Miskolc Mathematical Notes
Vol. 16 (2015), No. 2, pp. 1181–1189
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1044

CONVERGENCE AND SUBSEQUENTIAL CONVERGENCE OF REGULARLY GENERATED SEQUENCES

SEFA ANIL SEZER AND İBRAHIM ÇANAK

Received 03 November, 2013

Abstract. In this paper we recover convergence and subsequential convergence of a sequence of real numbers regularly generated by another sequence in some sequence spaces under certain conditions. We also give some information about the behavior of a sequence whose generator is given in terms of a moderately divergent sequence.

2010 *Mathematics Subject Classification:* 40A05; 40E05

Keywords: regularly generated sequences, slow oscillation, moderate oscillation, moderate divergence, subsequential convergence, weighted means

1. INTRODUCTION

Throughout this paper, $\mathbb{N}_0$ will denote the set of all nonnegative integers. Let $u = (u_n)$ be a sequence of real numbers and any term with a negative index be zero. Let $p = (p_n)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty, \ n \to \infty. \tag{1.1}$$

The n^{th} weighted mean of the sequence (u_n) is defined by

$$\sigma_{n,p}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \tag{1.2}$$

for all $n \in \mathbb{N}_0$.

The sequence (u_n) is said to be summable by the weighted mean method determined by the sequence p; in short, $(\overline{N}, p)$ summable to a finite number s if

$$\lim_{n\to\infty} \sigma_{n,p}(u) = s.$$

The difference between u_n and its n^{th} weighted mean $\sigma_{n,p}(u)$, which is called the weighted Kronecker identity, is given by

$$u_n - \sigma_{n,p}(u) = V_{n,p}(\Delta u), \tag{1.3}$$

where

$$V_{n,p}(\Delta u) := \frac{1}{P_n} \sum_{k=0}^{n} P_{k-1} \Delta u_k \tag{1.4}$$

and

$$\Delta u_n = u_n - u_{n-1}. \tag{1.5}$$

The $(\overline{N}, p)$ summability method is regular if and only if $P_n \to \infty$ as $n \to \infty$. If $p_n = 1$ for all $n \in \mathbb{N}_0$, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

A sequence (u_n) is slowly oscillating [14] if

$$\lim_{\lambda \to 1^+} \limsup_{n\to\infty} \max_{n+1 \le k \le [\lambda n]} | u_k - u_n | = 0, \tag{1.6}$$

where $[\lambda n]$ denotes the integer part of λn.

The space of all slowly oscillating sequences is denoted by $\mathcal{SO}$. Dik [9] proved that if a sequence (u_n) is slowly oscillating, then $(V_{n,1}(\Delta u))$ is bounded and slowly oscillating.

A generalization of slow oscillation is given as follows.

A sequence (u_n) is moderately oscillating [14] if

$$\limsup_{n\to\infty} \max_{n+1 \le k \le [\lambda n]} | u_k - u_n | < \infty \tag{1.7}$$

for $\lambda > 1$. The space of all moderately oscillating sequences is denoted by $\mathcal{MO}$.

Set

$$t_n = n \frac{p_n}{P_{n-1}}, \tag{1.8}$$

for $n \in \mathbb{N}_0$. We say that (u_n) is regularly generated by a sequence $\alpha = (\alpha_n)$ in some sequence space $\mathcal{A}$ and α is called a generator of (u_n) if

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{t_k}{k} \alpha_k. \tag{1.9}$$

The space of all sequences which are regularly generated by sequences in $\mathcal{A}$ is denoted by $U(\mathcal{A})$.

If (u_n) is regularly generated by a sequence (α_n) where $(\Delta\alpha_n) \in \mathcal{SO}$, we write $(u_n) \in U(\mathcal{SO}_\Delta)$. If (u_n) is regularly generated by a sequence (α_n) where $(\alpha_n) \in \mathcal{SO}$, we write $(u_n) \in U(\mathcal{SO})$.

A positive sequence (u_n) is O-regularly varying [12] if

$$\limsup_{n\to\infty} \frac{u_{[\lambda n]}}{u_n} < \infty \tag{1.10}$$

for $\lambda > 1$ and it is slowly varying if

$$\lim_{n\to\infty} \frac{u_{[\lambda n]}}{u_n} = 1. \tag{1.11}$$

It was proved by [11] that if a positive sequence (u_n) is O-regularly varying, then $(\log u_n)$ is slowly varying.

A positive sequence (u_n) is moderately divergent [13] if for every $\lambda > 1$

$$u_n = o(n^{\lambda-1}),\ n \to \infty \tag{1.12}$$

and

$$\sum_{n=1}^{\infty} \frac{u_n}{n^{\lambda}} < \infty. \tag{1.13}$$

We denote the space of all moderately divergent sequences by $\mathcal{MD}$. Note that every slowly oscillating sequence of positive numbers is moderately divergent.

The convergence of a sequence (u_n) implies that (u_n) is bounded and $\Delta u_n = o(1)$ as $n \to \infty$. But it is clear that the converse of this implication is not true in general. In the case where (u_n) is bounded with $\Delta u_n = o(1)$ as $n \to \infty$, we may not recover convergence of (u_n) but we may have convergence of some subsequences of (u_n). A new kind of convergence is defined as follows (See [8] for more details on subsequentially convergent sequences):

A sequence $u = (u_n)$ is said to be subsequentially convergent if there exists a finite interval $I(u)$ such that all accumulation points of (u_n) are in $I(u)$ and every point of $I(u)$ is an accumulation point of (u_n).

Recently, several results in terms of regularly generated sequences for different purposes have been obtained by Dik et al. [10], Çanak et al. [1], Çanak and Totur [3], Çanak et al. [2], Çanak et al. [7], Çanak and Totur [5] and many more. In this paper, we first recover convergence and subsequential convergence of a sequence which is regularly generated by another sequence in some sequence spaces under certain conditions. Secondly, we give some information about the behavior of a sequence whose generator is given in terms of a moderately divergent sequence.

2. The preliminary results

We need the following lemmas for the proof of our results.

Lemma 1 ([8]). *Let (u_n) be a bounded sequence of real numbers. If $\Delta u_n = o(1)$ as $n \to \infty$, then (u_n) converges subsequentially.*

Lemma 2. *If $(\sum_{k=1}^{n} t_k\alpha_k)$ is moderately oscillating, then $(\sum_{k=1}^{n} \frac{t_k}{k}\alpha_k)$ converges.*

Proof. Set $R_n := \exp\left(\left|\sum_{k=1}^{n} t_k\alpha_k\right|\right)$. Then we have

$$\frac{R_{[\lambda n]}}{R_n} \leq \exp\left(\left|\sum_{k=n+1}^{[\lambda n]} t_k\alpha_k\right|\right). \tag{2.1}$$

Taking limsup of both sides of (2.1) as $n \to \infty$ gives

$$\limsup_{n\to\infty} \frac{R_{[\lambda n]}}{R_n} \le \exp\left(\limsup_{n\to\infty}\left|\sum_{k=n+1}^{[\lambda n]} t_k\alpha_k\right|\right). \tag{2.2}$$

Since $(\sum_{k=1}^{n} t_k\alpha_k)$ is moderately oscillating, we have

$$\limsup_{n\to\infty} \frac{R_{[\lambda n]}}{R_n} \tag{2.3}$$

is finite for $\lambda > 1$. This says that (R_n) is O-regularly varying. Since (R_n) is O-regularly varying, $(\log R_n)$ is slowly varying. It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\left|\sum_{k=1}^{n} t_k\alpha_k\right|^p < \infty \tag{2.4}$$

for $p > 1$. This implies that

$$\sum_{n=1}^{\infty} \frac{t_n}{n}\alpha_n < \infty. \tag{2.5}$$

□

Lemma 3. *If* $(\sum_{k=1}^{n} \frac{t_k}{k}\alpha_k)$ *converges, then* $\sigma_{n,p}(\alpha) = o(1),\ n \to \infty$.

Proof. Set $\gamma_n := \sum_{k=1}^{n} \frac{t_k}{k}\alpha_k$. Then we obtain

$$\alpha_n = \frac{P_{n-1}}{p_n}\Delta\gamma_n \tag{2.6}$$

and

$$\sigma_{n,p}(\alpha) = V_{n,p}(\Delta\gamma) \tag{2.7}$$

for $n \in \mathbb{N}_0$. Since (γ_n) converges, it follows by the weighted Kronecker identity

$$\gamma_n - \sigma_{n,p}(\gamma) = V_{n,p}(\Delta\gamma) \tag{2.8}$$

that

$$V_{n,p}(\Delta\gamma) = o(1),\ \ n \to \infty.$$

This completes the proof. □

Lemma 4 ([6])**.** *Let* (p_n) *satisfy the condition*

$$1 \le \frac{P_n}{n} \to 1,\ \ n \to \infty. \tag{2.9}$$

If (u_n) *is slowly oscillating, then* $(V_{n,p}(\Delta u))$ *is slowly oscillating and bounded.*

Lemma 5 ([15])**.** *Let* (u_n) *be Cesàro summable to* s*. If* (u_n) *is slowly oscillating, then* (u_n) *converges to* s.

3. THE MAIN RESULTS

Theorem 1. *Suppose that*

$$\left(\sum_{k=1}^{n} t_k \alpha_k\right) \in \mathcal{MO}, \tag{3.1}$$

$$1 \le \frac{P_n}{n} \to 1, \ n \to \infty, \tag{3.2}$$

$$t_n = O(1), n \to \infty. \tag{3.3}$$

If $(u_n) \in U(\mathcal{SO}_\Delta)$, *then* (u_n) *converges subsequentially.*

Proof. Since $(u_n) \in U(\mathcal{SO}_\Delta)$, (u_n) can be written as

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{t_k}{k} \alpha_k, \tag{3.4}$$

where $(\Delta\alpha_n) \in \mathcal{SO}$. Moderate oscillation of $\left(\sum_{k=1}^{n} t_k \alpha_k\right)$ implies convergence of $(\gamma_n) = \left(\sum_{k=1}^{n} \frac{t_k}{k} \alpha_k\right)$ by Lemma 2 and $\sigma_{n,p}(\alpha) = o(1)$ as $n \to \infty$ by Lemma 3. Hence, (u_n) is $(\overline{N}, p)$ summable to the limit of (γ_n). By the condition (3.3), it follows that

$$\frac{\alpha_n}{n} \to 0, n \to \infty \tag{3.5}$$

by Lemma 3. Since $(\Delta\alpha_n) \in \mathcal{SO}$, we have that

$$\Delta\alpha_n \to 0, n \to \infty \tag{3.6}$$

by Lemma 5. Taking the backward difference of (3.4), we have

$$\Delta u_n = \Delta\alpha_n + \alpha_n \frac{p_n}{P_{n-1}} \tag{3.7}$$

for $n \in \mathbb{N}_0$.

It follows by (3.3), (3.5) and (3.6) that

$$\Delta u_n = o(1), n \to \infty. \tag{3.8}$$

To complete the proof, it suffices to prove that (u_n) is bounded. Applying Lemma 4 to $(v_n) = \left(\sum_{k=1}^{n} \alpha_k t_k\right)$, and taking $(v_n) \in \mathcal{SO}$ into account, we obtain $(V_{n,p}(\alpha t))$ is bounded and slowly oscillating, where $\alpha t = (\alpha_n t_n)$.

From the weighted Kronecker identity

$$S_n(\alpha) - \sigma_{n,p}(S(\alpha)) = V_{n,p}(\alpha) \tag{3.9}$$

where $S(\alpha) = (S_n(\alpha)) = (\sum_{k=0}^{n} \alpha_k)$, we have

$$\alpha_n - \frac{p_n}{P_{n-1}} V_{n,p}(\alpha) = \Delta V_{n,p}(\alpha). \tag{3.10}$$

Replacing α_n by $\alpha_n t_n$ in (3.10) and then dividing by t_n, we have

$$\alpha_n = \frac{V_{n,p}(\alpha t)}{n} + \frac{P_{n-1}}{np_n}\Delta V_{n,p}(\alpha t). \tag{3.11}$$

It follows from (3.11) that (α_n) is bounded. Hence, (u_n) is bounded. By Lemma 1, (u_n) is subsequentially convergent. □

Theorem 2. *Suppose that*

$$\left(\sum_{k=1}^{n} t_k\alpha_k\right) \in \mathcal{MO}, \tag{3.12}$$

$$1 \leq \frac{P_n}{n} \to 1, \ n \to \infty, \tag{3.13}$$

$$1 < \liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} < \limsup_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} < \infty, \quad \textit{for} \quad \lambda > 1, \tag{3.14}$$

$$1 < \liminf_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} < \limsup_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} < \infty, \quad \textit{for} \quad 0 < \lambda < 1, \tag{3.15}$$

$$t_n = O(1), \ n \to \infty. \tag{3.16}$$

If $(u_n) \in U(\mathcal{SO})$, *then* (u_n) *converges.*

Proof. Assume that $(u_n) \in U(\mathcal{SO})$. Then, (u_n) can be written as

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{t_k}{k}\alpha_k \tag{3.17}$$

where $(\alpha_n) \in \mathcal{SO}$. From (3.17), we have

$$V_{n,p}(\Delta u) = V_{n,p}(\Delta\alpha) + \sigma_{n,p}(\alpha). \tag{3.18}$$

Moderate oscillation of $\left(\sum_{k=1}^{n} t_k\alpha_k\right)$ implies convergence of $(\gamma_n) = \left(\sum_{k=1}^{n} \frac{t_k}{k}\alpha_k\right)$ by Lemma 2 and $\sigma_{n,p}(\alpha) = o(1)$ as $n \to \infty$ by Lemma 3. Therefore, (u_n) is $(\overline{N}, p)$ summable to the limit of (γ_n).

Since (α_n) is slowly oscillating, $(V_{n,p}(\Delta\alpha))$ is bounded and slowly oscillating by Lemma 4.

It follows from (3.18) that $(V_{n,p}(\Delta u)) \in \mathcal{SO}$ and bounded. Since (u_n) is $(\overline{N}, p)$ summable, (u_n) converges to $\lim_{n\to\infty} \sigma_{n,p}(u)$ by Theorem 6 in [4]. □

Theorem 3. *Suppose that* (u_n) *is regularly generated by* (α_n) *and*

$$\frac{p_n}{P_{n-1}} - \frac{p_{n+1}}{P_n} = O\left(\frac{1}{n^2}\right), n \to \infty. \tag{3.19}$$

If

$$\sum_{k=1}^{n} \alpha_k = n^{\gamma} m_n \tag{3.20}$$

for some $(m_n) \in \mathcal{MD}$ *and some* $\gamma \in (0,1)$, *then*

i) (u_n) *is* $(\overline{N}, p)$ *summable.*

ii) $u_n = \Delta(n^{\gamma} m_n) + \beta_n$, *where* $\beta_n = o(1), n \to \infty$.

iii) $u_n = o(n), n \to \infty$ *and* $\sum_{n=1}^{\infty} \frac{u_n}{n^2} < \infty$.

Proof. i) By Abel's partial summation formula, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{p_k}{P_{k-1}} \alpha_k &= \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} (S_k(\alpha) - S_{k-1}(\alpha)) \\
&= \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} S_k(\alpha) - \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} S_{k-1}(\alpha) \\
&= \sum_{k=1}^{n} \left(\frac{p_k}{P_{k-1}} S_k(\alpha) - \frac{p_{k+1}}{P_k} S_k(\alpha) \right) + \frac{p_n}{P_{n-1}} S_n(\alpha) - \frac{p_1}{P_0} S_0 \\
&= \frac{p_n}{P_{n-1}} S_n(\alpha) + \sum_{k=1}^{n-1} \left(\frac{p_k}{P_{k-1}} - \frac{p_{k+1}}{P_k} \right) S_k(\alpha)
\end{aligned} \tag{3.21}$$

Since $S_n(\alpha) = n^{\gamma} m_n$ for some $(m_n) \in \mathcal{MD}$, we have

$$\frac{p_n}{P_{n-1}} S_n(\alpha) = O\left(\frac{m_n}{n^{1-\gamma}} \right), \quad n \to \infty. \tag{3.22}$$

By moderate divergence of (m_n), we have

$$\frac{p_n}{P_{n-1}} S_n(\alpha) = o(1), \quad n \to \infty. \tag{3.23}$$

The second term on the right of (3.21) converges by (3.19). It follows from the representation

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} \alpha_k, \tag{3.24}$$

that (u_n) is $(\overline{N}, p)$ summable.

ii) Note that the sequence (β_n) defined by $\beta_n = \frac{t_n \alpha_n}{n}$ for $n \in \mathbb{N}_0$ converges to zero. From the representation and the condition (3.20) it follows that

$$u_n = \Delta(n^{\gamma} m_n) + \beta_n \tag{3.25}$$

where $\beta_n = \frac{t_n}{n} \alpha_n$.

iii) By ii), we have

$$u_n = n^{\gamma} m_n - (n-1)^{\gamma} m_{n-1} + \beta_n. \tag{3.26}$$

Dividing (3.26) by n, we have

$$\frac{u_n}{n} = \frac{m_n}{n^{1-\gamma}} - \frac{m_{n-1}}{(n-1)^{1-\gamma}} + \frac{\beta_n}{n}. \tag{3.27}$$

Since $(m_n) \in \mathcal{MD}$ and $\beta_n = o(1)$, we have

$$\frac{u_n}{n} = o(1), n \to \infty. \tag{3.28}$$

By (3.26), we obtain

$$\sum_{k=2}^{n} \frac{u_k}{k^2} = \sum_{k=2}^{n} \frac{m_k}{k^{2-\gamma}} - \sum_{k=2}^{n} \frac{m_{k-1}}{(k-1)^{2-\gamma}} + \sum_{k=2}^{n} \frac{\beta_k}{k^2}. \tag{3.29}$$

Taking the limit of both sides of (3.29) as $n \to \infty$, we obtain $\sum_{n=1}^{\infty} \frac{u_n}{n^2} < \infty$. □

REFERENCES

[1] İ. Çanak, M. Dik, and F. Dik, "Conditions for convergence and subsequential convergence," *Appl. Math. Lett.*, vol. 19, no. 10, pp. 1042–1045, 2006, doi: 10.1016/j.aml.2005.11.015.

[2] İ. Çanak, F. Hasekiler, and D. Kebapcı, "Some Tauberian theorems for regularly generated sequences," *Comput. Math. Appl.*, vol. 62, no. 21, pp. 4486–4491, 2011, doi: 10.1016/j.camwa.2011.10.027.

[3] İ. Çanak and Ü. Totur, "A note on Tauberian theorems for regularly generated sequences," *Tamkang J. Math.*, vol. 39, no. 2, pp. 187–191, 2008, doi: 10.5556/j.tkjm.39.2008.29.

[4] İ. Çanak and Ü. Totur, "Some Tauberian theorems for the weighted mean methods of summability," *Comput. Math. Appl.*, vol. 62, pp. 2609–2615, 2011, doi: 10.1016/j.camwa.2011.07.066.

[5] İ. Çanak and Ü. Totur, "Some Tauberian conditions for Cesàro summability method," *Math. Slovaca*, vol. 62, no. 2, pp. 271–280, 2012, doi: 10.2478/s12175-012-0008-y.

[6] İ. Çanak and Ü. Totur, "Tauberian theorems for the (J, p) summability method," *Appl. Math. Lett.*, vol. 25, no. 10, pp. 1430–1434, 2012, doi: 10.1016/j.aml.2011.12.017.

[7] İ. Çanak, Ü. Totur, and M. Dik, "On Tauberian theorems for (A, k) summability method," *Math. Slovaca*, vol. 61, no. 6, pp. 993–1001, 2011, doi: 10.2478/s12175-011-0064-8.

[8] F. Dik, "Tauberian theorems for convergence and subsequential convergence with moderately oscillatory behavior," *Math. Morav.*, vol. 5, pp. 19–56, 2001.

[9] M. Dik, "Tauberian theorems for sequences with moderately oscillatory control modulo," *Math. Morav.*, vol. 5, pp. 57–94, 2001.

[10] M. Dik, F. Dik, and İ. Çanak, "Classical and neoclassical Tauberian theorems for regularly generated sequences," *Far East J. Math. Sci. (FJMS)*, vol. 13, no. 2, pp. 233–240, 2004.

[11] D. Natsis, *Convergence of Fourier series and representations of Fourier and Fourier-Stieltjes coefficients*, ser. Ph. D. Dissertation. University of Missouri-Rola, 1991.

[12] Č. V. Stanojević, "O-regularly varying convergence moduli of Fourier and Fourier-Stieltjes series," *Math. Ann.*, vol. 279, no. 1-2, pp. 103–115, 1987, doi: 10.1007/BF01456193.

[13] Č. V. Stanojević, "Fourier and Fourier-Stieltjes series with compacticity property." Moscow: Steklov Mathematical Institute, 1990.

[14] Č. V. Stanojević, *Analysis of Divergence: Control and Management of Divergent Process*, ser. Graduate Research Seminar Lecture Notes, 1998.

[15] Č. V. Stanojević, *Analysis of Divergence: Applications to the Tauberian theory*, ser. Graduate Research Seminars, 1999.

Authors' addresses

Sefa Anıl Sezer
Istanbul Medeniyet University, Department of Mathematics, 34720 Istanbul, Turkey
Current address: Ege University, Department of Mathematics, 35100 Izmir, Turkey
E-mail address: sefaanil.sezer@medeniyet.edu.tr

İbrahim Çanak
Ege University, Department of Mathematics, 35100 Izmir, Turkey
E-mail address: ibrahimcanak@yahoo.com