\lambda)}{(n-2)(n-1)}\right)^{\frac{n-4}{n}}=K_0(n-4)^2\left\{4G(\lambda)-4\lambda G'(\lambda)+\frac{n-2}{n-1}\lambda^2G''(\lambda)\right\}.$$

By introducing the function

$$G_0(\lambda)=\left(\frac{K_0}{C}\right)^{\frac{n}{4}}(G(\lambda)-\lambda G'(\lambda)),\ \lambda>0,$$

the latter relation is equivalent to the ODE

$$\left(-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}}=C(n-4)^2\left\{4G_0(\lambda)-\frac{n-2}{n-1}\lambda G_0'(\lambda)\right\},\quad \lambda>0. \tag{3}$$

**Step 2.** *ODI via* $(\mathbf{SSI})_C$. Let $x_0\in M$ be the point for which the distance Laplacian growth condition holds and let $F:(0,\infty)\to\mathbb{R}$ be defined by

$$F(\lambda)=\int_M\frac{\mathrm{d}v_g}{(\lambda+\rho^2)^{n-2}}.$$

Since $(M,g)$ has nonnegative Ricci curvature, the Bishop-Gromov comparison theorem asserts that $\mathrm{vol}_g[B(x_0,t)]\le\omega_nt^n$ for every $t>0$; thus, by the layer cake representation and a change of variables, it turns out that

$$\begin{aligned}F(\lambda)&=\int_0^\infty\mathrm{vol}_g\left\{x\in M:\frac{1}{(\lambda+\rho(x)^2)^{n-2}}>s\right\}\mathrm{d}s\\&=2(n-2)\int_0^\infty\mathrm{vol}_g[B(x_0,t)]\frac{t}{(\lambda+t^2)^{n-1}}\mathrm{d}t\\&\le 2(n-2)\omega_n\int_0^\infty\frac{t^{n+1}}{(\lambda+t^2)^{n-1}}\mathrm{d}t\\&=G(\lambda).\end{aligned}\tag{4}$$

Thus $0<F(\lambda)<\infty$ for every $\lambda>0$, and $F$ is smooth. In a similar way,

$$F'(\lambda)=-2(n-2)(n-1)\int_0^\infty\mathrm{vol}_g[B(x_0,t)]\frac{t}{(\lambda+t^2)^n}\mathrm{d}t,\tag{5}$$

$$F''(\lambda)=2(n-2)(n-1)n\int_0^\infty\mathrm{vol}_g[B(x_0,t)]\frac{t}{(\lambda+t^2)^{n-1}}\mathrm{d}t,$$

and for every $\lambda>0$,

$$-\infty<G'(\lambda)\le F'(\lambda)<0\ \text{ and }\ 0<F''(\lambda)\le G''(\lambda)<\infty.\tag{6}$$

Let $\lambda>0$ be fixed; we observe that the function

$$w_\lambda=\left(\lambda+\rho^2\right)^{\frac{4-n}{2}}$$

can be approximated by elements from $C_0^\infty(M)$; in particular, by using an approximation procedure, one can use the function $w_\lambda$ as a test-function in $(\mathbf{SSI})_C$. Accordingly,

$$\left(\int_M|w_\lambda|^{2^\sharp}\mathrm{d}v_g\right)^{\frac{2}{2^\sharp}}\le C\int_M(\Delta_gw_\lambda)^2\,\mathrm{d}v_g,\ \forall\lambda>0.\tag{7}$$

A chain rule and the eikonal equation $|\nabla_g\rho|=1$ shows that

$$(\Delta_gw_\lambda)^2=(n-4)^2(\lambda+\rho^2)^{-n}\left(\lambda+(3-n)\rho^2+(\lambda+\rho^2)\rho\Delta_g\rho\right)^2.$$

Since the Ricci curvature is nonnegative on $(M,g)$, we first have the distance Laplacian comparison $\rho\Delta_g\rho \le n-1$. Thus,

$$\lambda + (3-n)\rho^2 + (\lambda+\rho^2)\rho\Delta_g\rho \le 2\rho^2 + n\lambda\,,\ \forall\lambda>0. \tag{8}$$

On the other hand, by the distance Laplacian growth condition, i.e., $\rho\Delta_g\rho \ge n-5$, we obtain that

$$-(2\rho^2+n\lambda) \le \lambda + (3-n)\rho^2 + (\lambda+\rho^2)\rho\Delta_g\rho\,,\ \forall\lambda>0. \tag{9}$$

Consequently, by (8) and (9), we have that

$$|\lambda + (3-n)\rho^2 + (\lambda+\rho^2)\rho\Delta_g\rho| \le 2\rho^2 + n\lambda\,,\ \forall\lambda>0.$$

Thus, it turns out that

$$(\Delta_g w_\lambda)^2 \le (n-4)^2(\lambda+\rho^2)^{-n}\left(2\rho^2+n\lambda\right)^2\,.$$

According to the latter estimate, relation (7) can be written in terms of the function $F$ as

$$\left(\frac{F''(\lambda)}{(n-2)(n-1)}\right)^{\frac{n-4}{n}} \le C(n-4)^2\left\{4F(\lambda) - 4\lambda F'(\lambda) + \frac{n-2}{n-1}\lambda^2F''(\lambda)\right\}\,.$$

By defining the function

$$F_0(\lambda) = F(\lambda) - \lambda F'(\lambda),$$

the latter relation is equivalent to the ordinary differential inequality

$$\left(-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}} \le C(n-4)^2\left\{4F_0(\lambda) - \frac{n-2}{n-1}\lambda F_0'(\lambda)\right\}\,,\quad \lambda>0. \tag{10}$$

**Step 3.** *Comparison of $G$ and $F$ near the origin.* We claim that

$$\liminf_{\lambda\to0}\frac{F(\lambda)-\lambda F'(\lambda)}{G(\lambda)-\lambda G'(\lambda)} \ge 1\,.$$

To see this, fix $\varepsilon>0$ arbitrarily small. Since

$$\lim_{t\to0}\frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n} = 1\,,$$

there exists a $\delta>0$ such that $\mathrm{vol}_g[B(x_0,t)] \ge (1-\varepsilon)\omega_n t^n$ for all $t\in(0,\delta]$. Thus, by (4) and (5), we have

$$\begin{aligned} F(\lambda) &\ge 2(n-2)\int_0^\delta \mathrm{vol}_g[B(x_0,t)]\frac{t}{(\lambda+t^2)^{n-1}}\mathrm{d}t \\ &\ge 2(n-2)\omega_n(1-\varepsilon)\int_0^\delta \frac{t^{n+1}}{(\lambda+t^2)^{n-1}}\mathrm{d}t \\ &= 2(n-2)\omega_n\lambda^{\frac{4-n}{2}}(1-\varepsilon)\int_0^{\delta\lambda^{-\frac12}}\frac{s^{n+1}}{(1+s^2)^{n-1}}\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} -\lambda F'(\lambda) &\ge 2(n-2)(n-1)\lambda\int_0^\delta \mathrm{vol}_g[B(x_0,t)]\frac{t}{(\lambda+t^2)^{n}}\mathrm{d}t \\ &\ge 2(n-2)(n-1)\omega_n\lambda(1-\varepsilon)\int_0^\delta \frac{t^{n+1}}{(\lambda+t^2)^{n}}\mathrm{d}t \\ &= 2(n-2)(n-1)\omega_n\lambda^{\frac{4-n}{2}}(1-\varepsilon)\int_0^{\delta\lambda^{-\frac12}}\frac{s^{n+1}}{(1+s^2)^{n}}\mathrm{d}s. \end{aligned}$$

Combining this estimates with relation (2), we obtain

$$\liminf_{\lambda\to 0}\frac{F(\lambda)-\lambda F'(\lambda)}{G(\lambda)-\lambda G'(\lambda)}\geq 1-\varepsilon\,.$$

Letting $\varepsilon\to 0$, we get the required claim.

**Step 4.** *Global comparison of $G_0$ and $F_0$.* We claim that

$$F_0(\lambda)\geq G_0(\lambda),\ \forall\lambda>0. \tag{11}$$

First of all, by Step 3 and the fact that $C>K_0$, we have

$$\begin{aligned}\liminf_{\lambda\to 0}\frac{F_0(\lambda)}{G_0(\lambda)} &= \left(\frac{C}{K_0}\right)^{\frac{n}{4}}\liminf_{\lambda\to 0}\frac{F(\lambda)-\lambda F'(\lambda)}{G(\lambda)-\lambda G'(\lambda)}\\ &\geq \left(\frac{C}{K_0}\right)^{\frac{n}{4}}\\ &> 1\,.\end{aligned}$$

Thus, for sufficiently small $\delta_0>0$, one has

$$F_0(\lambda)\geq G_0(\lambda),\ \forall\lambda\in(0,\delta_0). \tag{12}$$

In fact, we shall prove that $\delta_0$ can be arbitrarily large in (12) which ends the proof of (11). By contradiction, let us assume that $F_0(\lambda_0)<G_0(\lambda_0)$ for some $\lambda_0>0$; clearly, $\lambda_0>\delta_0$. Due to (12), we may set

$$\lambda_s=\sup\{\lambda<\lambda_0;\ F_0(\lambda)=G_0(\lambda)\}\,.$$

Then, $\lambda_s<\lambda_0$ and for any $\lambda\in[\lambda_s,\lambda_0]$, one has $F_0(\lambda)\leq G_0(\lambda)$. For $\lambda>0$, we define the function $\varphi_\lambda:(0,\infty)\to\mathbb{R}$ by

$$\varphi_\lambda(t)=t^{\frac{n-4}{n}}-C(n-2)^2(n-4)^2\lambda^2t\,.$$

We notice that $\varphi_\lambda$ is non-decreasing in $(0,t_\lambda]$, where

$$t_\lambda=\frac{\lambda^{-\frac{n}{2}}}{(Cn(n-4)(n-2)^2)^{\frac{n}{4}}}\,.$$

On one hand, a straightforward computation shows that for every $\lambda>0$, one has

$$0<-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}=\left(\frac{K_0}{C}\right)^{\frac{n}{4}}\frac{G''(\lambda)}{(n-2)(n-1)}<t_\lambda.$$

On the other hand, relation (6) and the assumption $C\leq\frac{n+2}{n-2}K_0$ imply that for every $\lambda>0$,

$$0<-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}=\frac{F''(\lambda)}{(n-2)(n-1)}\leq\frac{G''(\lambda)}{(n-2)(n-1)}\leq t_\lambda.$$

We claim that

$$F_0'(\lambda)\geq G_0'(\lambda)\,,\ \forall\lambda\in[\lambda_s,\lambda_0]. \tag{13}$$

Since $F_0(\lambda)\leq G_0(\lambda)$ for every $\lambda\in[\lambda_s,\lambda_0]$, by relations (10) and (3) we have that

$$\begin{aligned}\varphi_\lambda\left(-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}\right) &= \left(-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}}+C(n-4)^2\frac{n-2}{n-1}\lambda F_0'(\lambda)\\ &\leq 4C(n-4)^2F_0(\lambda)\\ &\leq 4C(n-4)^2G_0(\lambda)\\ &= \left(-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}}+C(n-4)^2\frac{n-2}{n-1}\lambda G_0'(\lambda)\\ &= \varphi_\lambda\left(-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}\right),\qquad\forall\lambda\in[\lambda_s,\lambda_0].\end{aligned}$$

By the monotonicity of $\varphi_\lambda$ on $(0,t_\lambda]$, relation (13) follows at once. In particular, the function $F_0-G_0$ is non-decreasing on the interval $[\lambda_s,\lambda_0]$. Consequently, we have

$$0=F_0(\lambda_s)-G_0(\lambda_s)\le F_0(\lambda_0)-G_0(\lambda_0)<0,$$

a contradiction, which shows the validity of (11).

**Step 5.** *Global volume non-collapsing property concluded.* Inequality (11) can be rewritten into

$$\int_0^\infty \left(\mathrm{vol}_g[B(x_0,t)]-b\omega_n t^n\right)\frac{((n-1)\lambda+t^2)t}{(\lambda+t^2)^n}\mathrm{d}t\ge 0\,,\ \forall\lambda>0, \tag{14}$$

where

$$b=(C^{-1}K_0)^{\frac{n}{4}}\,.$$

The Bishop-Gromov comparison theorem implies that the function $t\mapsto\frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n}$ is non-increasing on $(0,\infty)$; thus, the asymptotic volume growth

$$\limsup_{t\to\infty}\frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n}=b_0$$

is finite (and independent of the base point $x_0$).

We shall prove that $b_0\ge b$. By contradiction, let us suppose that $b_0=b-\varepsilon_0$ for some $\varepsilon_0>0$. Thus, there exists a number $N_0>0$ such that

$$\frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n}\le b-\frac{\varepsilon_0}{2}\,,\ \forall t\ge N_0\,. \tag{15}$$

For simplicity of notation, let

$$f(\lambda,t)=\frac{((n-1)\lambda+t^2)t}{(\lambda+t^2)^n},\quad \lambda,t>0.$$

Substituting (15) into (14) and by using the Bishop-Gromov comparison theorem, we obtain for every $\lambda>0$ that

$$\begin{aligned}
0 &\le \int_0^\infty \left(\mathrm{vol}_g[B(x_0,t)]-b\omega_n t^n\right)f(\lambda,t)\mathrm{d}t\\
&\le \int_0^{N_0}\mathrm{vol}_g[B(x_0,t)]f(\lambda,t)\mathrm{d}t+(b-\frac{\varepsilon_0}{2})\omega_n\int_{N_0}^\infty t^nf(\lambda,t)\mathrm{d}t-b\omega_n\int_0^\infty t^nf(\lambda,t)\mathrm{d}t\\
&\le \omega_n\int_0^{N_0}t^nf(\lambda,t)\mathrm{d}t-b\omega_n\int_0^{N_0}t^nf(\lambda,t)\mathrm{d}t-\frac{\varepsilon_0}{2}\omega_n\int_{N_0}^\infty t^nf(\lambda,t)\mathrm{d}t\\
&= \omega_n(1-b+\frac{\varepsilon_0}{2})\int_0^{N_0}t^nf(\lambda,t)\mathrm{d}-\frac{\varepsilon_0}{2}\omega_n\int_0^\infty t^nf(\lambda,t)\mathrm{d}t.
\end{aligned}$$

Note that for every $\lambda>0$, one has

$$\begin{aligned}
I_1(\lambda) &= \int_0^\infty t^nf(\lambda,t)\mathrm{d}t=\lambda^{\frac{4-n}{2}}\int_0^\infty s^nf(1,s)\mathrm{d}s\\
&= \frac{2^{1-n}\pi^{\frac12}(n^2-4n+6)\Gamma(\frac{n}{2}+1)}{(n-2)(n-4)\Gamma(\frac{n+1}{2})}\lambda^{\frac{4-n}{2}},
\end{aligned}$$

and

$$\begin{aligned}
I_2(\lambda) &= \int_0^{N_0}t^nf(\lambda,t)\mathrm{d}t=\int_0^{N_0}t^{n+1}\frac{(n-1)\lambda+t^2}{(\lambda+t^2)^n}\mathrm{d}t\\
&\le (n-1)N_0^{n+1}\lambda^{-n+1}+N_0^{n+3}\lambda^{-n}.
\end{aligned}$$

Consequently, the above estimates show that for every $\lambda>0$,

$$M_0\lambda^{\frac{4-n}{2}}\le M_1\lambda^{-n+1}+M_2\lambda^{-n},$$

where $M_0, M_1, M_2 > 0$ are independent on $\lambda > 0$. It is clear that the latter inequality is not valid for large values of $\lambda > 0$, i.e., we arrived to a contradiction. Accordingly, for every $r > 0$,

$$\frac{\text{vol}_g[B(x_0,r)]}{\omega_n r^n} \geq \limsup_{t\to\infty} \frac{\text{vol}_g[B(x_0,t)]}{\omega_n t^n} = b_0 \geq b = (C^{-1}K_0)^{\frac{n}{4}}.$$

Since the asymptotic volume growth of $(M, g)$ is independent of the point $x_0$, we obtain the desired property, which completes the proof of Theorem 1.1. □

**Remark 2.1.** Note that relation (9) is equivalent to the distance Laplacian growth condition. Indeed, a simple computation in Step 2 led us to relation (9) through the distance Laplacian growth condition. Conversely, if $\lambda \to 0$ in (9), we obtain precisely that $\rho\Delta_g\rho \geq n - 5$.

*Proof of Theorem 1.2.* (i) Due to Anderson [1] and Li [18], if $\text{vol}_g[B(x,r)] \geq k_0\omega_n r^n$ for every $r > 0$, then $(M, g)$ has finite fundamental group $\pi_1(M)$ and its order is bounded above by ${k_0}^{-1}$. By Theorem 1.1 (ii) the property follows directly. In particular, if $C < 2^{\frac{4}{n}}K_0$, then the order of $\pi_1(M)$ is strictly less than 2, thus $M$ is simply connected.

(ii) First of all, due to Munn [22, Table 5] and a direct computation, for every $n \geq 5$ one has

$$\alpha_{MP}(1,n)^{-\frac{4}{n}} = 2^{\frac{4}{n}} < \frac{n+2}{n-2}.$$

Thus, since $\alpha_{MP}(\cdot, n)$ is increasing, the values $\alpha_{MP}(k,n)^{-\frac{4}{n}}K_0$ are within the range where Theorem 1.1 (ii) applies, $k \in \{1, ..., n\}$.

Now, let us assume that $C < \alpha_{MP}(k_0,n)^{-\frac{4}{n}}K_0$ for some $k_0 \in \{1, ..., n\}$. By Theorem 1.1 (ii) we have the following estimate for the asymptotic volume growth of $(M, g)$:

$$\lim_{t\to\infty} \frac{\text{vol}_g[B(x,t)]}{\omega_n t^n} \geq \left(\frac{K_0}{C}\right)^{\frac{n}{4}} > \alpha_{MP}(k_0,n) \geq ... \geq \alpha_{MP}(1,n).$$

Therefore, due to Munn [22, Theorem 1.2], one has that $\pi_1(M) = ... = \pi_{k_0}(M) = 0$.

(iii) If $C < \alpha_{MP}(n,n)^{-\frac{4}{n}}K_0$, then $\pi_1(M) = ... = \pi_n(M) = 0$. Standard topological argument implies -based on Hurewicz's isomorphism theorem,- that $M$ is contractible.

(iv) If $C = K_0$ then by Theorem 1.1 (ii) and the Bishop-Gromov volume comparison theorem follows that $\text{vol}_g[B(x,r)] = \omega_n r^n$ for every $x \in M$ and $r > 0$. Now, the equality in Bishop-Gromov theorem implies that $(M, g)$ is isometric to the Euclidean space $\mathbb{R}^n$. The converse is trivial. □

## 3. Final remarks

We conclude the paper with some remarks and further questions:

(a) If $(M, g)$ is a complete $n-$dimensional Riemannian manifold and $x_0 \in M$ is arbitrarily fixed, we notice that

$$\rho\Delta_g\rho = n - 1 + \rho\frac{J'(u,\rho)}{J(u,\rho)} \quad \text{a.e. on } M,$$

where $\rho(x) = \rho(x, x_0)$, $x = \exp_{x_0}(\rho(x)u)$ for some $u \in T_{x_0}M$ with $|u| = 1$, and $J$ is the density of the volume form in normal coordinates, see Gallot, Hulin and Lafontaine [10, Proposition 4.16]. On one hand, if the Ricci curvature on $(M, g)$ is nonnegative, one has $J'(u, \rho) \leq 0$. On the other hand, the distance Laplacian growth condition $\rho\Delta_g\rho \geq n - 5$ is equivalent to

$$\frac{J'(u,\rho)}{J(u,\rho)} \geq -\frac{4}{\rho},$$

which is a curvature restriction on the manifold $(M, g)$. We are wondering if the latter condition can be removed from our results, which plays a crucial role in our arguments; see also Remark 2.1. Examples of Riemannian manifolds verifying the distance Laplacian growth condition (that are isometrically immersed into $\mathbb{R}^N$ with $N$ large enough) can be found in Carron [4].

(b) The requirement $C \leq \frac{n+2}{n-2}K_0$ is needed to explore the monotonicity of the function $\varphi_\lambda$ on $(0, t_\lambda]$, see Step 4 in the proof of Theorem 1.1. Although this condition is widely enough to obtain quantitative results, cf. Theorem 1.2, we still believe that it can be somehow removed.

(c) Let $(M, g)$ be an $n$-dimensional complete open Riemannian manifold with nonnegative Ricci curvature and fix $k \in \mathbb{N}$ such that $n > 2k$. Let us consider for some $C > 0$ the $k$-th order Sobolev inequality

$$\left( \int_M |u|^{\frac{2n}{n-2k}} \mathrm{d}v_g \right)^{\frac{n-2k}{n}} \leq C \int_M (\Delta_g^{k/2} u)^2 \mathrm{d}v_g, \ \forall u \in C_0^\infty(M), \qquad (\mathbf{SI})_C^k$$

where

$$\Delta_g^{k/2} u = \begin{cases} \Delta_g^{k/2} u & \text{if } \ k \text{ is even,} \\ |\nabla_g (\Delta_g^{(k-1)/2} u)| & \text{if } \ k \text{ is odd.} \end{cases}$$

Clearly, $(\mathbf{SI})_C^1 = (\mathbf{FSI})_C$ and $(\mathbf{SI})_C^2 = (\mathbf{SSI})_C$. It would be interesting to establish $k$-th order counterparts of Theorems 1.1&1.2 with $k \geq 3$, noticing that the optimal Euclidean $k$-th order Sobolev inequalities are well known with the optimal constant

$$\Lambda_k = \left[ \pi^k n(n-2k) \Pi_{i=1}^{k-1} (n^2 - 4i^2) \right]^{-1} \left( \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right)^{2k/n},$$

and the unique class of extremal functions (up to translations and multiplications)

$$u_\lambda(x) = (\lambda + |x^2|)^{\frac{2k-n}{2}}, \ x \in \mathbb{R}^n,$$

see Cotsiolis and Tavoularis [5], Liu [21]. Once we use $w_\lambda = (\lambda + \rho^2)^{\frac{2k-n}{2}}$ as a test-function in $(\mathbf{SI})_C^k$, after a multiple application of the chain rule we have to estimate in a sharp way the terms appearing in $\Delta_g^{k/2} w_\lambda$, similar to the eikonal equation $|\nabla_g \rho| = 1$ and the distance Laplacian comparison $\rho \Delta_g \rho \leq n-1$, respectively. In the second-order case this fact is highlighted in relation (8). Furthermore, higher-order counterparts of the distance Laplacian growth condition $\rho \Delta_g \rho \geq n - 5$ should be found, (see relation (9) for the second order case), assuming this condition cannot be removed, see (a).

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