



## ON THE LOCAL KERNEL BASED APPROXIMATION OF HIGHLY OSCILLATORY INTEGRALS

M. UDDIN, Z. MINULLAH, A. ALI, AND KAMRAN

*Received 02 January, 2014*

*Abstract.* A local kernel based interpolation scheme for computing oscillatory integrals is presented. David Levin [17] transformed the integration problem into a systems of ODEs/PDEs and suggested collocation methods for solving the resultant ODEs/PDEs numerically. We extended the approach of David Levin and propose a local kernel based interpolation scheme for solving the resultant ODEs/PDEs. The proposed numerical scheme is capable of computing the oscillatory integrals for large data points in the domain of integrations.

2010 *Mathematics Subject Classification:* 65C30; 65G20; 65L80

*Keywords:* local interpolation schemes, radial kernels basis, oscillatory integrals

### 1. INTRODUCTION

In this section, we describe a local meshless method for integrals of the form

$$I = \int_D f(\mathbf{x}) e^{iq(\mathbf{x})} d\mathbf{x}, \quad (1.1)$$

where  $f$  is non oscillatory,  $D$  is a finite interval  $[a, b]$  in one-dimensional case  $\mathbf{x} = x$  and  $|q'(\mathbf{x})| \gg (b-a)^{-1}$ , while a rectangle  $[a, b] \times [c, d]$  in two-dimensional case  $\mathbf{x} = (x, y)$  and  $q_x(x, y) \gg (b-a)^{-1}$  and  $q_y(x, y) \gg (d-c)^{-1}$ . The function  $q(x)$  is called the oscillator of (1.1). If we write  $q(x) = \omega g(x)$  intuitively, the value  $I$  of the integral will decreases with the increment in frequency parameter  $\omega$ , because increasing integral oscillations cancel out.

The standard integration technique are unable to compute highly oscillatory integrals, when the frequency is significantly larger than the number of quadrature points. Many integration techniques have been developed for computing highly oscillatory integrals. [4, 7, 11–14, 17, 18, 20, 21]. Highly oscillatory integral (1.1) occurs in a wide range of practical problems and applications ranging from celestial mechanics to nonlinear optics, plasma transport, fluid dynamics, Schrodinger spectra, computerized tomography [13]. The methods developed in [2, 7, 18] for evaluating rapidly oscillatory integrals are applicable to any integral of the form (1.1) and performed very well when  $q' = W$  a constant frequency. For varying frequency  $q'$  David Levin

[17] has shown that if the function  $f$  is of the form

$$\mathcal{L}p(x) \equiv p'(x) + iq'(x)p(x) = f(x), \quad a \leq x \leq b, \quad (1.2)$$

then the integral could be evaluated directly as

$$I = \int_a^b \frac{d}{dx}(p(x)e^{iq(x)})dx = p(b)e^{iq(b)} - p(a)e^{iq(a)}, \quad (1.3)$$

and for the two-dimensional case the formulations in equations (1.2)-(1.3) are given as

$$\mathcal{L}p(x, y) \equiv p_{xy} + iq_y p_x + iq_x p_y + (iq_{xy} - q_x q_y)p = f(x, y), \quad (1.4)$$

and the integral could readily be obtained as

$$\begin{aligned} I &= \int_a^b \int_c^d \frac{\partial^2}{\partial x \partial y}(pe^{iq})dxdy \\ &= p(b, d)e^{iq(b, d)} - p(a, d)e^{iq(a, d)} - p(b, c)e^{iq(b, c)} + p(a, c)e^{iq(a, c)}. \end{aligned} \quad (1.5)$$

The general solution of this equation is so that in general is as oscillatory as the integrand in (1.1). If the functions  $f$  and  $q'$  are slowly oscillatory, then there also exists a slowly oscillatory solution  $p_0$  of (1.2) [17]. By looking for this  $p_0$  we avoid all the difficulties caused by the rapid oscillations. Writing the general solution of (1.1) in the form

$$p(x) = p_0(x) + ce^{-iq(x)}, \quad (1.6)$$

and using (1.3) we get

$$I = p_0(b)e^{iq(b)} - p_0(a)e^{iq(a)}. \quad (1.7)$$

Therefore,  $p_0(a)$  and  $p_0(b)$  are actually all we need. A problem arises since we do not have any initial condition for  $p_0$ . Furthermore, even if  $p_0(a)$  were known, any forward integration scheme for (1.2) will develop high oscillations due to round-off errors. However, the rapidly oscillatory homogeneous solution  $ce^{-iq(x)}$  can be excluded automatically by solving (1.2) by collocation using slowly oscillatory basis functions. By this method  $p_0$  is singled out without specifying any boundary conditions.

**Theorem 1.** *If  $f$  and  $q$  are slowly oscillatory and  $|q'(x)| \gg 1$  then the differential equation  $\mathcal{L}p(x) \equiv p'(x) + iq'(x)p(x) = f(x)$ , has a particular solution  $p_0$  which is slowly oscillatory in comparison with the rapidly oscillatory homogeneous solution  $e^{iq}$ .*

The resultant system of ODEs/PDEs recovered in the Levin method can be approximated by using suitable interpolation technique with carefully selected basis functions. To avoid the difficulties of large linear systems, we could divide the domain of integration, or we could use some compactly supported basis in the interpolation technique. An other way to use some localized interpolation technique. In the

present study we construct a local kernel based interpolation scheme for approximating highly oscillatory integrals.

## 2. COLLOCATION WITH KERNEL BASIS FUNCTION

A collocation with radial kernels (RBFs) of multivariate data points  $x_k, k = 1, \dots, N \in \Omega$  takes the form

$$p(x) = \sum_{k=1}^n \lambda_k \phi(\|x - x_k\|). \quad (2.1)$$

Here  $\|\cdot\|$  denotes the standard Euclidean vector norm,  $\phi_\varepsilon(r)$  is some radial kernel, where the entries of the system matrix  $\mathbf{A}$  are  $\mathbf{A}_{i,j} = \phi(\|x_i - x_j\|), i = 1, \dots, N, j = 1, \dots, N$ . Similarly the kernel basis interpolant of the function  $f(x)$  in the David Levin method is defined as

$$\mathcal{L}p(x) = \sum_{j=1}^N \lambda_j \phi'(\|x - x_j\|) + i\omega q'(x) \sum_{j=1}^N \lambda_j \phi(\|x - x_j\|), \quad j = 1, \dots, N. \quad (2.2)$$

The coefficients  $\lambda_k$  are determined in such a way that  $\mathcal{L}p(x_j) = f(x_j)$  at interpolation nodes  $x_1, \dots, x_N \in \Omega$ , we have

$$\mathbf{B}^v = \mathbf{f}. \quad (2.3)$$

Computing the above linear system at the evaluations points  $x_1, \dots, x_M \in \Omega$ , the solution  $p(x)$  is given as

$$\mathbf{p} = \mathbf{M}\mathbf{B}^{-1}\mathbf{f} = \mathbf{W}\mathbf{f}. \quad (2.4)$$

The approximation to the integrals (1.3) and (1.5) are

$$I_N = p(b)e^{iq(b)} - p(a)e^{iq(a)}, \quad (2.5)$$

and

$$I_N = p(b,d)e^{iq(b,d)} - p(a,d)e^{iq(a,d)} - p(b,c)e^{iq(b,c)} + p(a,c)e^{iq(a,c)}, \quad (2.6)$$

respectively. In this Kansa's global approach [15] the differentiation matrix  $\mathbf{W}$  is full dense, and is very hard to solve the above linear system, when the data points increases, especially in multi-dimensions.

## 3. LOCAL KERNEL BASIS FUNCTION INTERPOLATION

Here we construct a local variant of Kansa's method for solving the ODEs/PDEs in David Levin method. The present approximation scheme is based on the local methods in [31]. For a given set of interpolation nodes  $x_1, \dots, x_N \in \Omega$ , these nodes can be chosen any where in the domain. In the present interpolation scheme at a given node  $x_i \in \Omega$ , the local interpolant takes the form

$$p(x_i) = \sum_{j=1}^n \lambda_j \phi(\|x_i - x_j\|), \quad x_j \in \Omega_i \subset \Omega, \quad (3.1)$$

where  $\Omega_i$  is a small local domain contains the  $n$  neighboring nodes corresponding to the nodes  $x_i$ . Hence to each local domain  $\Omega_i, i = 1, \dots, N \in \Omega$ , the  $N$  number of  $n \times n$  linear systems,

$$\mathbf{p}^i = \mathbf{A}^{i\omega}, \quad i = 1, \dots, N. \quad (3.2)$$

The entries of the matrix  $\mathbf{A}^i$  are  $a_{kj}^i = \phi(\|x_k - x_j\|)$ ,  $x_k, x_j \in \Omega_i, i = 1, \dots, N$ . The matrix  $\mathbf{A}^i$  is called local interpolation matrix, and each system have to be solved for the expansion coefficients.

Now to approximate  $\mathcal{L}p(x)$ , where  $\mathcal{L}$  is a linear differential operator, we have

$$\mathcal{L}p(x_i) = \sum_{j=1}^n \lambda_j \mathcal{L}\phi(\|x_i - x_j\|), \quad x_j \in \Omega_i \subset \Omega. \quad (3.3)$$

We write equation (3.3) as a dot product of two vectors, as

$$\mathcal{L}p(x_i) = \mathbf{v}^i \cdot \mathbf{v}^i, \quad \mathbf{v}^i = \mathcal{L}\phi(\|x_i - x_j\|), \quad x_j \in \Omega_i. \quad (3.4)$$

The expansion coefficients from (3.3) is substituted in (3.4), we get

$$\mathcal{L}p(x_i) = \mathbf{v}^i (\mathbf{A}^i)^{-1} \mathbf{p}^i = \mathbf{w}^i \mathbf{p}^i, \quad (3.5)$$

where,

$$\mathbf{w}^i = \mathbf{v}^i (\mathbf{A}^i)^{-1}, \quad (3.6)$$

is the weight corresponding to center  $x_i$ . Hence for all centers locations, we have

$$\mathcal{L}p = \mathbf{W}\mathbf{p}, \quad (3.7)$$

consequently, the local interplant of the function  $f$  is

$$\mathcal{L}\mathbf{p} = \mathbf{W}\mathbf{p} = \mathbf{f}. \quad (3.8)$$

The solution  $\mathbf{p}$

$$\mathbf{p} = \mathbf{W}^{-1}\mathbf{f}. \quad (3.9)$$

Here the differentiation matrix  $\mathbf{W}$  is sparse, each row of the matrix  $\mathbf{W}$  contains  $n$  non-zeros elements and  $N-n$  zeros elements, where  $n$  is the number of elements in the local domain  $\Omega_i$ .

### 3.1. Choosing a good value of shape parameter

In our local scheme, we used the multiquadrics,  $\phi_\varepsilon(r) = \sqrt{1 + (\varepsilon r)^2}$ , as well as the Gaussian kernel  $\phi_\varepsilon(r) = e^{-\varepsilon^2 r^2}$ , as the basis functions in our local interpolant. As usual these kernel functions contain a parameter  $\varepsilon$  and the solution accuracy greatly depends on this parameter. In the literature a variety of algorithm are available for choosing good value of shape parameters [6, 9, 10, 23, 27, 29, 30]. In our computation we used the following algorithm based on the uncertainty principle [24] related to the kernel based interpolation theory for choosing good value of parameter.

**Algorithm**

$\kappa = 1$   
 $10^{13} < \kappa < 10^{15}$   
 while  $\kappa < \kappa_{min}$  and  $\kappa > \kappa_{max}$   
 $\mathbf{L}, \mathbf{S}, \mathbf{U} = svd(\mathbf{A}^i)$   
 $\kappa = \max(D)/\min(D)$   
 if  $\kappa < \kappa_{min}$ ,  $\varepsilon = \varepsilon - \delta\varepsilon$   
 if  $\kappa > \kappa_{max}$ ,  $\varepsilon = \varepsilon + \delta\varepsilon$   
 $\varepsilon$  (good) =  $\varepsilon$ .

Here svd is the singular value decomposition of the interpolation matrix  $\mathbf{A}^i$ .  $\mathbf{L}, \mathbf{U}$  are  $n \times n$  orthogonal matrices and  $\mathbf{D}$  is  $n \times n$  diagonal matrix contains the  $n$  singular values of  $\mathbf{A}^i$ , and  $\kappa = \max(D)/\min(D)$  is the condition number of the matrix  $\mathbf{A}^i$ . The good value of shape parameter is obtained from the above algorithm, then the svd is used to compute  $(\mathbf{A}^i)^{-1} = (\mathbf{LSU}^T)^{-1} = \mathbf{US}^{-1}\mathbf{L}^T$  (see [5]).

**Theorem 2.** *If  $\Phi(r) = \phi(\sqrt{r})$  is completely monotone but not constant on  $[0, \infty)$ , then for any points  $x_k$  in  $R^d$ , the system matrix  $\mathbf{A}$  in the kernel interpolation is positive definite.*

**Definition 1.** A function  $\Phi(r)$  is completely monotone on  $[0, \infty)$ , if

- (1)  $\Phi(r) \in C[0, \infty)$ .
- (2)  $\Phi(r) \in C^\infty[0, \infty)$ .
- (3)  $(-1)^k \frac{d^k}{dr^k} \Phi(r) \geq 0$ , for  $r > 0$  and  $k = 0, 1, 2, \dots$

**Theorem 3.** *If  $\phi(r) \in C[0, \infty)$  with  $\phi(0) > 0$  and  $\phi(\rho) < 0$  for some  $\rho > 0$ , then there is an upper limit on the dimension  $d$  for which the interpolation problem is nonsingular for all points distributions.*

Various choices of radial kernels have been used in the literature. The existence and uniqueness of the interpolants  $p(x)$  have been discussed in the literature [1, 19, 22] for the radial kernels listed in Table 1.

**Lemma 1** ([32]). *Suppose  $q(x)$  is real-valued and smooth in  $(a, b)$ , and that  $q^k(x) \geq 1$  for all  $x \in (a, b)$  for a fixed value of  $k$ . Then*

$$\left| \int_a^b e^{i\omega q(x)} \right| \leq c(k)/\omega^{1/k}, \tag{3.10}$$

*holds when (i)  $k \geq 2$ , or (ii)  $k = 1$  and  $q'(x)$  is monotonic. The bound  $c(k)$  is independent of  $q(x)$  and  $\omega$ ,  $c(k) = 5 \cdot 2^{k-1} - 2$ .*

TABLE 1. Various types of finitely and infinitely smooth radial basis functions.

Duchon spline	$\phi(r)=r^{2k} \log r, k \in \mathbb{N},$
Wendland	$\phi(r)=(1-r)_+^k p(r),$ p is a polynomial, $k \in \mathbb{N}$
Matern	$\phi(r)=\frac{2^{1-\nu}}{\Gamma(\nu)} r^\nu K_\nu(r), \nu > 0$
Gaussian (GA)	$\phi(r)=e^{-(\varepsilon r)^2}$
Multiquadric (MQ)	$\phi(r)=(1 + (\varepsilon r)^2)^{1/2}$
Generalized multiquadric (GMQ)	$\phi(r)=(1 + (\varepsilon r)^2)^{\nu/2}, \nu \neq 0$

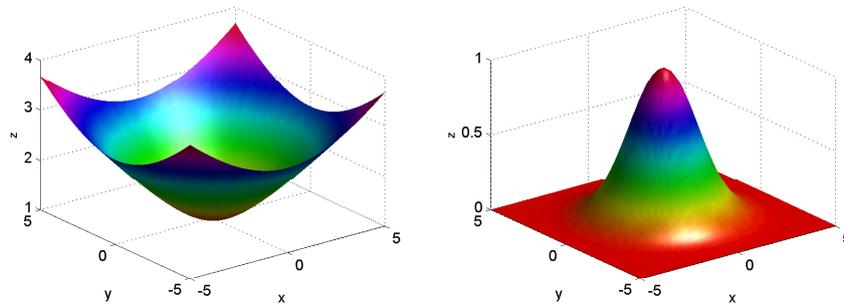


FIGURE 1. Plots of MQ and GA radial basis functions in the domain  $[-5, 5]$ , with shape parameter  $\varepsilon = 0.5$

**Lemma 2** ([32]). *Under the assumption on  $q(x)$  in Lemma 3.3, then*

$$\left| \int_a^b e^{i\omega q(x)} g(x) dx \right| \leq c(k)/\omega^{1/k} \left( |g(b)| + \int_a^b |g'(x)| dx \right). \quad (3.11)$$

**Theorem 4.** *Suppose that  $f(x)$  and  $q(x)$  are suitably smooth and  $q'(x) \neq 0$  for all  $x$  in  $[a, b]$ . Then the quadrature (1.3) at arbitrary interpolation nodes  $x_1, \dots, x_N$  satisfies*

$$E = |I - I_N| \leq \frac{3(N + 1) \|F^N(x)\|_\infty (b - a)^N}{\omega N!}, \quad (3.12)$$

where  $F(x) = f(x) - \mathcal{L}p(x)$ .

Proof. Let  $\mathcal{L}p(x)$  be the radial kernel interpolant of the function  $f(x)$  defined by

$$\mathcal{L}p(x) = \sum_{j=1}^N \lambda_j \phi'(\|x - x_j\|) + i\omega q'(x) \sum_{j=1}^N \lambda_j \phi(\|x - x_j\|), \quad j = 1, \dots, N, \quad (3.13)$$

where the interpolation nodes  $x_1, \dots, x_N \in [a, b]$ , then

$$E = |I - I_N| = \left| \int_a^b (f(x) - \mathcal{L}p(x)) e^{i\omega q(x)} dx \right| = \left| \int_a^b F(x) e^{i\omega q(x)} dx \right|, \quad (3.14)$$

where,  $F(x) = f(x) - \mathcal{L}p(x)$  satisfy the interpolation condition  $F(x_j) = 0$ ,  $j = 1, \dots, N$ . So there exist  $x_j \in (x_j, x_{j+1})$ , such that  $F'(x_j) = 0$ ,  $j = 1, \dots, N-1$ . Then  $F(x)$  and  $F'(x)$  are represented by Lagrange interpolating polynomials

$$F(x) = \frac{F^N(\zeta_1)}{N!} \prod_{j=1}^N (x - x_j) \quad (3.15)$$

and

$$F'(x) = \frac{F^N(\zeta_2)}{(N-1)!} \prod_{j=1}^{N-1} (x - x_j) \quad (3.16)$$

for some  $\zeta_1, \zeta_2 \in [a, b]$ , then for  $k = 1$ ,  $c(k) = 3$

$$\begin{aligned} E = |I - I_N| &= \left| \int_a^b (f(x) - \mathcal{L}p(x)) e^{i\omega q(x)} dx \right| = \left| \int_a^b F(x) e^{i\omega q(x)} dx \right| \\ &\leq c(k)/\omega^{1/k} \left( |F(b)| + \int_a^b |F'(x)| dx \right) \\ &\leq \frac{3(N+1)(b-a)^N \|F^N(x)\|_\infty}{\omega N!}. \end{aligned} \quad (3.17)$$

Various types of radial kernels contain a shape parameter  $\varepsilon$  in the form  $\phi_\varepsilon = \phi(\varepsilon r)$ . This types of basis function  $\phi_\varepsilon$  becomes increasingly flat when  $\varepsilon$  is very small. In such a case we have good approximation properties for both interpolation problems and solving elliptic partial differential equations (see [8, 16, 25]). However the condition number of the system matrix  $\mathbf{A}$  is, quite large when  $\varepsilon$  is very small (see [26]). This conflict situation is known as *uncertainty principle* [24]. This uncertainty principle holds only if the direct approach uses the system matrix  $\mathbf{A}$  to compute the expansion coefficients. It was investigated in [3] that, for infinitely smooth basic functions  $\phi(\|\cdot\|)$ , such as multiquadrics the limit as  $\varepsilon \rightarrow 0$ , limiting interpolants often exist and take the form of polynomials. In the 1-D case, it is proved in [3] that with simple conditions on the basis function, the radial kernels interpolants converge to the Lagrange interpolating polynomial. In 2-D case, the limiting behavior of the kernel based interpolants as  $\varepsilon \rightarrow 0$  to be a multivariate polynomial [28].

Under the assumption that when  $\varepsilon \rightarrow 0$ , the kernel based interpolant  $\mathcal{L}p(x)$  converges to the Lagrange interpolating polynomial. In other words  $(\mathcal{L}p(x))^N \equiv 0$ ,

which imply  $\|F^N(x)\|_\infty = \|f^N(x)\|_\infty$ , and from (3.17) we have,

$$E = |I - I_N| \leq \frac{3(N+1)(b-a)^N \|f^N(x)\|_\infty}{\omega N!}. \quad (3.18)$$

#### 4. APPLICATION OF THE METHOD AND NUMERICAL EXPERIMENTS

In this section, the proposed method is applied to check the accuracy and validity of the present method for computing the oscillatory integrals.

**Problem 1:** Here we consider the integrals of the form

$$I = \int_0^1 \sin x e^{i\omega(x+Cx^2)} dx, \text{ and } J = \int_{100}^{200} (1 + \ln x) e^{i(x \ln x)} dx. \quad (4.1)$$

We compute the integrals  $I$  and  $J$  by the present method using two types of radial kernels, the multiquadric and the Gaussian. The results are compared with Romberg integration technique and with the Levin method [17] (see Tables 2-4). The good accuracy over the Romberg integration technique is achieved. The results of the present method is also well agreed with method of Levin. The error bound developed for the present scheme is tested, and the good agreement between the actual error and the error bound is observed.

TABLE 2. Exact value of the real part of the integral  $I = \int_0^1 \sin x e^{i\omega(x+Cx^2)} dx = 4.59859e-004$ , when  $\omega = 500$ ,  $C = 1$ , Problem 4.1.

N	L method (MQ)	L method (GA)	RI	Levin method [17]
5	4.5793e-004	8.1457e-004	3.04379076e-002	4.60098e-004
10	5.8445e-005	1.5156e-004	2.81033592e-003	4.59863e-004

TABLE 3. Absolute error and the error bound (3.17),  $I = \int_0^1 \sin x e^{i\omega(x+Cx^2)} dx$ , when  $C = 1$ , Problem 4.1.

$\omega$	10	$10^2$	$10^3$	$10^4$
$ I - I_N $	8.1902e-002	5.9239e-003	5.1740e-004	9.8644e-005
Error bound(N=5)	1.5000e-002	1.5000e-003	1.5000e-004	1.5000e-005
$\varepsilon(\text{MQ})$	0.0300	0.0300	0.0300	0.0300
$\kappa$	3.2626e+016	3.2626e+016	3.2626e+016	3.2626e+016

**Problem 2:** Here we consider the integral of the form

$$I = \int_0^1 \int_0^1 \cos(x+y) e^{i\omega(x+y+C(x^2+y^2))} dx, \quad (4.2)$$

TABLE 4. Exact value of the real part of the integral  $J = \int_{100}^{200} (1 + \ln x) e^{i(x \ln x)} dx = -1.77429e+00$ , when  $\omega = 1$ , Problem 4.1.

N,n		L method (MQ)	L method (GA)
500,70	$I$	-1.5377e+000	-1.3643e+000
	$\tau$	2.006854	2.081579

We compute the integral for  $N = 25$  equally spaced nodes in the domain of integration  $[0, 1] \times [0, 1]$ , and  $n = 8$  nodes in each local domain  $\Omega_i$ , for the choice  $\omega = 100$  and  $C = 1$ . We used two radial kernels the multiquadric and the Gaussian. The results of the present method are well agreed with the results of the Levin method [17].

TABLE 5. Exact value of the real part of the integral  $I = -6.92072e-009$   $[0, 1]^2$ ,  $\omega = 10^4$ ,  $C = 1$ , corresponding to Problem 4.2.

N,n		L method (MQ)	L method (GA)	Levin method [17]
25,8	$I$	-7.2313e-008	-7.5778e-009	-6.9208e-009

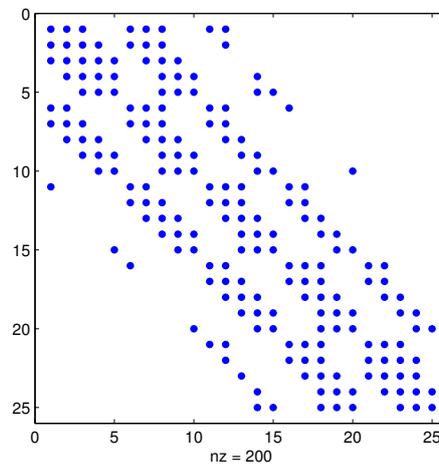


FIGURE 2. Sparsity of the differentiation matrix  $W$  in Problem 4.2, where out of 625 entries only 200 are non zeros and the rest are zeros.

## 5. CONCLUSIONS

In this work the local kernel based interpolation scheme is constructed for evaluating the oscillatory integrals. The error bound of the present scheme is also developed. The proposed scheme is based on the work of [17, 31]. The differentiation matrices in the proposed kernel based method are sparse in nature. Most of the interpolation schemes including Levin method [17] faces problems for large number of nodes in the domain of integration due to the dense interpolation matrices. The current numerical scheme is capable of handling thousand of nodes in the domain of integration. In fact we are solving small size interpolation problems of size  $n \leq N$  corresponding to each domain  $\Omega_i$ .

## ACKNOWLEDGEMENT

The authors are very grateful to the anonymous referees for their constructive comments.

## REFERENCES

- [1] M. D. Buhmann, *Radial Basis Functions: Theory and Implementations*. Cambridge: Cambridge University Press, 2003.
- [2] P. J. David and P. Rabinowitz, *Methods of numerical integration*. New York: Academic Press, 1975.
- [3] T. Driscoll and B. Fornberg, "Interpolation in the limit of increasingly flat radial basis functions," *Comput. Math. Appl.*, vol. 43, pp. 413–422, 2002, doi: [10.1016/S0898-1221\(01\)00295-4](https://doi.org/10.1016/S0898-1221(01)00295-4).
- [4] G. A. Evans, "An alternative methods for irregular oscillatory integrals," *Int. J. Comput. Math.*, vol. 53, pp. 185–193, 1994, doi: [10.1080/00207169408804303](https://doi.org/10.1080/00207169408804303).
- [5] G. A. Evans, *Practical Numerical Integration*. first ed.: SIAM, 1997.
- [6] G. E. Fasshauer and Z. J. G., "On choosing optimal shape parameters for rbf approximation," *Numer. Algorithms*, vol. 45, pp. 345–368, 2007, doi: [10.1007/s11075-007-9072-8](https://doi.org/10.1007/s11075-007-9072-8).
- [7] L. N. G. Filon, "On a quadrature formula for trigonometric integrals," *Proc. R. Soc. Edinb.*, vol. 49, pp. 38–47, 1928, doi: [10.1017/S0370164600026262](https://doi.org/10.1017/S0370164600026262).
- [8] B. Fornberg, G. Wright, and E. Larsson, "Some observations regarding interpolants in the limit of flat radial basis functions," *Comput. Math. Appl.*, vol. 47 (1), pp. 37–55, 2004, doi: [10.1016/S0898-1221\(04\)90004-1](https://doi.org/10.1016/S0898-1221(04)90004-1).
- [9] F. Franke, "Scattered data interpolation: tests of some methods," *Math. Comp.*, vol. 38, pp. 181–200, 1982, doi: [10.1090/S0025-5718-1982-0637296-4](https://doi.org/10.1090/S0025-5718-1982-0637296-4).
- [10] R. L. Hardy, "Multiquadric equations of topography and other irregular surfaces," *Journal of Geophysical Research*, vol. 76, pp. 1905–1915, 1971, doi: [10.1029/JB076i008p01905](https://doi.org/10.1029/JB076i008p01905).
- [11] D. Huybrechs and S. Vandewalle, *The construction of cubature rules for multivariate highly oscillatory integrals*. Leuven: Katholieke University Leuven, 2005. doi: [10.1090/S0025-5718-07-01937-0](https://doi.org/10.1090/S0025-5718-07-01937-0).
- [12] A. Iserles and S. P. Norsett, "On quadrature methods for highly oscillatory integrals and their implementation," *BIT*, vol. 44, pp. 755–772, 2004, doi: [10.1007/s10543-004-5243-3](https://doi.org/10.1007/s10543-004-5243-3).
- [13] A. Iserles and S. P. Norsett, "Quadrature methods formultivariate highly oscillatory integrals using derivatives," *Math. Comput.*, vol. 75, pp. 1233–1259, 2006, doi: [10.1090/S0025-5718-06-01854-0](https://doi.org/10.1090/S0025-5718-06-01854-0).

- [14] L. Y. K., "On the computation of oscillatory integrals," *Proc. Camb. Philos. Soc.*, vol. 50, pp. 269–277, 1954, doi: [10.1017/S0305004100029327](https://doi.org/10.1017/S0305004100029327).
- [15] E. J. Kansa, "Multiquadrics - a scattered data approximations scheme with application to computational fluid dynamics, parti. surface approximations and partial derivative estimates," *Comput. Math. Appl.*, vol. 19, pp. 127–145, 1990, doi: [10.1016/0898-1221\(90\)90270-T](https://doi.org/10.1016/0898-1221(90)90270-T).
- [16] E. Larsson and B. Fornberg, "A numerical study of some radial basis function based solution methods for elliptic pdes," *Comput. Math. Appl.*, vol. 46, pp. 891–902, 2003, doi: [10.1016/S0898-1221\(03\)90151-9](https://doi.org/10.1016/S0898-1221(03)90151-9).
- [17] D. Levin, "Procedures for computing one-and-two dimensional integrals of functions with rapid irregular oscillations," *Math. Comput.*, vol. 38, pp. 531–538, 1982, doi: [10.1090/S0025-5718-1982-0645668-7](https://doi.org/10.1090/S0025-5718-1982-0645668-7).
- [18] I. M. Longman, "A method for numerical evaluation of finite integrals of oscillatory functions," *Math. Comput.*, vol. 14, pp. 53–59, 1960.
- [19] C. A. Micchelli, "Interpolation of scattered data: Distance matrices and conditionally positive definite functions," *Constr. Approx.*, vol. 2, pp. 11–22, 1986.
- [20] S. Olver, "Moment-free numerical integration of highly oscillatory functions," *IMA. J. Numer. Anal.*, vol. 26, pp. 213–227, 2006.
- [21] R. Piessens and F. Poleunis, "A numerical method for the integration of oscillatory functions," *BIT*, vol. 11, pp. 317–327, 1971, doi: [10.1007/BF01931813](https://doi.org/10.1007/BF01931813).
- [22] M. J. D. Powell, *The theory of radial basis function approximation in 1990, In Advances in Numerical Analysis, Volume II: Wavelets, Subdivision Algorithms and Radial Functions.* Oxford, UK: Oxford University Press, 1992.
- [23] S. Rippa, "An algorithm for selecting a good value for the parameter  $c$  in radial basis function interpolation," *Adv. Comput. Math.*, vol. 11, pp. 193–210, 1999, doi: [10.1023/A:1018975909870](https://doi.org/10.1023/A:1018975909870).
- [24] R. Schaback, "Error estimates and condition numbers for radial basis function interpolation," *Adv. Comput. Math.*, vol. 3, pp. 251–264, 1995, doi: [10.1007/BF02432002](https://doi.org/10.1007/BF02432002).
- [25] R. Schaback, "Limit problems for interpolation by analytic radial basis functions," *J. Comput. Appl. Math.*, vol. 212, pp. 127–149, 2008, doi: [10.1016/j.cam.2006.11.023](https://doi.org/10.1016/j.cam.2006.11.023).
- [26] R. Schaback and H. Wendland, "Kernel techniques: from machine learning to meshless methods," *Acta Numer.*, vol. 15, pp. 543–639, 2006, doi: [10.1017/S0962492906270016](https://doi.org/10.1017/S0962492906270016).
- [27] M. Scheuerer, "An alternative procedure for selecting a good value for the parameter  $c$  in rbf-interpolation," *Adv. Comput. Math.*, vol. 34, pp. 105–126, 2011, doi: [10.1007/s10444-010-9146-3](https://doi.org/10.1007/s10444-010-9146-3).
- [28] G. Song, J. Riddle, G. E. Fasshauer, and F. J. Hickernell, "Multivariate interpolation with increasingly flat radial basis functions of finite smoothness," *Adv. Comput. Math.*, vol. 36, pp. 485–501, 2012, doi: [10.1007/s10444-011-9192-5](https://doi.org/10.1007/s10444-011-9192-5).
- [29] M. Uddin, "Rbf-ps scheme for solving the equal width equation," *Appl. Math. and Comput.*, vol. 222, pp. 619–631, 2013, doi: [10.1016/j.amc.2013.07.031](https://doi.org/10.1016/j.amc.2013.07.031).
- [30] M. Uddin, "On the selection of a good value of shape parameter in solving time-dependent partial differential equations using rbf approximation method," *Appl. Math. Model.*, vol. 38, pp. 135–144, 2014, doi: [10.1016/j.apm.2013.05.060](https://doi.org/10.1016/j.apm.2013.05.060).
- [31] R. Vertnik and B. Sarler, "Meshless local radial basis function collocation method for convective-diffusive solid-liquid phase change problems," *Int. J. Numer. Methods Heat Fluid Flow*, vol. 16, pp. 617–640, 2006, doi: [10.1108/09615530610669148](https://doi.org/10.1108/09615530610669148).
- [32] S. Xiang, "Efficient filon-type methods for  $\int_a^b f(x)e^{i\omega g(x)} dx$ ," *Numer. Math.*, vol. 105, pp. 633–658, 2007, doi: [10.1007/s00211-006-0051-0](https://doi.org/10.1007/s00211-006-0051-0).

*Authors' addresses***M. Uddin**

University of Engineering and Technology, Department of Basic Sciences, 25000 Peshawar, Pakistan

*E-mail address:* marjan@uetpeshawar.edu.pk (corresponding author)

**Z. Minullah**

University of Engineering and Technology, Department of Basic Sciences, 25000 Peshawar, Pakistan

*E-mail address:* zminullah@yahoo.com

**A. Ali**

University of Engineering and Technology, Department of Basic Sciences, 25000 Peshawar, Pakistan

*E-mail address:* amjad\_puet@yahoo.com

**Kamran**

University of Engineering and Technology, Department of Basic Sciences, 25000 Peshawar, Pakistan

*E-mail address:* kamrank44@gmail.com