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LINEAR OPERATOR ASSOCIATED WITH THE GENERALIZED FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract. In this note, we define a generalized linear operator using the generalized fractional differential operator. By employing this operator together withe the Cesàro partial sums, we impose starlike class of analytic functions depending on the subordination relation in the unit disk. We shall show that the functions in this class imply that the Libera-Pascu integral operator is also in the class. Moreover, we discus some other properties of convex functions such as convolution and inclusion properties.

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1. INTRODUCTION

In the geometric function theory (GFT) much concern is driven to different operators mapping the class of the univalent functions and its subclasses into themselves. Numerous authors obtained sufficient conditions that guarantee such mappings can be held. All these operators have convolution structures (Hadamard product) with special functions such as Gauss hypergeometric function [3], the Meijer G- and Fox H-functions [9]. In our investigation, we use the generalized Fox-Wright functions to obtain a new generalized operator.

Newly, fractional calculus in complex domain has established delightful implementations in (GFT). The conventional ideas of fractional operators and their generalizations have been employed in realizing, for example, coefficient estimates, distortion inequalities, the characterization properties and convolution structures for various subclasses of analytic functions and the doings in the research monographs.

In [22], Srivastava and Owa, gave definitions for the left-sided fractional integrals and derivatives in the complex z-plane $\mathbb{C}$ as follows:

Definition 1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \;\; 0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \;\; \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

In [4], the author have derived a formula for the generalized fractional integral as follows :

$$I_z^{\alpha,\mu} g(z) = \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\mu+1} - \zeta^{\mu+1})^{\alpha-1} \zeta^\mu g(\zeta) d\zeta, \tag{1.1}$$

where α and $\mu \neq -1$ are real numbers and the function $g(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$. When $\mu = 0$, we arrive at the standard Srivastava-Owa fractional integral. Further information can be found in [4]. Corresponding to the fractional integral operator, the fractional differential operator is

$$D^{\alpha,\mu} g(z) := \frac{(\mu+1)^\alpha}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{\zeta^\mu g(\zeta)}{(z^{\mu+1} - \zeta^{\mu+1})^\alpha} d\zeta; \;\; 0 \le \alpha < 1, \tag{1.2}$$

where the function $g(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$. We have

$$D^{\alpha,\mu} z^\nu = \frac{(\mu+1)^{\alpha-1} \Gamma\left(\frac{\nu}{\mu+1} + 1\right)}{\Gamma\left(\frac{\nu}{\mu+1} + 1 - \alpha\right)} z^{(1-\alpha)(\mu+1)+\nu-1}.$$

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^\infty a_n z^n, \quad z \in \mathbb{U}. \tag{1.3}$$

Also, let $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{C}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent, starlike and convex in $\mathbb{U}$. It is well known that; if the function

$f(z)$ given by (1.3) is in the class $\mathcal{S}$, then $|a_n| \le n, \quad n \in \mathbb{N}\backslash\{1\}$. Moreover, if the function $f(z)$ given by (1.3) is in the class $\mathcal{C}$, then $|a_n| \le 1, \quad n \in \mathbb{N}$.
In our present investigation, we shall also make use the Fox-Wright generalization ${}_q\Psi_p[z]$ of the hypergeometric ${}_qF_p$ function defined by [20]

$$
\begin{aligned}
{}_q\Psi_p\left[\begin{array}{r}(\alpha_1, A_1), \ldots, (\alpha_q, A_q); \\ z \\ (\beta_1, B_1), \ldots, (\beta_p, B_p);\end{array}\right] &= {}_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z] \\
&:= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1)\ldots\Gamma(\alpha_q + nA_q)}{\Gamma(\beta_1 + nB_1)\ldots\Gamma(\beta_p + nB_p)} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q}\Gamma(\alpha_j + nA_j)}{\prod_{j=1}^{p}\Gamma(\beta_j + nB_j)} \frac{z^n}{n!},
\end{aligned}
$$

where $A_j > 0$ for all $j = 1, \ldots, q$, $B_j > 0$ for all $j = 1, \ldots, p$ and $1 + \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \ge 0$ for suitable values $|z| < 1$ and α_i, β_j are complex parameters. It is well known that

$$
{}_q\Psi_p\left[\begin{array}{r}(\alpha_1, 1), \ldots, (\alpha_q, 1); \\ z \\ (\beta_1, 1), \ldots, (\beta_p, 1);\end{array}\right] = \Lambda^{-1}\, {}_qF_p(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p; z),
$$

where $\Lambda := \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}$ and ${}_qF_p$ is the generalized hypergeometric function. Now by making use the operator (1.2), we introduce the following extension operator $\Phi^{\alpha,\mu} : \mathcal{A} \to \mathcal{A}$:

$$
\begin{aligned}
&\Phi^{\alpha,\mu} f(z) \qquad (1.4) \\
&:= \frac{\Gamma\left(\frac{1}{\mu+1} + 1 - \alpha\right)}{(\mu+1)^{\alpha-1}\Gamma\left(\frac{1}{\mu+1} + 1\right)} z^{\alpha(1+\mu)-\mu} D^{\alpha,\mu} f(z) \\
&= \frac{\Gamma\left(\frac{1}{\mu+1} + 1 - \alpha\right)}{(\mu+1)^{\alpha-1}\Gamma\left(\frac{1}{\mu+1} + 1\right)} z^{\alpha(1+\mu)-\mu} D^{\alpha,\mu}\left(z + \sum_{n=2}^{\infty} a_n z^n\right) \\
&= \frac{\Gamma\left(\frac{1}{\mu+1} + 1 - \alpha\right)}{(\mu+1)^{\alpha-1}\Gamma\left(\frac{1}{\mu+1} + 1\right)} z^{\alpha(1+\mu)-\mu}\left[\frac{(\mu+1)^{\alpha-1}\Gamma\left(\frac{1}{\mu+1} + 1\right)}{\Gamma\left(\frac{1}{\mu+1} + 1 - \alpha\right)} z^{\mu+1-\alpha(\mu+1)}\right. \\
&\left. + \sum_{n=2}^{\infty} \frac{(\mu+1)^{\alpha-1}\Gamma\left(\frac{n}{\mu+1} + 1\right)}{\Gamma\left(\frac{n}{\mu+1} + 1 - \alpha\right)} a_n z^{n+\mu-\alpha(\mu+1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= z + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{1}{\mu+1}+1-\alpha\right)}{\Gamma\left(\frac{1}{\mu+1}+1\right)} \frac{\Gamma\left(\frac{n}{\mu+1}+1\right)}{\Gamma\left(\frac{n}{\mu+1}+1-\alpha\right)} a_n z^n \\
&:= z + \sum_{n=2}^{\infty} \phi_n^{\alpha,\mu} a_n z^n .
\end{aligned}
\tag{1.5}
$$

Obviously, when $\mu = 0$ we have the extension fractional differential operator defined in [13] ([5] for recent work), which contains the Carlson and Shaffer operator. In term of the Fox-Wright generalized function,

$$
\begin{aligned}
\Phi^{\alpha,\mu} f(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma\left(\frac{1}{\mu+1}+1-\alpha\right)}{\Gamma\left(\frac{1}{\mu+1}+1\right)} \frac{\Gamma(\frac{n}{\mu+1}+1)}{\Gamma\left(\frac{n}{\mu+1}+1-\alpha\right)} \frac{a_n}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\Gamma\left(\frac{1}{\mu+1}+1-\alpha\right)}{\Gamma\left(\frac{1}{\mu+1}+1\right)} \frac{\Gamma\left(\frac{n}{\mu+1}+1\right)}{\Gamma\left(\frac{n}{\mu+1}+1-\alpha\right)} \frac{a_n}{n!} z^n \\
&= \frac{\Gamma\left(\frac{1}{\mu+1}+1-\alpha\right)}{\Gamma\left(\frac{1}{\mu+1}+1\right)} {}_2\Psi_1 \left[\begin{array}{c} (1,1),(1,\frac{1}{\mu+1}); \\ z \\ (1-\alpha,\frac{1}{\mu+1}) \ ; \end{array} \right] * f(z) \\
&= \frac{\Gamma\left(\frac{1}{\mu+1}+1-\alpha\right)}{\Gamma\left(\frac{1}{\mu+1}+1\right)} {}_2\Psi_1[z] * f(z) \\
&:= \Psi(\alpha,\mu;z) * f(z),
\end{aligned}
\tag{1.6}
$$

where $a_0 = 0, a_1 = 1$ and $*$ is the Hadamard product. Note that $\Phi^{0,0} f(z) = f(z)$. Furthermore, one can easily define a linear fractional operator involving (1.4) as follows:

$$
D^{n,\alpha,\mu} f(z) = [\underbrace{\Psi(\alpha,\mu;z) * \ldots * \Psi(\alpha,\mu;z)}_{n-times}] * f(z).
\tag{1.7}
$$

Recently, various results, as convolution and inclusion properties, distortion theorem, extreme points, coefficient estimates etc., are proposed by many authors for the operators due to Srivastava involving the Wright function, generalized hypergeometric function and Meijer's G-functions. These operators are Dziok-Srivastava [25] and [10], Srivastava-Wright, Srivastava-Owa operators (see [8] and [21]), Cho-Kwon-Srivastava operator [14], [19], Cho-Saigo-Srivastava operator operator [6], Jung-Kim-Srivastava operator [2] and for recent work [7], [16], [17] and [18].

From the partial sum

$$s_k(z) = \sum_{n=1}^{k} a_n z^n, \ z \in \mathbb{U},$$

the Cesàro means $\sigma_k(z, f)$ of $f(z) = \sum_{n=1}^{\infty} a_n z^n$ given by

$$\begin{aligned}
\sigma_k(z, f) &= \frac{1}{k} \sum_{n=1}^{k} s_n(z) \\
&= \frac{1}{k} [s_1(z) + \ldots + s_k(z)] \\
&= \frac{1}{k} \left[a_1 z + (a_1 z + a_2 z^2) + \ldots + (a_1 z + \ldots + a_k z^k) \right] \\
&= \frac{1}{k} \left[k a_1 z + (k-1) a_2 z^2 + \ldots + a_k z^k \right] \\
&= \sum_{n=1}^{k} \frac{k-n+1}{k} a_n z^n \\
&= f(z) * \left[\sum_{n=1}^{k} \frac{k-n+1}{k} z^n \right] \\
&= f(z) * g_k(z),
\end{aligned}$$

where

$$g_k(z) = \sum_{n=1}^{k} \frac{k-n+1}{k} z^n.$$

Definition 3 (Subordination Principle)**.** For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinated to $g(z)$ in $\mathbb{U}$ and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z)), z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Definition 4 (Differential subordination)**.** Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

Definition 5. A function $f \in \mathcal{A}$ is called $\lambda-$starlike; $\lambda \geq 0$, with respect to Cesàro means if it satisfies

$$\frac{(1-\lambda)zf'(z)+\lambda z(zf'(z))'}{(1-\lambda)\sigma_k(z,f)+\lambda z\sigma_k'(z,f)} \prec q(z),$$

where $q(0)=1$ and $\Re q'(z)>0$. Denote this class by $S\sigma_k(\lambda,q)$. Note that the class $S\sigma_k(0,q)$ takes the form

$$\frac{zf'(z)}{\sigma_k(z,f)} \prec q(z).$$

Some classes of partial sums are suggested and studied in [1].

Definition 6. A function $f \in \mathcal{A}$ is in the class $S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ if it satisfies

$$\frac{(1-\lambda)z\big(D^{n,\alpha,\mu}f(z)\big)'+\lambda z[z\big(D^{n,\alpha,\mu}f(z)\big)']'}{(1-\lambda)\sigma_k(z,D^{n,\alpha,\mu}f)+\lambda z(\sigma_k'(z,D^{n,\alpha,\mu}f))} \prec q(z),$$

where $q(0)=1$ and $\Re q'(z)>0$.

We need the following preliminaries in the sequel. The Libera-Pascu integral operator $L_a:\mathcal{A}\to\mathcal{A}$ defined by

$$F(z):=L_af(z)=\frac{1+a}{z^a}\int_0^z f(t)t^{a-1}\mathrm{d}t,\ a\in\mathbb{C},\ \Re(a)\geq 0.$$

For $a=1$ we obtain the Libera integral operator, for $a=0$ we obtain the Alexander integral operator and in the case $a=1,2,3,\dots$ we obtain the Bernardi integral operator.

Lemma 1 ([11]). *Let h be convex univalent in $\mathbb{U}$ and θ,ϕ be analytic in domain D. Let p be analytic in $\mathbb{U}$, with $h(0)=\theta(p(0))$ and $p(\mathbb{U})\subset D$. If the differential equation*

$$\theta[q(z)]+zq'(z)\phi[q(z)]=h(z)$$

has a univalent solution in $\mathbb{U}$ that satisfies $q(0)=p(0)$ and $\theta[q(z)]\prec h(z)$ then the differential subordination

$$\theta[p(z)]+zp'(z)\phi[p(z)]\prec h(z)$$

implies that $p(z)\prec q(z)$. The function q is the best dominant.

Lemma 2 ([16]). *Let h and g be in the classes $\mathcal{C}$ and $\mathcal{S}^*$ respectively. Then, for every analytic function $\mathcal{F}$ with $\mathcal{F}(0)=1$, we have*

$$\frac{h(z)*g(z)\mathcal{F}(z)}{h(z)*g(z)}\in\overline{co}\mathcal{F}(\mathbb{U}),\quad z\in\mathbb{U},$$

where $\overline{co}$ denotes the closed convex hull.

Lemma 3 ([12]). *If $f \in \mathcal{A}$ satisfies*

$$|f''(z)| \leq \frac{\sqrt{5}}{5} = 0.4472..., \quad (z \in U),$$

then $f \in \mathcal{C}$.

Our aim is to show that if $f \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ then $L_a f(z) \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$. Furthermore, we shall consider some relation of differential subordination and convolution.

2. Libera-Pascu class

Our main results are provided in this section.

Theorem 1. *Let $f \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$. Denote $\widehat{g} := D^{n,\alpha,\mu} g$ and $F(z) := L_a f(z)$. Define two analytic functions*

$$\varphi(z) := (1-\lambda)\sigma_k(z,\widehat{F}) + \lambda z \sigma_k'(z,\widehat{F})$$

and

$$p(z) := \frac{(1-\lambda)z\widehat{F}'(z) + \lambda z[z\widehat{F}'(z)]'}{\varphi(z)}.$$

Assume that for a convex univalent function h the fractional differential equation

$$\frac{q(z)[(1-\lambda)\varphi(z)+\lambda z\varphi'(z)]}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})} + \frac{\lambda z q'(z)\varphi(z)}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})} = h(z)$$

has a univalent solution $q \in U$ satisfying $p(0) = q(0)$. Then the subordination

$$\frac{p(z)[(1-\lambda)\varphi(z)+\lambda z\varphi'(z)]}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})} + \frac{\lambda z p'(z)\varphi(z)}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})} \prec$$

$$\frac{q(z)[(1-\lambda)\varphi(z)+\lambda z\varphi'(z)]}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})} + \frac{\lambda z q'(z)\varphi(z)}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})}$$

implies $p(z) \prec q(z)$ and q is the best dominant.

Proof. Since $f \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$, then

$$\frac{(1-\lambda)z\widehat{f}'(z)+\lambda z[z\widehat{f}'(z)]'}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})} \prec q(z).$$

From the definition of the Libera-Pascu integral operator we have

$$(1+a)f(z) = aF(z) + zF'(z),$$

by using the linear operator $D^{n,\alpha,\mu}g = \widehat{g}$, we have

$$(1+a)\widehat{f}(z) = a\widehat{F}(z) + \widehat{zF'(z)}, \quad \Re(a) \geq 0.$$

By making use the first derivative of the last assertion, we obtain

$$z\widehat{f}'(z) = \frac{a}{(1+a)} z\widehat{F}'(z) + \frac{1}{(1+a)} z[\widehat{zF'(z)}]'$$

and using the fact that

$$\widehat{zF'(z)} = z\widehat{F}'(z),$$

yields

$$\begin{aligned} z\widehat{f}'(z) &= \frac{a}{(1+a)} z\widehat{F}'(z) + \frac{1}{(1+a)} z[z(\widehat{zF'(z)})]' \\ &:= (1-\lambda)z\widehat{F}'(z) + \lambda z[z(\widehat{zF'(z)})]' \\ &= p(z)[(1-\lambda)\sigma_k(z,\widehat{F}) + \lambda z\sigma_k'(z,\widehat{F})] \\ &:= p(z)\varphi(z). \end{aligned}$$

A computation implies that

$$\begin{aligned} \frac{(1-\lambda)z\widehat{f}'(z) + \lambda z[z\widehat{f}'(z)]'}{(1-\lambda)\sigma_k(z,\widehat{f}) + \lambda z\sigma_k'(z,\widehat{f})} &= \frac{p(z)[(1-\lambda)\varphi(z) + \lambda z\varphi'(z)]}{(1-\lambda)\sigma_k(z,\widehat{f}) + \lambda z\sigma_k'(z,\widehat{f})} \\ &\quad + \frac{\lambda z p'(z)\varphi(z)}{(1-\lambda)\sigma_k(z,\widehat{f}) + \lambda z\sigma_k'(z,\widehat{f})} \\ &:= p(z)\psi(z) + zp'(z)\phi(z) \\ &:= \theta[p(z)] + zp'(z)\phi(z), \end{aligned}$$

where θ and ϕ are analytic in $\mathbb{U}$. It is clear that $\theta[q(z)] \prec h(z)$. Hence in view of Lemma 1, we have $p(z) \prec q(z)$ and q is the best dominant. $\square$

Immediately, we have the following result:

Corollary 1. *Let the assumptions of Theorem 1 hold. Then the Libera-Pascu integral operator* $F(z) \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$.

3. Convex class

In this section, we discuss the class $S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ when $f \in \mathcal{C}$. We need the following preliminaries:

Lemma 4. *Assume that* $\alpha_i > 0, \beta_j > 0; i = 1,\dots,q, j = 1,\dots,p; q \le p+1$. *If for* $0 \le \alpha < 1$,

$$\frac{\Gamma\left(\frac{1}{\mu+1} + 1 - \alpha\right)}{\Gamma\left(\frac{1}{\mu+1} + 1\right)} {}_2\Psi_1 \left[\begin{array}{c} (2,1), (1 + \frac{1}{\mu+1}, \frac{1}{\mu+1}); \\ 1 \\ (1 - \alpha + \frac{1}{\mu+1}, \frac{1}{\mu+1}) \ ; \end{array} \right] < 2,$$

then the operator (1.6) *maps a convex function* $f(z)$ *into a convex function that is* $\Phi^{\alpha,\mu}:\mathcal{C}\to\mathcal{C}$.

Proof. Assume that $f\in\mathcal{C}$, $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and the operator (1.4) such that

$$\Phi^{\alpha,\mu}f(z)=z+\sum_{n=2}^{\infty}b_nz^n,$$

where

$$b_n=\phi_n^{\alpha,\mu}a_n.$$

The following condition is satisfied (see [10]) :

$$\rho:=\sum_{n=2}^{\infty}n|b_n|=\sum_{n=2}^{\infty}n\phi_n^{\alpha,\mu}|a_n|<1.$$

By using the fact that $|a_n|\le 1$ we estimate ρ as follows

$$\begin{aligned}\rho&=\sum_{n=2}^{\infty}n\phi_n^{\alpha,\mu}|a_n|\le\sum_{n=2}^{\infty}n\phi_n^{\alpha,\mu}\\&=\sum_{n=2}^{\infty}\frac{n}{(1)_n}[\phi_n^{\alpha,\mu}(1)_n]=\sum_{n=2}^{\infty}\frac{n}{(1)_n}\varrho(n)\\&<1,\end{aligned}\tag{3.1}$$

where

$$\varrho(n)=\frac{(1)_n\Gamma(\frac{1}{\mu+1}+1-\alpha)}{\Gamma(\frac{1}{\mu+1}+1)}\frac{\Gamma(\frac{n}{\mu+1}+1)}{\Gamma(\frac{n}{\mu+1}+1-\alpha)}.$$

Since

$$\frac{n}{(1)_n}=\frac{1}{(1)_{n-1}}$$

therefore, the estimate (3.1) takes the formula

$$
\begin{aligned}
&\rho \\
&\leq \sum_{n=2}^{\infty} \frac{n}{(1)_n} \varrho(n) = \sum_{n=2}^{\infty} \frac{1}{(1)_{n-1}} \varrho(n) = \sum_{n=2}^{\infty} \frac{\varrho(n)}{(1)_{n-1}} \\
&= \sum_{n=2}^{\infty} \frac{(1)_n \Gamma(\frac{1}{\mu+1}+1-\alpha)}{\Gamma(\frac{1}{\mu+1}+1)} \frac{\Gamma(\frac{n}{\mu+1}+1)}{\Gamma(\frac{n}{\mu+1}+1-\alpha)} \frac{1}{(1)_{n-1}} \\
&= \frac{\Gamma(\frac{1}{\mu+1}+1-\alpha)}{\Gamma(\frac{1}{\mu+1}+1)} \sum_{n=1}^{\infty} \frac{(1)_{n+1} \Gamma(\frac{n+1}{\mu+1}+1)}{\Gamma(\frac{n+1}{\mu+1}+1-\alpha)} \frac{1}{(1)_n} \\
&= \frac{\Gamma(\frac{1}{\mu+1}+1-\alpha)}{\Gamma(\frac{1}{\mu+1}+1)} \sum_{n=1}^{\infty} \frac{\Gamma(n+2) \Gamma(\frac{n+1}{\mu+1}+1)}{\Gamma(\frac{n+1}{\mu+1}+1-\alpha)} \frac{1}{(1)_n} \\
&= \frac{\Gamma\left(\frac{1}{\mu+1}+1-\alpha\right)}{\Gamma\left(\frac{1}{\mu+1}+1\right)} \left\{ {}_2\Psi_1 \left[\begin{array}{c} (2,1),(1+\frac{1}{\mu+1},\frac{1}{\mu+1}); \\ 1 \\ (1-\alpha+\frac{1}{\mu+1},\frac{1}{\mu+1}) \ ; \end{array} \right] - \frac{\Gamma(\frac{1}{\mu+1}+1)}{\Gamma(\frac{1}{\mu+1}+1-\alpha)} \right\} \\
&< 1.
\end{aligned}
$$

Hence, we estimate ρ in terms of the Fox-Wright function ${}_2\Psi_1$ at $z = 1$. This completes the proof. □

Now we put

$$
\begin{aligned}
\theta(z) :&= z + \sum_{n=2}^{\infty} \phi_n^{\alpha,\mu} z^n \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(\frac{1}{\mu+1}+1-\alpha)}{\Gamma(\frac{1}{\mu+1}+1)} \frac{\Gamma(\frac{n}{\mu+1}+1)}{\Gamma(\frac{n}{\mu+1}+1-\alpha)} z^n
\end{aligned} \tag{3.2}
$$

consequently we obtain the function

$$
\begin{aligned}
\Theta(z) :&= \theta^{(-1)}(z) \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(\frac{1}{\mu+1}+1)}{\Gamma(\frac{1}{\mu+1}+1-\alpha)} \frac{\Gamma(\frac{n}{\mu+1}+1-\alpha)}{\Gamma(\frac{n}{\mu+1}+1)} z^n \\
&:= z + \sum_{n=2}^{\infty} \psi_n^{\alpha,\mu} z^n .
\end{aligned}
$$

We have the following result:

Lemma 5. *If* $|\psi_n^{\alpha,\mu}| \leq \frac{1}{\sqrt{5}n(n-1)2^{n-1}}$, $n \geq 2$, *then* $\Theta(z) \in \mathcal{C}$.

Proof. By the assumption, we conclude that

$$\begin{aligned}|\Theta''(z)| &\le \sum_{n=2}^{\infty} n(n-1)|\psi_n^{\alpha,\mu}| \\ &\le \frac{1}{\sqrt{5}}\sum_{n=2}^{\infty}\frac{1}{2^{n-1}} \\ &\le \frac{\sqrt{5}}{5}.\end{aligned}$$

Thus, in virtue of Lemma 3 yields $\Theta(z) \in \mathcal{C}$. □

Now we provide our main result in this section.

Theorem 2. *Let* $f \in \mathcal{C}$ *and* $\Phi^{\alpha,\mu} f(z) \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$, $\Re(q) > 0$. *Then* $f \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$.

Proof. Let $f \in \mathcal{C}$ and $\Phi^{\alpha,\mu} f(z) \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$, $\Re(q) > 0$. To be brief we will write $\widehat{g} := D^{n,\alpha,\mu} g$, and $\Phi f(z) := \Phi^{\alpha,\mu} f(z)$. Using (1.6) and (1.7), we can write

$$\frac{(1-\lambda)z\widehat{f}'(z)+\lambda z[z\widehat{f}'(z)]'}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})}$$

in terms of $\Phi f(z)$ as follows :

$$\begin{aligned}\widehat{f}(z) &= \Theta(z) * \widehat{\Phi f}(z), \\ z\widehat{f}'(z) &= \Theta(z) * z\widehat{\Phi f}'(z), \\ z[z\widehat{f}'(z)]' &= z[\Theta(z) * z\widehat{\Phi f}'(z)]' \\ &= \Theta(z) * z[z\widehat{\Phi f}'(z)]'\end{aligned}$$

thus using Lemma 2 and Lemma 5, yields

$$\begin{aligned}&\frac{(1-\lambda)z\widehat{f}'(z)+\lambda z[z\widehat{f}'(z)]'}{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})} \\ &= \frac{(1-\lambda)\Theta(z) * z\widehat{\Phi f}'(z)+\lambda\Theta(z) * z[z\widehat{\Phi f}'(z)]'}{(1-\lambda)\sigma_k(z,\Theta(z) * \widehat{\Phi f}(z))+\lambda z\sigma_k'(z,\Theta(z) * \widehat{\Phi f}(z))} \\ &= \frac{\Theta(z) * \Big((1-\lambda)z\widehat{\Phi f}'(z)+\lambda z[z\widehat{\Phi f}'(z)]'\Big)}{\Theta(z) * \Big((1-\lambda)\sigma_k(z,\widehat{\Phi f}(z))+\lambda z\sigma_k'\big(z,\widehat{\Phi f}(z)\big)\Big)} \\ &:= \frac{\Theta(z) * g(z)\mathcal{F}(z)}{\Theta(z) * g(z)},\end{aligned}$$

where

$$g(z) := (1-\lambda)\sigma_k(z, \widehat{\Phi f}(z)) + \lambda z \sigma_k'\big(z, \widehat{\Phi f}(z)\big)$$

and

$$\mathcal{F}(z) := \frac{(1-\lambda)\widehat{\Phi f}\,'(z) + \lambda z[z\widehat{\Phi f}\,'(z)]'}{(1-\lambda)\sigma_k(z, \widehat{\Phi f}(z)) + \lambda z \sigma_k'\big(z, \widehat{\Phi f}(z)\big)}.$$

Hence

$$\begin{aligned}
&\frac{(1-\lambda)z\widehat{f}\,'(z) + \lambda z[z\widehat{f}\,'(z)]'}{(1-\lambda)\sigma_k(z,\widehat{f}) + \lambda z \sigma_k'(z,\widehat{f})}\\
&\in \overline{co}\left(\frac{(1-\lambda)\widehat{\Phi f}\,'(z) + \lambda z[z\widehat{\Phi f}\,'(z)]'}{(1-\lambda)\sigma_k(z, \widehat{\Phi f}(z)) + \lambda z \sigma_k'(z, \widehat{\Phi f}(z))}\right)\\
&\subseteq q(\mathbb{U}).
\end{aligned}$$

Therefore, $f \in S\sigma_k^{n,\alpha,\mu}(\lambda, q)$. □

Immediately we obtain the following Corollaries:

Corollary 2. *Let $f \in \mathcal{C}$. Then*

$$S\sigma_k^{n+1,\alpha,\mu}(\lambda, q) \subset S\sigma_k^{n,\alpha,\mu}(\lambda, q),$$

where $\Re(q) > 0$.

Proof. Let $f \in S\sigma_k^{n+1,\alpha,\mu}(\lambda, q)$. Since

$$\begin{aligned}
D^{n+1,\alpha,\mu} f(z) &= [\underbrace{\Psi(\alpha,\mu;z) * \ldots * \Psi(\alpha,\mu;z)}_{(n+1)-times}] * f(z)\\
&= [\underbrace{\Psi(\alpha,\mu;z) * \ldots * \Psi(\alpha,\mu;z)}_{n-times}] * \Psi(\alpha,\mu;z) * f(z)\\
&= D^{n,\alpha,\mu}\big(\Psi(\alpha,\mu;z) * f(z)\big)\\
&= \widehat{\Phi f}(z).
\end{aligned}$$

Then a computation implies that

$$\begin{aligned}
&\frac{(1-\lambda)z\big(D^{n+1,\alpha,\mu} f(z)\big)' + \lambda z[z\big(D^{n+1,\alpha,\mu} f(z)\big)']'}{(1-\lambda)\sigma_k(z, D^{n+1,\alpha,\mu} f) + \lambda z \sigma_k'(z, D^{n+1,\alpha,\mu} f)} \prec q(z)\\
&\Rightarrow \frac{(1-\lambda)z\widehat{\Phi f}\,'(z) + \lambda z[z\widehat{\Phi f}\,'(z)]'}{(1-\lambda)\sigma_k(z, \widehat{\Phi f}(z)) + \lambda z \sigma_k'(z, \widehat{\Phi f}(z))} \prec q(z)
\end{aligned}$$

that is $f \in \Phi f(z) \in S\sigma_k^{n,\alpha,\mu}(\lambda, q)$; hence in view of Theorem 2, we obtain that $f \in S\sigma_k^{n,\alpha,\mu}(\lambda, q)$. □

Corollary 3. *Let $f \in \mathcal{C}$ and $q(z) := \frac{1+z}{1-z}$. Then*

$$S\sigma_k^{n+1,\alpha,\mu}\left(\lambda, \frac{1+z}{1-z}\right) \subset S\sigma_k^{n,\alpha,\mu}\left(\lambda, \frac{1+z}{1-z}\right) \subset \ldots \subset S\sigma_k^{0,\alpha,\mu}\left(\lambda, \frac{1+z}{1-z}\right)$$
$$= S\sigma_k\left(\lambda, \frac{1+z}{1-z}\right).$$

Theorem 3. $f \in S\sigma_k^{n+1,\alpha,\mu}(\lambda, q) \Rightarrow \Phi^{\alpha,\mu} f \in S\sigma_k^{n,\alpha,\mu}(\lambda, q)$.

Proof. Let $f \in S\sigma_k^{n+1,\alpha,\mu}(\lambda, q)$. Since

$$D^{n+1,\alpha,\mu} f(z) = \widehat{\Phi f}(z),$$

then a calculation implies

$$\begin{aligned}
&\frac{(1-\lambda)z\widehat{\Phi f}'(z) + \lambda z[z\widehat{\Phi f}'(z)]'}{(1-\lambda)\big(\sigma_k(z, \widehat{\Phi f})\big) + \lambda z(\sigma_k'(z, \widehat{\Phi f}))} \\
&= \frac{(1-\lambda)z\big(D^{n+1,\alpha,\mu} f(z)\big)' + \lambda z[z\big(D^{n+1,\alpha,\mu} f(z)\big)']'}{(1-\lambda)\big(\sigma_k(z, D^{n+1,\alpha,\mu} f)\big) + \lambda z(\sigma_k'(z, D^{n+1,\alpha,\mu} f))} \\
&\prec q(z).
\end{aligned}$$

Hence $\Phi^{\alpha,\mu} f \in S\sigma_k^{n,\alpha,\mu}(\lambda, q)$. □

In the following theorem, we prove the class $S\sigma_k^{n,\alpha,\mu}(\lambda, q)$ is closed under convolution with convex function.

Theorem 4. *Let $f \in S\sigma_k^{n,\alpha,\mu}(\lambda, q)$ and ϑ be a convex function with real coefficients in $\mathbb{U}$. Then $f * \vartheta \in S\sigma_k^{n,\alpha,\mu}(\lambda, q)$.*

Proof. Applying Lemma 2 and using the convolution properties, we have

$$\begin{aligned}
&\frac{(1-\lambda)z\big(D^{n,\alpha,\mu}(f(z) * \vartheta(z))\big)' + \lambda z[z\big(D^{n,\alpha,\mu}(f(z) * \vartheta(z))\big)']'}{(1-\lambda)\sigma_k(z, D^{n,\alpha,\mu}(f * \vartheta)) + \lambda z\sigma_k'(z, D^{n,\alpha,\mu}(f * \vartheta))} \\
&= \frac{\vartheta(z) * \Big((1-\lambda)z(\widehat{f}'(z) + \lambda z[z\widehat{f}'(z)]'\Big)}{\vartheta(z) * \Big((1-\lambda)\sigma_k(z, \widehat{f}(z)) + \lambda z\sigma_k'\big(z, \widehat{f}(z)\big)\Big)} \\
&:= \frac{\vartheta(z) * g(z)\mathcal{F}(z)}{\vartheta(z) * g(z)},
\end{aligned}$$

where

$$g(z) := (1-\lambda)\sigma_k(z, \widehat{f}(z)) + \lambda z\sigma_k'\big(z, \widehat{f}(z)\big)$$

and

$$\mathcal{F}(z) := \frac{(1-\lambda)z\widehat{f}'(z) + \lambda z[z\widehat{f}'(z)]'}{(1-\lambda)\sigma_k(z, \widehat{f}(z)) + \lambda z\sigma_k'\big(z, \widehat{f}(z)\big)}.$$

Hence

$$\begin{aligned}
&\frac{(1-\lambda)z\big(D^{n,\alpha,\mu}(f(z)*\vartheta(z))\big)'+\lambda z[z\big(D^{n,\alpha,\mu}(f(z)*\vartheta(z))\big)']'}{(1-\lambda)\sigma_k(z,D^{n,\alpha,\mu}(f*\vartheta))+\lambda z\sigma_k'(z,D^{n,\alpha,\mu}(f*\vartheta))}\\
&\in \overline{co}\left\{\frac{(1-\lambda)z\widehat{f}\,'(z)+\lambda z[z\widehat{f}\,'(z)]'}{(1-\lambda)\sigma_k(z,\widehat{f}(z))+\lambda z\sigma_k'\big(z,\widehat{f}(z)\big)}\right\}\\
&\subseteq q(\mathbb{U}).
\end{aligned}$$

Therefore, $f*\vartheta\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$. □

4. Class $\sigma_k^{n,\alpha,\mu}$

In this section, we proceed to show that the function

$$f\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)\Rightarrow \sigma_k(z,\widehat{f}(z))\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$$

and

$$f\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)\Rightarrow \sigma_k(z,\widehat{\Phi f}(z))\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$$

whenever $q\in\mathcal{C}$.

Theorem 5. *Let $q\in\mathcal{C}$. If $f\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ then $\sigma_k(z,\widehat{f}(z))\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$.*

Proof. By the definition of $\sigma_k(z,f)$ we obtain the following facts:

$$\begin{gathered}
z\sigma_k'(z,f)=zf'(z)*g_k(z),\\
z[z\sigma_k'(z,f)]'=z(zf'(z))'*g_k(z),\\
\sigma_k(z,\sigma_k(z,f))=\sigma_k(z,f*g_k)=(f*g_k)*g_k=\sigma_k(z,f)*g_k
\end{gathered}$$

and since $q\in\mathcal{C}$ we pose that

$$\begin{aligned}
&\frac{(1-\lambda)z\sigma_k'(z,\widehat{f})+\lambda z[z\sigma_k'(z,\widehat{f})]'}{(1-\lambda)\sigma_k(z,\sigma_k(z,\widehat{f}))+\lambda z\sigma_k'(z,\sigma_k(z,\widehat{f}))}\\
&=\frac{(1-\lambda)z\widehat{f}\,'(z)*g_k(z)+\lambda z[z\widehat{f}\,'(z)]'*g_k(z)}{(1-\lambda)\big(\sigma_k(z,\widehat{f})*g_k(z)\big)+\lambda z\sigma_k'(z,\widehat{f})*g_k(z)}\\
&=\frac{\{(1-\lambda)z\big(\widehat{f}\,'(z)+\lambda z[z\widehat{f}\,'(z)]'\}*g_k(z)}{\{(1-\lambda)\sigma_k(z,\widehat{f})+\lambda z\sigma_k'(z,\widehat{f})\}*g_k(z)}\\
&\prec q(z).
\end{aligned}$$

Hence $\sigma_k(z,\widehat{f}(z))\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$. □

Similarly, we have the following result:

Theorem 6. *Let $q\in\mathcal{C}$. If $f\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ then $\sigma_k(z,\widehat{\Phi f}(z))\in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$.*

Proof. By using the relation

$$\widehat{\Phi f}(z) = \widehat{f}(z) * \theta(z),$$

where $\theta(z)$ is defined in (3.2), implies

$$\begin{aligned}
&\frac{(1-\lambda)z\sigma_k'(z,\widehat{\Phi f}) + \lambda z[z\sigma_k'(z,\widehat{\Phi f})]'}{(1-\lambda)\sigma_k(z,\sigma_k(z,\widehat{f})) + \lambda z\sigma_k'(z,\sigma_k(z,\widehat{\Phi f}))} \\
&= \frac{(1-\lambda)z\widehat{\Phi f}'(z) * g_k(z) + \lambda z[z\widehat{\Phi f}'(z)]' * g_k(z)}{(1-\lambda)\big(\sigma_k(z,\widehat{\Phi f}) * g_k(z)\big) + \lambda z\sigma_k'(z,\widehat{\Phi f} * g_k(z))} \\
&= \frac{\big\{(1-\lambda)z\widehat{\Phi f}'(z) + \lambda z[z\widehat{\Phi f}'(z)]'\big\} * g_k(z)}{\big\{(1-\lambda)\sigma_k(z,\widehat{\Phi f}) + \lambda z\sigma_k'(z,\widehat{\Phi f})\big\} * g_k(z)} \\
&= \frac{\big\{(1-\lambda)z\widehat{f}'(z) + \lambda z[z\widehat{f}'(z)]'\big\} * g_k(z) * \theta(z)}{\big\{(1-\lambda)\sigma_k(z,\widehat{f}) + \lambda z\sigma_k'(z,\widehat{f})\big\} * g_k(z) * \theta(z)} \\
&\prec q(z).
\end{aligned}$$

Since $\theta(z)$, $g_k(z)$ and $q(z)$ are convex in the unit disk, therefore, $\sigma_k(z,\widehat{\Phi f}(z)) \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$. □

5. Conclusion

It is well known that the classical Cesàro means retains the zero free property of the derivatives of bounded convex functions in the unit disk [23]. Furthermore, for univalent functions $f \in \mathcal{A}$, the partial sums f_k in general are not univalent in the unit disk $\mathbb{U}$; though they are univalent for $|z| < 1/4$ as shown by Szegö (1928) (see [24]). Robertson (1936) showed that when f is univalent in $\mathbb{U}$; then also all the Cesàro sums are univalent in the unit disk $\mathbb{U}$ (see[15]). Therefor from above, we have illustrated classes of univalent functions involving the Cesàro partial sums instead of the usual partial sums to preserve the geometry properties of the function. We have considered a linear fractional operator defined by using the generalized Srivastava-Owa fractional differential operator. We have imposed that for the function $f \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ the Cesàro partial sums $\sigma_k(z,D^{n,\alpha,\mu}\Phi^{\alpha,\mu}f(z))$ and $\sigma_k(z,D^{n,\alpha,\mu}f(z))$ are also in the class $S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ (Theorem 5 & Theorem 6). Moreover, we have proved that for the function $f \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$, the Libera-Pascu integral operator $F(z) \in S\sigma_k^{n,\alpha,\mu}(\lambda,q)$ (Theorem 1). For convex function f, we have shown that the fractional differential operator maps a convex function into a convex function ($\Phi^{\alpha,\mu} : \mathcal{C} \to \mathcal{C}$) (Lemma 2). Depending on this property, we have illustrated some convolution relations (Corollary 2 & Corollary 3).

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