COEFFICIENT ESTIMATES FOR A NEW SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper, we introduce and investigate a new subclass $\mathcal{H}^{h,p}_\Sigma (\lambda; \Theta)$ of analytic and bi-univalent functions in the open unit disk $U$. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$ 

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $U$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $U$. In fact, the Koebe one-quarter theorem [7] ensures that the image of $U$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1} (f(z)) = z \quad (z \in U)$$

and

$$f \left( f^{-1} (w) \right) = w \quad \left( |w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right).$$

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In fact, the inverse function $f^{-1}$ is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.$$ 

Denote by $f * \Theta$ the Hadamard product (or convolution) of the functions $f$ and $\Theta$, that is, if $f$ is given by (1.1) and $\Theta$ is given by

$$\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0),$$

then

$$(f * \Theta)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$  

For two functions $f$ and $\Theta$, analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $\Theta$ in $\mathbb{U}$, and write

$$f(z) < \Theta(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = \Theta(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) < \Theta(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = \Theta(0) \quad \text{and} \quad f(\mathbb{U}) \subset \Theta(\mathbb{U}).$$

Furthermore, if the function $\Theta$ is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) < \Theta(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = \Theta(0) \quad \text{and} \quad f(\mathbb{U}) \subset \Theta(\mathbb{U}).$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions in the class $\Sigma$, see [13] (see also [2]). In fact, the aforesaid work of Srivastava et al. [13] essentially revived the investigation of various subclasses of the bi-univalent function class $\Sigma$ in recent years; it was followed by such works as those by El-Ashwah [8], Frasin and Aouf [9], Aouf et al. [1], and others (see, for example, [3–6, 10–12, 14]).

Definition 1. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min \left\{ \Re (h(z)) , \Re (p(z)) \right\} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1.$$  

Also let the function $f$, defined by (1.1), be in the analytic function class $A$ and let $\Theta \in \Sigma$. We say that

$$f \in \mathcal{H}^h_p (\lambda ; \Theta) \quad (\lambda \geq 1).$$
if the following conditions are satisfied:

\[ f \in \Sigma \quad \text{and} \quad (1 - \lambda) \left( \frac{f \ast (\Theta)}{z} \right) + \lambda (f \ast (\Theta))' (z) \in h (U) \quad (z \in U) \quad (1.4) \]

and

\[ (1 - \lambda) \left( \frac{(f \ast (\Theta)^{-1}}{w} \right) + \lambda \left( (f \ast (\Theta)^{-1})' (w) \in p (U) \quad (w \in U), \quad (1.5) \]

where the function \((f \ast (\Theta)^{-1}\) is defined by

\[ (f \ast (\Theta)^{-1} (w) = w - a_2 b_2 w^2 + \left(2a_2^2 b_2^2 - a_3 b_3\right) w^3 \]

\[ - \left(5a_2^3 b_2^3 - 5a_2 b_2 a_3 b_3 + a_4 b_4\right) w^4 + \cdots. \quad (1.6) \]

Remark 1. If we let

\[ \Theta (z) = \frac{z}{1 - z} = z + \sum_{n=2}^{\infty} z^n, \]

then the class \( \mathcal{H}_\Sigma^{h, p} (\lambda; \Theta) \) reduces to the class denoted by \( \mathcal{B}_\Sigma^{h, p} (\lambda) \) which is the subclass of the functions \( f \in \Sigma \) satisfying

\[ (1 - \lambda) \left( \frac{f (z)}{z} \right) + \lambda f' (z) \in h (U) \]

and

\[ (1 - \lambda) \frac{g (w)}{w} + \lambda g' (w) \in p (U), \]

where the function \( g \) is defined by

\[ g (w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \quad (1.7) \]

This class is introduced and studied by Xu et al. [15]. Also we get the function class

\[ \mathcal{H}_\Sigma^{h, p} \left(1; \frac{z}{1 - z}\right) = \mathcal{H}_\Sigma^{h, p} \]

introduced and studied by Xu et al. [14].

Remark 2. If we let

\[ h (z) = \left( \frac{1 + z}{1 - z} \right)^{\alpha} \quad \text{and} \quad p (z) = \left( \frac{1 - z}{1 + z} \right)^{\alpha} \quad (0 < \alpha \leq 1), \]

then the class \( \mathcal{H}_\Sigma^{h, p} (\lambda; \Theta) \) reduces to the class denoted by \( \mathcal{B}_\Sigma (\theta, \alpha, \lambda) \) which is the subclass of the functions \( f \in \Sigma \) satisfying

\[ \left| \arg \left\{ (1 - \lambda) \left( \frac{f \ast (\Theta)}{z} \right) + \lambda (f \ast (\Theta))' (z) \right\} \right| < \frac{\alpha \pi}{2} \]

and

\[ \left| \arg \left\{ (1 - \lambda) \left( \frac{(f \ast (\Theta)^{-1}}{w} \right) + \lambda \left( (f \ast (\Theta)^{-1})' (w) \right\} \right| < \frac{\alpha \pi}{2}. \]
where the function \((f * \Theta)^{-1}\) is defined by (1.6). This class is introduced and studied by El-Ashwah [8]. In this class,

(i) if we set

\[ \Theta(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n, \]

then the class \(B_{\Sigma}(\Theta, \alpha, \lambda)\) reduces to the class denoted by \(B_{\Sigma}(\alpha, \lambda)\) introduced and studied by Frasin and Aouf [9]. Also we have the class

\[ B_{\Sigma}(\alpha, 1) = \mathcal{H}_{\Sigma}^{\alpha} \]

introduced and studied by Srivastava et al. [13].

(ii) if we set

\[ \Theta(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n} \frac{1}{n!} z^n, \]

then the class \(B_{\Sigma}(\Theta, \alpha, \lambda)\) reduces to the class denoted by \(T_{q,s}^{\Sigma} [a_1; b_1, \alpha, \lambda]\) introduced and studied Aouf et al. [1].

Remark 3. If we let

\[ h(z) = \frac{1 + (1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1 - (1-2\beta)z}{1+z}, \]

then the class \(\mathcal{H}_{\Sigma}^{h,p}(\lambda; \Theta)\) reduces to the class denoted by \(B_{\Sigma}(\Theta, \beta, \lambda)\) which is the subclass of the functions \(f \in \Sigma\) satisfying

\[ \Re \left\{ (1-\lambda) \frac{(f * \Theta)(z)}{z} + \lambda (f * \Theta)'(z) \right\} > \beta \]

and

\[ \Re \left\{ (1-\lambda) \frac{(f * \Theta)^{-1}(w)}{w} + \lambda \left((f * \Theta)^{-1}\right)'(w) \right\} > \beta, \]

where the function \((f * \Theta)^{-1}\) is defined by (1.6). This class is introduced and studied by El-Ashwah [8]. In this class,

(i) if we set

\[ \Theta(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n, \]

then the class \(B_{\Sigma}(\Theta, \beta, \lambda)\) reduces to the class denoted by \(B_{\Sigma}(\beta, \lambda)\) introduced and studied by Frasin and Aouf [9]. Also we have the class

\[ B_{\Sigma}(\beta, 1) = \mathcal{H}_{\Sigma}^{\beta} \]

introduced and studied by Srivastava et al. [13].
(ii) if we set
\[ \Theta(z) = z + \sum_{n=2}^{\infty} \frac{(a_1^n \cdots a_q^n)}{(b_1^n \cdots b_q^n)} \frac{1}{n!} z^n, \]
then the class \( B_{\Sigma} (\Theta, \beta, \lambda) \) reduces to the class denoted by \( T_{q, \beta} [a_1; b_1, \beta, \lambda] \) introduced and studied by Aouf et al. [1].

2. A SET OF GENERAL COEFFICIENT ESTIMATES

In this section, we state and prove our general results involving the bi-univalent function class \( H_{h;p} \) given by Definition 1.

**Theorem 1.** Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (1.1) be in the function class \( H_{h;p} \) with
\[ \Theta \in \Sigma \quad \text{and} \quad \Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0). \]
Then
\[ |a_2| \leq \frac{1}{b_2} \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\lambda)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{4(1+2\lambda)}} \right\} \]
and
\[ |a_3| \leq \frac{1}{b_3} \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\lambda)^2} + \frac{|h''(0)| + |p''(0)|}{4(1+2\lambda)}, \frac{|h''(0)|}{2(1+2\lambda)} \right\}. \]

**Proof.** First of all, we write the argument inequalities in (1.4) and (1.5) in their equivalent forms as follows:
\[ (1-\lambda) \frac{(f * \Theta)(z)}{z} + \lambda (f * \Theta)'(z) = h(z) \quad (z \in \mathbb{U}) \]
and
\[ (1-\lambda) \frac{1}{w} \left( f * \Theta \right)^{-1}(w) + \lambda \left( f * \Theta \right)^{-1}'(w) = p(w) \quad (w \in \mathbb{U}), \]
respectively, where \( h(z) \) and \( p(w) \) satisfy the conditions of Definition 1. Furthermore, the functions \( h(z) \) and \( p(w) \) have the following Taylor-Maclaurin series expansions:
\[ h(z) = 1 + h_1 z + h_2 z^2 + \cdots \]
and
\[ p(w) = 1 + p_1 w + p_2 w^2 + \cdots, \]
respectively. Now, upon equating the coefficients of
\[ (1 - \lambda) \frac{(f * \Theta)(z)}{z} + \lambda (f * \Theta)'(z) \]
with those of \( h(z) \) and the coefficients of
\[ (1 - \lambda) \frac{(f * \Theta)^{-1}(w)}{w} + \lambda \left( (f * \Theta)^{-1} \right)'(w) \]
with those of \( p(w) \), we get
\[ (1 + \lambda) b_2 a_2 = h_1, \quad (2.3) \]
\[ (1 + 2\lambda) b_3 a_3 = h_2, \quad (2.4) \]
\[ -(1 + \lambda) b_2 a_2 = p_1 \]
and
\[ -(1 + 2\lambda) b_3 a_3 + 2(1 + 2\lambda) b_2^2 a_2^2 = p_2. \quad (2.5) \]
From (2.3) and (2.5), it follows that
\[ h_1 = -p_1 \quad (2.7) \]
and
\[ 2(1 + \lambda)^2 b_2^2 a_2^2 = h_1^2 + p_1^2. \quad (2.8) \]
Also from (2.4) and (2.6), we find that
\[ 2(1 + 2\lambda) b_2^2 a_2^2 = h_2 + p_2. \quad (2.9) \]
Therefore, we find from the equations (2.8) and (2.9) that
\[ |a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(1 + \lambda)^2 b_2^2} \]
and
\[ |a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{4(1 + 2\lambda) b_2^2}, \]
respectively. So we get the desired estimate on the coefficient \( |a_2| \) as asserted in (2.1).

Next, in order to find the bound on the coefficient \( |a_3| \), by subtracting (2.6) from (2.4), we obtain
\[ 2(1 + 2\lambda) b_3 a_3 - 2(1 + 2\lambda) b_2^2 a_2^2 = h_2 - p_2. \quad (2.10) \]
Upon substituting the value of \( a_2^2 \) from (2.8) into (2.10), it follows that
\[ a_3 = \frac{h_1^2 + p_1^2}{2(1 + \lambda)^2 b_3} + \frac{h_2 - p_2}{2(1 + 2\lambda) b_3}. \]
We thus find that
\[
|a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(1 + \lambda)^2 b_3} + \frac{|h''(0)| + |p''(0)|}{4(1 + 2\lambda) b_3}.
\]

On the other hand, upon substituting the value of \(a_2^2\) from (2.9) into (2.10), it follows that

\[
a_3 = \frac{h_2}{(1 + 2\lambda) b_3}.
\]

Consequently, we have

\[
|a_3| \leq \frac{|h''(0)|}{2(1 + 2\lambda) b_3}.
\]

This evidently completes the proof of Theorem 1.

\[\square\]

Remark 4. If we take

\[
\Theta(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n
\]

in Theorem 1, then we have the estimates obtained by Srivastava et al. [12, Corollary 1] which is an improvement of the estimates given by Xu et al. [15, Theorem 3]. In addition the above condition on \(\Theta\), if we set \(\lambda = 1\), then we get [14, Theorem 3].

If we let

\[
h(z) = \left( \frac{1+z}{1-z} \right)^{\alpha} \quad \text{and} \quad p(z) = \left( \frac{1-z}{1+z} \right)^{\alpha} \quad (0 < \alpha \leq 1)
\]

in Theorem 1, then we have Corollary 1 below.

Corollary 1. Let the function \(f(z)\) given by the Taylor-Maclaurin series expansion (1.1) be in the function class

\[
B_{2\Sigma}(\Theta, \alpha, \lambda) \quad (\lambda \geq 1, \ 0 < \alpha \leq 1)
\]

with

\[
\Theta \in \Sigma \quad \text{and} \quad \Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0).
\]

Then

\[
|a_2| \leq \frac{1}{b_2} \left\{ \begin{array}{ll}
\frac{2\alpha}{1+\lambda}, & \lambda \geq 1 + \sqrt{2} \\
\sqrt{\frac{2}{1+2\lambda}} \alpha, & 1 \leq \lambda < 1 + \sqrt{2}
\end{array} \right.
\]

and

\[
|a_3| \leq \frac{2\alpha^2}{(1 + 2\lambda) b_3}.
\]
Remark 5. It is easy to see, for the coefficient $|a_2|$, that

$$\frac{2\alpha}{1+\lambda} \leq \frac{2\alpha}{\sqrt{(1+\lambda)^2 + \alpha(1+2\lambda - \lambda^2)}} \quad \left(0 < \alpha \leq 1, \lambda \geq 1 + \sqrt{2}\right)$$

and

$$\sqrt{\frac{2}{1+2\lambda}} \alpha \leq \frac{2\alpha}{\sqrt{(1+\lambda)^2 + \alpha(1+2\lambda - \lambda^2)}} \quad \left(0 < \alpha \leq 1, 1 \leq \lambda < 1 + \sqrt{2}\right).$$

On the other hand, for the coefficient $|a_3|$, we have

$$\frac{2\alpha^2}{1+2\lambda} \leq \frac{2\alpha}{1+2\lambda} \leq \frac{4\alpha^2}{(1+\lambda)^2} + \frac{2\alpha}{1+2\lambda} \quad \left(0 < \alpha \leq 1, \lambda \geq 1\right).$$

Thus, clearly, Corollary 1 is an improvement of the estimates obtained by El-Ashwah [8, Theorem 1].

Remark 6. (i) If we take

$$\Theta(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$$

in Corollary 1, then we have an improvement of the estimates obtained by Frasin and Aouf [9, Theorem 2.2]. In addition the above condition on $\Theta$, if we set $\lambda = 1$, then we get an improvement of the estimates obtained by Srivastava et al. [13, Theorem 1].

(ii) If we take

$$\Theta(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n n!} \frac{1}{z^n}$$

in Corollary 1, then we have an improvement of the estimates obtained by Aouf et al. [1, Theorem 4].

If we let

$$h(z) = \frac{1+(1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1-(1-2\beta)z}{1+z} \quad (0 \leq \beta < 1)$$

in Theorem 1, then we have Corollary 2 below.

**Corollary 2.** Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class

$$B_{\Sigma}(\Theta, \beta, \lambda) \quad (\lambda \geq 1, 0 \leq \beta < 1)$$
with
\[ \Theta \in \Sigma \quad \text{and} \quad \Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0). \]

Then
\[ |a_2| \leq \frac{1}{b_2} \min \left\{ \frac{2(1 - \beta)}{1 + \lambda}, \sqrt{\frac{2(1 - \beta)}{1 + 2\lambda}} \right\} \]
and
\[ |a_3| \leq \frac{2(1 - \beta)}{(1 + 2\lambda)b_3}. \]

Remark 7. Corollary 2 is an improvement of the estimates obtained by El-Ashwah [8, Theorem 2].

Remark 8. (i) If we take
\[ \Theta(z) = \frac{z}{1 - z} = z + \sum_{n=2}^{\infty} z^n \]
in Corollary 2, then we have an improvement of the estimates obtained by Frasin and Aouf [9, Theorem 3.2]. In addition the above condition on \( \Theta \), if we set \( \lambda = 1 \), then we get an improvement of the estimates obtained by Srivastava et al. [13, Theorem 2].

(ii) If we take
\[ \Theta(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n n!} z^n \]
in Corollary 2, then we have an improvement of the estimates obtained by Aouf et al. [1, Theorem 8].

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