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STURM-LIOUVILLE PROBLEMS WITH FINITELY MANY POINT $\delta-$INTERACTIONS AND EIGEN-PARAMETER IN BOUNDARY CONDITION

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Abstract. This paper deals with the Sturm-Liouville equation with a finite number of point $\delta-$interactions and eigenvalue parameter contained in the boundary condition. Sturm–Liouville problem with discontinuities at one or two points and its different variants have already been investigated. In this study we extend these results to a finite number of point $\delta-$interactions case. The crucial part of this study is the using graph demonstration to obtain asymptotic representation of solutions.

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1. INTRODUCTION

We consider the boundary problem (BVP) for the differential equation

$$\ell y := -y'' + q(x)y = \lambda y \tag{1.1}$$

on $[a,x_1)\cup(x_1,x_2)\cup\cdots\cup(x_{n-1},x_n)\cup(x_n,b]$ and boundary condition at $x=a$

$$L_1(y) := \alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{1.2}$$

with transmission conditions at discontinuous points $x_i,\ i=\overline{1,n}$

$$U_i(y) := y(x_i-0) = y(x_i+0) = y(x_i), \tag{1.3}$$

$$V_i(y) := y'(x_i+0) - y'(x_i-0) = \gamma_i y(x_i) \tag{1.4}$$

and the eigenparameter-dependent boundary condition at $x=b$

$$L_2(y) := \lambda[\beta_1' y(b) + \beta_2' y'(b)] + [\beta_1 y(b) - \beta_2 y'(b)] = 0, \tag{1.5}$$

where $q(x)$ is real-valued function and continuous in $L_1[a,b]$. We assume that α_i, γ_i, β_i, β_i', $i=1,2$ are real numbers, satisfying $|\alpha_1|+|\alpha_2|\neq 0$, λ is a complex spectral parameter. Throughout this paper, we assume that

$$r := \beta_1'\beta_2 - \beta_1\beta_2' > 0. \tag{1.6}$$

Notice that, we can understand problem (1.1), (1.3) and (1.4) as one of the treatments of the equation

$$-y''+(p(x)+q(x))\,y=\lambda y, \qquad x\in(a,b), \tag{1.7}$$

when $p(x)=\sum_{i=1}^{n}\gamma_i\delta(x-x_i)$, where $\delta(x)$ is the Dirac function, (see [2]).

Sturmian theory is one of the most extensively developing fields in theoretical and applied mathematics. Particularly, there has been increasing interest in the spectral analysis of BVPs with eigenvalue-dependent boundary conditions. There are quite substantial literatures on such problems. Here we mention the results of [5], [6], [7], [8], [10], [14], [18] and the corresponding references cited therein.

BVPs with discontinuities inside the interval and eigenvalue contained in the boundary conditions often appear in many branches of natural sciences. We note that Sturm-Liouville problems with eigen-dependent boundary conditions and with transmission conditions have been investigated in [1], [3], [9], [13], [11], [12]. Furthermore Green's formula for impulsive differential equation has been studied in [16] and [17].

In this paper we consider Sturm-Liouville problem with eigenparameter depend boundary condition and $\delta-$interactions at finite number of interior points. The aim of this article is to carry results in [9] to the case of finitely many $\delta-$interactions and the main difference of this study is the using graph theory (see [4]) for the complicated asymptotic formulas.

2. THE HILBERT SPACE CONSTRUCTION AND SOME PROPERTIES OF THE SPECTRUM

In the Hilbert space $\mathcal{H}:=L_2[a,b]\oplus\mathbb{C}$ of two component vectors, we define an inner product by

$$\langle f,g\rangle_{\mathcal{H}}:=\int_a^b f_1(x)\bar{g}_1(x)dx+\frac{1}{r}f_2\overline{g}_2 \tag{2.1}$$

for

$$f=\begin{pmatrix} f_1(x)\\ f_2\end{pmatrix},\quad g=\begin{pmatrix} g_1(x)\\ g_2\end{pmatrix}$$

where $f_1(x),\,g_1(x)\in L_2(a,b),\ f_2,\ g_2\in\mathbb{C}$, the constant r is defined in (1.6). For convenience we introduce

$$R_b(y):=\beta_1 y(b)-\beta_2 y'(b),$$

$$R'_b(y):=\beta'_1 y(b)-\beta'_2 y'(b).$$

In the Hilbert space $\mathcal{H}$ we define the operator $\mathbf{L}$

$$\mathbf{L}:\mathcal{H}\to\mathcal{H}$$

with domain

$$D(\mathbf{L}):\left\{ f\in\mathcal{H} \;\middle|\; \begin{array}{c} f_1,\ f_1'\in AC\left([a,x_1)\cup\bigcup_{i=1}^{n-1}(x_i,x_{i+1})\cup(x_n,b]\right),\\ \ell f_1\in L_2[a,b],\ L_1(f_1)=0,\\ U_i(f_1)=V_i(f_1)=0\ for\ i=\overline{1,n},\ f_2=R_b'(f_1) \end{array}\right\}$$

and operator rule

$$\mathbf{L}(f):=\begin{pmatrix} \ell f_1 \\ -R_b(f_1)\end{pmatrix}.$$

Here $AC\,(.)$ denotes the set of all absolutely continuous functions on related interval. In particular, those functions will have limits at the boundary points x_i.

It is clear that the eigenvalues of the operator $\mathbf{L}$ and the boundary value problem (1.1)-(1.5) are same and the eigenfunctions of (1.1)-(1.5) coincide with the first components of corresponding eigenelements of the operator $\mathbf{L}$.

Theorem 1. *The operator* $\mathbf{L}$ *is symmetric.*

Proof. Let $f,g\in D(\mathbf{L})$. From the inner product defined in (2.1), we obtain

$$\begin{aligned}\langle Lf,g\rangle_{\mathcal{H}}-\langle f,Lg\rangle_{\mathcal{H}} &= [W(f,\overline{g};x_1-0)-W(f,\overline{g};a)]\\ &+[W(f,\overline{g};x_2-0)-W(f,\overline{g};x_1+0)]\\ &+\sum_{p=2}^{n-1}\left[W(f,\overline{g};x_{p+1}-0)-W(f,\overline{g};x_p+0)\right]\\ &+[W(f,\overline{g};b)-W(f,\overline{g};x_n+0)]\\ &+\frac{1}{r}\left[R_b'(f)R_b(\overline{g})-R_b(f)R_b'(\overline{g})\right]\end{aligned}\tag{2.2}$$

where $W(f,g;x)=f(x)g'(x)-f'(x)g(x)$ is the wronkskian of the functions f and g. Since f and $\overline{g}$ satisfy the same boundary condition (1.2) and from the transmission conditions (1.3) and (1.4), we get

$$W(f,\overline{g};a)=0,\tag{2.3}$$

$$W(f,\overline{g};x_i-0)=W(f,\overline{g};x_i+0),\quad (i=\overline{1,n}),\tag{2.4}$$

$$R_b'(f)R_b(\overline{g})-R_b(f)R_b'(\overline{g})=rW(f,\overline{g};b).\tag{2.5}$$

Substituting (2.3)-(2.5) in (2.2) we obtain $\langle\mathbf{L}f,g\rangle_{\mathcal{H}}=\langle f,\mathbf{L}g\rangle_{\mathcal{H}}$ for $f,g\in D(\mathbf{L})$. So $\mathbf{L}$ is symmetric. □

Corollary 1. *All eigenvalues of the problem (1.1)-(1.5) are real and if λ_1 and λ_2 are two different eigenvalues of the problem (1.1)-(1.5), then the corresponding eigenfunctions y_1 and y_2 are orthogonal in the sense of*

$$\int_a^b y_1(x)y_2(x)dx+\frac{1}{r}R_b'(y_1)R_b'(y_2)=0.$$

To define a solution of (1.1)-(1.5), we need the following lemma. The proof of this assertion reproduces that of Theorem 1.1 in [15, p. 14] or [12].

Lemma 1. *Let $q(x)$ be a real-valued, continuous function and let $f(\lambda)$ and $g(\lambda)$ are given entire functions. Then for any $\lambda \in \mathbb{C}$, the equation*

$$-y'' + q(x)y = \lambda y, \qquad x \in [a,b]$$

has a unique solution $y = y(x,\lambda)$ such that

$$y(a) = f(\lambda), \;\; y'(a) = g(\lambda) \qquad (\text{or } y(b) = f(\lambda), \;\; y'(b) = g(\lambda))$$

and for each $x \in [a,b]$, $y(x,\lambda)$ is an entire function of λ.

Now we define two solutions of the equation (1.1) as follows:

$$\phi_\lambda(x) = \begin{cases} \phi_{1\lambda}(x), & x \in [a,x_1) \\ \phi_{2\lambda}(x), & x \in (x_1,x_2) \\ \quad\vdots & \quad\vdots \\ \phi_{n\lambda}(x), & x \in (x_{n-1},x_n) \\ \phi_{(n+1)\lambda}(x), & x \in (x_n,b] \end{cases}, \quad \chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), & x \in [a,x_1) \\ \chi_{2\lambda}(x), & x \in (x_1,x_2) \\ \quad\vdots & \quad\vdots \\ \chi_{n\lambda}(x), & x \in (x_{n-1},x_n) \\ \chi_{(n+1)\lambda}(x), & x \in (x_n,b] \end{cases}$$

Let $\phi_{1\lambda}(x) = \phi_1(x,\lambda)$ be a solution of the equation (1.1) on $[a,x_1)$ which satisfies the initial conditions at the point a

$$y(a) = \alpha_2, \qquad y'(a) = -\alpha_1 \tag{2.6}$$

From Lemma 1, we can define a solution $\phi_{i+1}(x,\lambda)$ of the equation (1.1) on $[x_i, x_{i+1}]$, $(i = \overline{1,n-1})$ by means of the solution $\phi_i(x,\lambda)$ by the nonstandard initial conditions

$$y(x_i+0) = \phi_{i\lambda}(x_i-0), \qquad y'(x_i+0) = \phi'_{i\lambda}(x_i-0) + \gamma_i\phi_{i\lambda}(x_i). \tag{2.7}$$

In the same manner, we define a solution $\phi_{n+1}(x,\lambda)$ of the equation (1.1) on $[x_n,b]$ by the nonstandard initial conditions

$$y(x_n+0) = \phi_{n\lambda}(x_n-0), \qquad y'(x_n+0) = \phi'_{n\lambda}(x_n-0) + \gamma_i\phi_{n\lambda}(x_n). \tag{2.8}$$

Therefore $\phi(x,\lambda)$ satisfies the equation (1.1) on the interval $[a,x_1) \cup (x_1,x_2) \cup \cdots \cup (x_{n-1},x_n) \cup (x_n,b]$ and the boundary condition (1.2) and the transmission conditions (1.3), (1.4).

Similarly let $\chi_{(n+1)\lambda}(x) = \chi_{n+1}(x,\lambda)$ be a solution of (1.1) on $(x_n,b]$ which satisfies the initial conditions

$$y(b) = \beta_2'\lambda + \beta_2, \qquad y'(b) = \beta_1'\lambda + \beta_1. \tag{2.9}$$

Likewise let define the solution $\chi_i(x,\lambda)$, on $[x_{i-1},x_i]$, $(\, i = \overline{n,2})$ satisfies the conditions

$$y(x_i-0) = \chi_{(i+1)\lambda}(x_i+0), \quad y'(x_i-0) = \chi'_{(i+1)\lambda}(x_i+0) - \gamma_i\chi_{(i+1)\lambda}(x_i). \tag{2.10}$$

Finally we define a solution $\chi_1(x,\lambda)$ on $[a,x_1)$ satisfies the conditions

$$y(x_1-0)=\chi_{2\lambda}(x_1+0), \quad y'(x_1-0)=\chi'_{2\lambda}(x_1+0)-\gamma_i\chi_{2\lambda}(x_1). \tag{2.11}$$

Hence $\chi(x,\lambda)$ satisfies the equation (1.1) on $[a,x_1)\cup(x_1,x_2)\cup\cdots\cup(x_{n-1},x_n)\cup(x_n,b]$, the boundary condition (1.5) and the transmission conditions (1.3), (1.4).

Let define the wronskians of ϕ_i and χ_i for x in the intervals $[a,x_1)$, (x_1,x_2), ..., (x_{n-1},x_n), $(x_n,b]$ respectively:

$$\begin{aligned} w_i(\lambda) &= W_\lambda(\phi_i,\chi_i,x) \\ &= \phi_i(x,\lambda)\chi_i'(x,\lambda)-\phi_i'(x,\lambda)\chi_i(x,\lambda), \quad i=\overline{1,n+1}. \end{aligned} \tag{2.12}$$

Lemma 2. *For each* $\lambda\in\mathbb{C}$ $w(\lambda):=w_1(\lambda)=w_2(\lambda)=w_3(\lambda)=\cdots=w_{n+1}(\lambda)$.

Proof. From (2.7), (2.10), (1.3) and (1.4)

$$\begin{aligned} W_\lambda(\phi_1,\chi_1;x_1-0) &= \phi_1(x_1-0,\lambda)\chi_1'(x_1-0,\lambda)-\phi_1'(x_1-0,\lambda)\chi_1(x_1-0,\lambda) \\ &= \phi_2(x_1+0,\lambda)[\chi_2'(x_1+0,\lambda)-\gamma_1\chi_2(x_1,\lambda)] \\ &\quad -[\phi_2'(x_1+0,\lambda)-\gamma_1\phi_2(x_1,\lambda)]\chi_2(x_1+0,\lambda) \\ &= W_\lambda(\phi_2,\chi_2;x_1+0)-\gamma_1[\phi_2(x_1+0,\lambda)\chi_2(x_1,\lambda) \\ &\quad -\phi_2(x_1,\lambda)\chi_2(x_1+0,\lambda)] \\ &= W_\lambda(\phi_2,\chi_2;x_1+0) \end{aligned}$$

Since the wronskians are independent of x (see [15, pp.7]), we get

$$W_\lambda(\phi_2,\chi_2;x_1+0)=W_\lambda(\phi_2,\chi_2;x_2-0)$$

and similar calculation gives

$$W_\lambda(\phi_2,\chi_2;x_2-0)=W_\lambda(\phi_3,\chi_3;x_2+0).$$

Consequently we arrive at

$$\begin{aligned} W_\lambda(\phi_1,\chi_1;x_1-0) &= W_\lambda(\phi_2,\chi_2;x_1+0)=W_\lambda(\phi_3,\chi_3;x_2+0)=\cdots \\ &= W_\lambda(\phi_{n+1},\chi_{n+1};x_n+0) \end{aligned}$$

□

Theorem 2. *The eigenvalues of the problem (1.1)-(1.5) consist the zeros of the function* $w(\lambda)$.

Proof. For the proof, we will follow the technique in [9]. Suppose that μ is the zero of $w(\lambda)$. Then the wronskian of $\phi_1(x,\mu)$ and $\chi_1(x,\mu)$ is zero, so that $\chi_1(x,\mu)$ is a constant multiple of $\phi_1(x,\mu)$, say

$$\chi_1(x,\mu)=k\phi_1(x,\mu), \quad x\in[a,x_1]$$

for some $k\neq 0$. Therefore $\chi_1(x,\mu)$ satisfies the boundary condition (1.2) and this means that $\chi_1(x,\mu)$ is an eigenfunction for the eigenvalue μ.

For the converse, let $\Phi(x)$ be any eigenfunction corresponding to eigenvalue μ. Then the function $\Phi(x)$ may be represented in the form

$$\Phi(x)=\begin{cases} k_1\phi_1(x,\mu)+k_2\chi_1(x,\mu), & x\in[a,x_1) \\ k_3\phi_2(x,\mu)+k_4\chi_2(x,\mu), & x\in(x_1,x_2) \\ \vdots & \vdots \\ k_{2n-1}\phi_n(x,\mu)+k_{2n}\chi_n(x,\mu), & x\in(x_{n-1},x_n) \\ k_{2n+1}\phi_{n+1}(x,\mu)+k_{2n+2}\chi_{n+1}(x,\mu), & x\in(x_n,b] \end{cases}$$

where at least one of the constants k_j, $(j=\overline{1,2n+2})$ is not zero. By using the initial conditions (2.6)-(2.11), the equations

$$L_1(\Phi)=0;\ \ U_i(\Phi)=0,\ V_i(\Phi)=0,\ i=\overline{1,n};\ \ L_2(\Phi)=0 \tag{2.13}$$

give a system of linear equations in the variables k_j, $(j=\overline{1,2n+2})$ and the coefficient matrix of this system is

$$\begin{bmatrix} 0 & w_1(\mu) & 0 & 0 & \cdots & & 0 & 0 \\ & N_1 & M_1 & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ 0 & 0 & N_2 & M_2 & & & & \\ 0 & 0 & & & & & & \\ & & & N_3 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & & & & \\ & & & & \ddots & M_{n-1} & 0 & 0 \\ & & & & & & 0 & 0 \\ 0 & 0 & & & & & & \\ 0 & 0 & & & & N_n & M_n & \\ 0 & 0 & & \cdots & & 0 \quad 0 & w_{n+1}(\mu) & 0 \end{bmatrix}_{(2n+2)\times(2n+2)} \tag{2.14}$$

where M_i and N_i are 2×2 matrices defined as follows

$$M_i=\begin{bmatrix} -\phi_{(i+1)\mu}(x_i+0) & -\chi_{(i+1)\mu}(x_i+0) \\ -\phi'_{(i+1)\mu}(x_i+0) & -\chi'_{(i+1)\mu}(x_i+0) \end{bmatrix},\quad i=\overline{1,n},$$

$$N_i=\begin{bmatrix} \phi_{i\mu}(x_i-0) & \chi_{i\mu}(x_i-0) \\ \phi'_{i\mu}(x_i-0) & \chi'_{i\mu}(x_i-0) \end{bmatrix},\quad i=\overline{1,n}.$$

The determinant of the matrix (2.14) is $-\prod_{i=1}^{n} w_i(\mu).w^{n+1}(\mu)$ which must be zero in order to the system (2.13) has a nontrivial solution and hence $w(\mu)=0$. □

3. ASYMPTOTIC FORM OF SOLUTIONS AND GRAPH REPRESENTATION

In this section, we shall derive the asymptotic formulas for the characteristic function $w(\lambda)$ of (1.1), (1.5) in four different cases. The main difficulty is derivation of asymptotic formulas for the solutions of (1.1), (1.5). Because these formulas fastly lead to a very complicated equations for large values of n. A convenient way to manage the resulting asymptotic representation of solutions is by a graph analogy. For further reading for graph based theory, see, for example, [4]. We start with some lemmas.

Lemma 3. *Let $\lambda = \rho^2$ and $\phi(x,\lambda)$ be solution of the equation (1.1). Then the following integral equations hold for $k = 0,1$*

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \alpha_2\frac{d^k}{dx^k}(\cos\rho(x-a)) - \alpha_1\frac{1}{\rho}\frac{d^k}{dx^k}(\sin\rho(x-a)) + \frac{1}{\rho}\int_a^{x_1}\frac{d^k}{dx^k}(\sin\rho(x-t))q(t)\phi_{1\lambda}(t)dt, \tag{3.1}$$

$$\frac{d^k}{dx^k}\phi_{(i+1)\lambda}(x) = \phi_{i\lambda}(x_i-0)\frac{d^k}{dx^k}(\cos\rho(x-x_i)) + \frac{1}{\rho}\phi'_{i\lambda}(x_i-0)\frac{d^k}{dx^k}(\sin\rho(x-x_i)) + \frac{1}{\rho}\int_{x_i}^{x}\frac{d^k}{dx^k}(\sin\rho(x-t))q(t)\phi_{(i+1)\lambda}(t)dt,\ \ i = \overline{1,n}. \tag{3.2}$$

Proof. The last terms in (3.1) and (3.2) are equal to

$$\frac{1}{\rho}\int_a^{x_1}\frac{d^k}{dx^k}(\sin\rho(x-t))\{\rho^2\phi_{1\lambda}(t)+\phi''_{1\lambda}(t)\}dt,$$

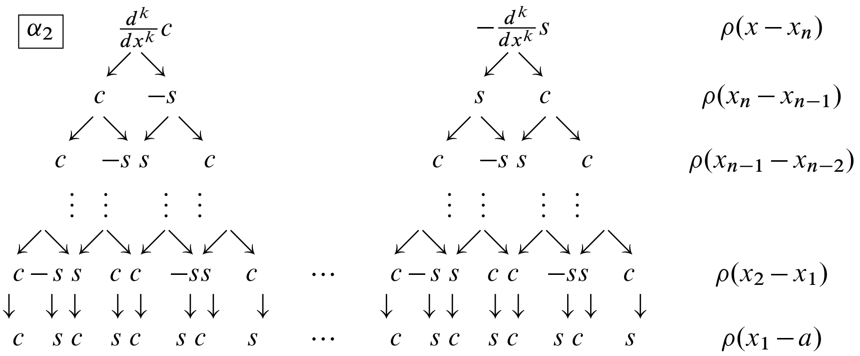
$$\frac{1}{\rho}\int_{x_i}^{x}\frac{d^k}{dx^k}(\sin\rho(x-t))\{\rho^2\phi_{(i+1)\lambda}(t)+\phi''_{(i+1)\lambda}(t)\}dt,\ \ i = \overline{1,n}$$

by the equation (1.1), respectively. On integrating by parts twice we obtain (3.1) and (3.2). □

Lemma 4. *Let $\tau := Im\rho$ and $\alpha_2 \neq 0$. As $|\lambda| \to \infty$, the asymptotic formula for $\phi_{1\lambda}(x)$ is*

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \alpha_2 \frac{d^k}{dx^k}(\cos\rho(x-a)) + O\left(|\rho|^{k-1} e^{|\tau|(x-a)}\right) \tag{3.3}$$

and the asymptotic formula for $\phi_{(i+1)\lambda}(x)$, $i = \overline{1,n}$, is obtained from the following tree:

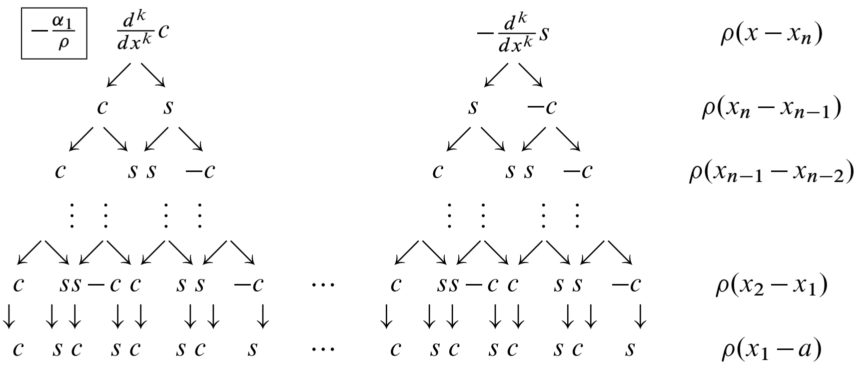


(3.4)

where c and s denote cosine and sine functions respectively. The roots consist of two nodes $\frac{d^k}{dx^k}c$ and $-\frac{d^k}{dx^k}s$. The children of roots are written by repeating the processing s −s s c. The last children are written by repeating the processing c s. After constructing trees for any $\frac{d^k}{dx^k}\phi_{i\lambda}(x)$, $i = \overline{2,n+1}$ using rules above to get its formula, first we write each terms by multiplying all notes on the branch from root to least children and then sum these terms. Then, while if $\alpha_2 = 0$

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = -\frac{\alpha_1}{\rho}\frac{d^k}{dx^k}(\sin\rho(x-a)) + O\left(|\rho|^{k-2} e^{|\tau|(x-a)}\right) \tag{3.5}$$

and



(3.6)

Each result obtained from trees holds uniformly for the intervals $a \leq x \leq x_1$, $x_i \leq x \leq x_{i+1}$, $i = \overline{1,n-1}$ and $x_n \leq x \leq b$ respectively.

Proof. The proof for $i = 1,2$ and 3 without graph representation were given in [9] and [15]. Here, we propose an alternative proof in general sense by using graph demonstration. As can be seen in studies [9], [15] the continuation of this process, although theoretically clear, soon leads to very complicated formulas. It is therefore advantageous to use a graphical representation. Indeed, we can construct a link about the array of the functions. Below, we exhibit the trees corresponding to each $\frac{d^k}{dx^k}\phi_{i\lambda}(x)$, $i = \overline{1,n+1}$ and give a systematic pattern. We will write $\frac{d^k}{dx^k}\phi_{i\lambda}(x)$ for $i = 1,2,3$ from [9, Lemma 3.2]. For $\alpha_2 \neq 0$ and $i = 1$ we have

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \alpha_2 \frac{d^k}{dx^k}(\cos\rho(x-a)) + O\left(|\rho|^{k-1} e^{|\tau|(x-a)}\right).$$

The tree for this formula will consist of only one node will be named by root:

$$\boxed{\alpha_2} \qquad \frac{d^k}{dx^k}c \qquad \rho(x-a)$$

In order to work with trees we introduce the following notations: $\boxed{\alpha_2}$ is common factor of all terms, $\frac{d^k}{dx^k}$ shows the derivative of terms on root and $\rho(x-a)$ is at values of functions which are on the same line with it. For $i = 2$,

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = \alpha_2 \left\{ \frac{d^k}{dx^k}(\cos\rho(x-x_1))\cos\rho(x_1-a) - \frac{d^k}{dx^k}(\sin\rho(x-x_1))\sin\rho(x_1-a) + O\left(|\rho|^{k-1} e^{|\tau|[(x-x_1)+(x_1-a)]}\right)\right\}. \tag{3.7}$$

For this formula the tree will consist of two roots and their one child:

$$\begin{array}{cccc} \boxed{-\frac{\alpha_1}{\rho}} & \frac{d^k}{dx^k}c & -\frac{d^k}{dx^k}s & \rho(x-x_1) \\ & \downarrow & \downarrow & \\ & c & s & \rho(x_1-a) \end{array}$$

In this tree, the multiplication of terms on the vertical branch from root to its child gives the first term of (3.7) and similarly, second branch gives the second term of (3.7). Then the sum of these branches gives the formula (3.7). Finally, for $i = 3$

$$\frac{d^k}{dx^k}\phi_{3\lambda}(x) = \alpha_2 \left\{ \frac{d^k}{dx^k}(\cos\rho(x-x_2))\cos\rho(x_2-x_1)\cos\rho(x_1-a) - \frac{d^k}{dx^k}(\cos\rho(x-x_2))\sin\rho(x_2-x_1)\sin\rho(x_1-a) - \frac{d^k}{dx^k}(\sin\rho(x-x_2))\sin\rho(x_2-x_1)\cos\rho(x_1-a) \right.$$

$$
\begin{aligned}
&-\frac{d^k}{dx^k}(\sin\rho(x-x_2))\cos\rho(x_2-x_1)\sin\rho(x_1-a)\\
&+O\left(|\rho|^{k-1}e^{|\tau|[(x-x_2)+(x_2-x_1)+(x_1-a)]}\right)\Big\}
\end{aligned}
$$

and corresponds to:

$$
\begin{array}{cccccccc}
\boxed{\alpha_2} & & \frac{d^k}{dx^k}c & & & -\frac{d^k}{dx^k}s & & \rho(x-x_2)\\
& \swarrow & & \searrow & \swarrow & & \searrow & \\
& c & & -s & s & & c & \rho(x_2-x_1)\\
& \downarrow & & \downarrow & \downarrow & & \downarrow & \\
& c & & s & c & & s & \rho(x_1-a)
\end{array}
$$

The above trees help in giving a procedure to construct trees for $i \geq 4$. From using the rules above we generalize a tree for $\frac{d^k}{dx^k}\phi_{(n+1)\lambda}(x)$ as in (3.4) and by using the same technique we obtain a tree for $\frac{d^k}{dx^k}\phi_{(n+1)\lambda}(x)$ when $\alpha_2 = 0$ as in (3.6). $\square$

Theorem 3. *Let $\lambda = \rho^2$ and $\tau := Im\rho$. Then the estimates obtained from the following four trees for $w(\lambda)$ are valid.*

For $\beta_2' \neq 0,\ \alpha_2 \neq 0$, the estimate obtained from a tree which is the same as in (3.4) except common factor and root changed by

$$
\boxed{-\beta_2'\alpha_2\rho^3} \qquad s \qquad c \qquad \rho(b-x_n)
$$

For $\beta_2' \neq 0,\ \alpha_2 = 0$, the estimate obtained from a tree which is the same as in (3.6) except common factor and root changed by

$$
\boxed{-\beta_2'\alpha_1\rho^2} \qquad s \qquad c \qquad \rho(b-x_n)
$$

For $\beta_2' = 0,\ \alpha_2 \neq 0$, the estimate obtained from a tree which is the same as in (3.4) except common factor and root changed by

$$
\boxed{-\beta_1'\alpha_2\rho^2} \qquad c \qquad -s \qquad \rho(b-x_n)
$$

For $\beta_2' = 0,\ \alpha_2 = 0$, the estimate obtained from a tree which is the same as in (3.6) except common factor and root changed by

$$
\boxed{-\beta_1'\alpha_1\rho} \qquad c \qquad s \qquad \rho(b-x_n)
$$

Proof. The case $\beta_2' \neq 0,\ \alpha_2 \neq 0$ will be considered; similar proof works for other three cases. Since $w_{n+1}(\lambda)$ in (2.12) is independent of $x \in [a,b]$, from (2.9) we get

$$
w_{n+1}(\lambda) = (\beta_1'\lambda + \beta_1)\phi_{(n+1)\lambda}(b) - (\beta_2'\lambda + \beta_2)\phi'_{(n+1)\lambda}(b) \tag{3.8}
$$

Putting $x = b$ in estimates obtained from trees (3.4) and (3.6) for the asymptotic behavior of $\frac{d^k}{dx^k}\phi_{(n+1)\lambda}(x)$ and then substituting in (3.8) we obtain four different cases for the asymptotic behavior of $w_{n+1}(\lambda)$ and their trees as $|\lambda| \to \infty$. For $\beta_2' \neq 0$, $\alpha_2 \neq 0$, we briefly write

$$\begin{aligned} w_{n+1}(\lambda) = \beta_2'\alpha_2\rho^3 \{\sin\rho(b-x_n)[\ldots]_1 + \cos\rho(b-x_n)[\ldots]_2\} \\ + O\left(|\rho|^2 e^{|\tau|[(b-x_n)+(x_n-x_{n-1})+\cdots+(x_2-x_1)+(x_1-a)]}\right) \end{aligned} \tag{3.9}$$

where the terms in square brackets $[\ldots]_1$ and $[\ldots]_2$ are the same as the terms in estimate obtained from (3.4). Therefore the tree for (3.9) is similar to the tree (3.4) except common factor and root. The common factor and root are replaced by

$$\boxed{-\beta_2'\alpha_2\rho^3} \qquad s \qquad c \qquad \rho(b-x_n)$$

□

Putting $\rho = i\tau$ $(\tau > 0)$ in these formula it follows that $w(\lambda) \neq 0$ for λ negative and sufficiently large. The eigenvalues of the problem (1.1)-(1.5) are bounded below in all cases.

4. Asymptotic behavior of the eigenvalues and eigenfunctions

Our object in this section is to obtain asymptotic estimates for eigenvalues of problem (1.1)-(1.5). We know from Lemma 2 and Theorem 2 that the eigenvalues coincide with the zeros of the entire function $w_{n+1}(\lambda)$. Since the eigenvalues are real and bounded below we can denote the eigenvalues by λ_n $(n = 0, 1, \ldots)$, where

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

Let $\beta_2' \neq 0$, $\alpha_2 \neq 0$. In the formula (3.9) and so in the corresponding tree, after successive trigonometric operations from last children to the root we arrive at

$$w_{n+1}(\lambda) = \beta_2'\alpha_2\rho^3 \prod_{k=1}^{n} \sin\rho(b-a) + O\left(|\rho|^2 e^{|\tau|(b-a)}\right). \tag{4.1}$$

By applying Rouché's theorem on a sufficiently large contour it follows that $w_{n+1}(\lambda)$ has the same number of zeros inside the contour as

$$\beta_2'\alpha_2\rho^3 \prod_{k=1}^{n} \sin\rho(b-a).$$

Hence, if $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, are the zeros of $w_{n+1}(\lambda)$ and $\rho_m^2 = \lambda$, we have for sufficiently large m

$$\rho_m = \frac{(m-1)\pi}{b-a} + O\left(\frac{1}{m}\right). \tag{4.2}$$

With the same idea, we obtain: For $\beta_2' \neq 0$, $\alpha_2 = 0$ and for $\beta_2' = 0$, $\alpha_2 \neq 0$,

$$\rho_m = \frac{(m-1/2)\pi}{b-a} + O\left(\frac{1}{m}\right). \tag{4.3}$$

For $\beta_2' = 0,\ \alpha_2 = 0$,

$$\rho_m = \frac{m\pi}{b-a} + O\left(\frac{1}{m}\right). \tag{4.4}$$

Let $\phi(x,\lambda_m)$ is an eigenfunction corresponding to eigenvalue λ_m and $\beta_2' \neq 0,\ \alpha_2 \neq 0$. After inserting (4.2) into (3.3) and (3.4) and taking into account the definition of solutions defined in §2 we get

$$\phi(x,\lambda_m) = \alpha_2 \cos\left(\frac{(m-1)\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m}\right), \quad x \in [a,b],$$

and for $\beta_2' \neq 0,\ \alpha_2 = 0$

$$\phi(x,\lambda_m) = -\alpha_1 \frac{(b-a)}{\pi(m-1/2)} \sin\left(\frac{(m-1/2)\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m^2}\right), \quad x \in [a,b],$$

for $\beta_2' = 0,\ \alpha_2 \neq 0$

$$\phi(x,\lambda_m) = \alpha_2 \cos\left(\frac{(m-1/2)\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m}\right), \quad x \in [a,b],$$

for $\beta_2' = 0,\ \alpha_2 = 0$

$$\phi(x,\lambda_m) = -\alpha_1 \frac{(b-a)}{\pi m} \sin\left(\frac{m\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m^2}\right), \quad x \in [a,b].$$

REFERENCES

[1] Z. Akdoğan, M. Demirci, and O. Mukhtarov, "Discontinuous Sturm-Liouville problems with eigendependent boundary and transmissions conditions," *Acta Applicandae Mathematicae*, vol. 86, pp. 329–344, 2005, doi: 10.1007/s10440-004-7466-3.

[2] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed. London: AMS Chelsea Publishing, 2005.

[3] N. Altinisik, M. Kadakal, and O. S. Mukhtarov, "Eigenvalues and eigenfunctions of discontinuous Sturm-Liouville problems with eigenparameter-dependent boundary conditions," *Acta Mathematica Hungarica*, vol. 102, no. 1-2, pp. 159–175, 2004.

[4] C. Berge, *The Theory of Graphs and its Applications*. New York: John Wiley & Sons,, 1962.

[5] P. A. Binding and P. J. Browne, "Oscillation theory for indefinite Sturm-Liouville problems with eigenparameter-dependent boundary conditions," *Proc. Roy. Soc. Edin. A*, vol. 127, no. 6, pp. 1123–1136, 1997, doi: 10.1017/S0308210500026974.

[6] P. A. Binding, P. J. Browne, and B. A. Watson, "Strum-Liouville problems with boundary conditions rationally dependent on the eigenparameter I," *Proceedings of the Edinburgh Mathematical Society*, vol. 45, pp. 631–645, 2002, doi: 10.1017/S0013091501000773.

[7] C. T. Fulton, "Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions," *Proc. Roy. Soc. Edin. A*, vol. 77, pp. 293–308, 1977, doi: 10.1017/S030821050002521X.

[8] D. B. Hinton, "An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition," *Q. J. Math. Oxford*, vol. 2, no. 30, pp. 33–42, 1979, doi: 10.1093/qmath/30.1.33.

[9] M. Kadakal and O. S. Mukhtarov, "Sturm-Liouville problems with discontinuities at two points," *Computers and Mathematics with Applications*, vol. 54, pp. 1367–1379, 2007, doi: 10.1016/j.camwa.2006.05.032.

[10] N. B. Kerimov and K. R. Mamedov, "On a boundary value problem with a spectral parameter in the boundary conditions," *Sibirsk. Mat. Zh.*, vol. 40, no. 2, pp. 325–335, 1999.

[11] O. S. Mukhtarov, M. Kadakal, and F. S. Mukhtarov, "Eigenvalues and normalized eigenfunctions of discontinuous Sturm-Liouville problem with transmission conditions," *Rep. Math. Phys.*, vol. 54, pp. 41–56, 2004.

[12] O. S. Mukhtarov and E. Tunc, "Eigenvalue problems for Sturm-Liouville equations with transmission conditions," *Israel J. Math.*, vol. 144, pp. 367–380, 2004, doi: 10.1007/BF02916718.

[13] O. S. Mukhtarov and S. Yakubov, "Problems for ordinary differential equations with transmission conditions," *Applicable Analysis*, vol. 81, no. 5, pp. 1033–1064, 2002.

[14] A. A. Shkalikov, "Boundary-value problems for ordinary differential equations with a parameter in the boundary conditions," *Funkts. Anal. Prilozh.*, vol. 16, no. 4, pp. 92–93, 1982, doi: 10.1007/BF01077869.

[15] E. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, 2nd ed. London: Oxford University Press, 1962.

[16] O. Uğur and M. Akhmet, "Boundary value problems for higher order linear impulsive differential equations," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 1, pp. 139–156, 2006, doi: 10.1016/j.jmaa.2005.12.077.

[17] O. Uğur and M. Akhmet, "The Sturm-Liouville operator on the space of functions with discontinuity conditions," *Computers & Mathematics with Applications*, vol. 51, no. 6-7, pp. 889–896, 2006, doi: 10.1016/j.camwa.2005.11.034.

[18] J. Walter, "Regular eigenvalue problems with eigenvalue parameter in the boundary condition," *Mathematische Zeitschrift*, vol. 133, no. 4, pp. 301–312, 1973, doi: 10.1007/BF01177870.

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