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TAUBERIAN CONDITIONS UNDER WHICH λ-STATISTICAL CONVERGENCE FOLLOWS FROM STATISTICAL SUMMABILITY (V,λ)

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Abstract. In this paper, we will show Tauberian conditions under which $\lambda-$statistical convergence follows from $(V,\lambda)-$statistical convergence. Our results generalize the ones given in [3].

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1. INTRODUCTION

We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero (cf. [1, 6]), i.e. for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent but not conversely.

The idea of λ-statistical convergence was introduced in [5] as follows: Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, n-1 < \lambda_n$. The generalized de la Vallée-Poussin mean is defined by $T_n(x) =: \frac{1}{\lambda_n} \sum_{j\in I_n} x_j$, where $I_n = [n-\lambda_n+1, n]$. A sequence $x = (x_j)$ is said to be (V,λ)-summable to a number L (see [2]) if $T_n(x) \to L$ as $n \to \infty$. In this case L is called the λ-limit of x. And we say that $x = (x_n)$ is λ-statistical convergent to L, if

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

for every given $\epsilon > 0$. And will write $st_\lambda - \lim_n x_n = L$.

In paper of Mursaleen et al. [4], the definition of the statistically $\lambda-$ convergent sequences was given as follows:

A sequence $x = (x_n)$ is said to be statistically $\lambda-$convergent to L if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{n} |\{k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \tag{1.1}$$

holds. In this case we write that $st - \lim_n T_n = L$.

In what follows we will define the following type of the statistical convergence. A sequence $x = (x_n)$ is said to be $(V, \lambda)-$statistically convergent to L if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \tag{1.2}$$

holds. In this case we write that $st_\lambda - \lim_n T_n = L$.

In the sequel we will show conditions under which for every bounded sequence (x_k) the implication

$$st_\lambda - \lim_k x_k = L \quad \text{implies} \quad st_\lambda - \lim_k T_k = L$$

holds.

Theorem 1. *Let us suppose that (x_k) is a bounded sequence such that exists $st_\lambda - \lim_k x_k = L$, then it follows that $st_\lambda - \lim_k T_k = L$, but not conversely.*

Proof. Let us suppose that $st_\lambda - \lim_k x_k = L$. Let $\epsilon > 0$ be any given number, and $B_\epsilon = \{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}$. Then

$$|T_k(x) - L| = \left| \frac{1}{\lambda_k} \sum_{j \in I_k} x_j - L \right| = \left| \frac{1}{\lambda_k} \sum_{j \in I_k} (x_j - L) \right| \leq \left| \frac{1}{\lambda_k} \sum_{j \in B_\epsilon} (x_j - L) \right| \leq$$

$$\frac{1}{\lambda_k} (\sup_j |x_j - L|) \cdot B_\epsilon \to 0,$$

as $k \to \infty$. Hence $T_k(x) \to L$, as $k \to \infty$, respectively $st_\lambda - \lim_k T_k = L$.

Example 1. Let us consider that $\lambda_n = n$ and $x = (x_n)$ defined as follows:

$$x_n = \begin{cases} 1 & \text{if} \quad k \quad \text{is odd} \\ -1 & \text{if} \quad k \quad \text{is even} \end{cases}$$

Of course this sequence is not st_λ- convergent. On the other hand, x is (V, λ)-summable to 0 and hence $(V, \lambda)-$statistically convergent to 0. □

In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for the defined type of convergence. Exactly, we will prove under which conditions $\lambda-$statistical convergence follows from (V,λ)-statistically convergence. This method generalize the method given in [3], as it is shown by the following example.

Example 2. In case where $\lambda_n = n$, then $(V,\lambda)-$ summability method is the Cesaro summability method $(C,1)$ as given in [3].

2. MAIN RESULTS

Theorem 2. *Let (λ_n) be a sequence of real numbers defined as above and*

$$st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} > 1, \quad t > 1 \tag{2.1}$$

where t_n, denotes the integral parts of the $[tn]$ for every $n \in \mathbb{N}$, and let (T_k) be a sequence of real numbers such that $st_\lambda - \lim_k T_k = L$. Then (x_k) is $st_\lambda-$ convergent to the same number L if and only if the following to conditions holds:

$$\inf_{t>1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \leq -\epsilon \right\} \right| = 0 \tag{2.2}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} (x_k - x_j) \leq -\epsilon \right\} \right| = 0. \tag{2.3}$$

Remark 1. Let us suppose that $st_\lambda - \lim_k x_k = L$; $st_\lambda - \lim_k T_k = L$ and relation (2.1) satisfies, then for every $t > 1$, the following relation is valid:

$$st_\lambda - \lim_n \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} x_j = 0 \tag{2.4}$$

and in case where $0 < t < 1$,

$$st_\lambda - \lim_n \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} x_j = 0. \tag{2.5}$$

In what follows we will show some auxiliary lemmas which are going to be used in the sequel.

Lemma 1. *For the sequence of real numbers $\lambda = (\lambda_n)$, condition given by relation (2.1) is equivalent to this one:*

$$st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} > 1, \quad 0 < t < 1. \tag{2.6}$$

Proof. Let us suppose that relation (2.1) is valid, $0 < t < 1$ and $m = t_n = [t \cdot n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{t} > 1 \Rightarrow \frac{m}{t} = \frac{[t \cdot n]}{t} \leq n$$

now by the nondecreasing property of the sequence $\lambda = (\lambda_n)$, we get:

$$\frac{\lambda_n}{\lambda_{t_n}} \geq \frac{\lambda_{[\frac{m}{t}]}}{\lambda_m} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} \geq st_\lambda - \liminf_n \frac{\lambda_{[\frac{m}{t}]}}{\lambda_{t_n}} > 1.$$

Conversely, let use suppose that relation (2.6) is valid. Let $t > 1$ be given and let t_1 be chosen such that $1 < t_1 < t$. Set $m = t_n = [t \cdot n]$. From $0 < \frac{1}{t} < \frac{1}{t_1} < 1$, it follows that:

$$n \leq \frac{tn-1}{t_1} < \frac{[tn]}{t_1} = \frac{m}{t_1},$$

provided $t_1 \leq t - \frac{1}{n}$, which is a case where if n is large enough. Under this conditions we have:

$$\frac{\lambda_{t_n}}{\lambda_n} \geq \frac{\lambda_{t_n}}{\lambda_{\left[\frac{m}{t_1}\right]}} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} \geq st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_{\left[\frac{m}{t_1}\right]}} > 1.$$

□

Lemma 2. *Let us suppose that relation (2.1) is satisfied and let $x = (x_k)$ be a sequence of complex numbers which is $(V,\lambda)-$statistically convergent to L. Then for every $t > 0$,*

$$st_\lambda - \lim_n T_{t_n} = L.$$

Proof. Let us consider that $t > 1$, then from construction of the sequence $\lambda = (\lambda_n)$, we get:

$$\lim_n (n - \lambda_n) = \lim_n (t_n - \lambda_{t_n}), \tag{2.7}$$

and for every $\epsilon > 0$ we have:

$$\{k \in I_{t_n} : |T_{t_k} - L| \geq \epsilon\} \subset \{k \in I_n : |T_k - L| \geq \epsilon\} \cup$$
$$\left\{ k \in I_n : \frac{1}{\lambda_k} \sum_{j=k-\lambda_k+1}^{k} x_j \neq \frac{1}{\lambda_{t_k}} \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j \right\}.$$

Now proof of the lemma in this case follows from relation (2.7) and $st_\lambda - \lim_n T_{t_n} = L$.

(II) In this case we have that $0 < t < 1$. For $t_n = [t \cdot n]$, for any natural number n, we can conclude that (T_{t_n}) does not appears more than $[1 + t^{-1}]$ times in the sequence (T_n). In fact if there exists a integers k, l such that

$$n \leq t \cdot k < t(k+1) < \cdots < t(k+l-1) < n+1 \leq t(k+l),$$

then

$$n+t(l-1)\le t(k+l-1)<n+1\Rightarrow l<1+\frac{1}{t}.$$

And we have this estimation:

$$\frac{1}{\lambda_{t_n}}\left|\left\{k\in I_{t_n}:|T_{t_k}-L|\ge\epsilon\right\}\right|\le\left(1+\frac{1}{t}\right)\frac{1}{\lambda_{t_n}}\left|\{k\in I_n:|T_k-L|\ge\epsilon\}\right|\le$$

$$2(1+t)\frac{1}{\lambda_n}\left|\{k\in I_n:|T_k-L|\ge\epsilon\}\right|.$$

From the last relation it follows that $st_\lambda-\lim_n T_{t_n}=L$. □

Lemma 3. *Let us suppose that relation (2.1) is satisfied and let $x=(x_k)$ be a sequence of complex numbers which is $(V,\lambda)-$statistically convergent to L. Then for every $t>1$,*

$$st_\lambda-\lim_n(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n+1}^{t_n}x_j=L; \tag{2.8}$$

and for every $0<t<1$,

$$st_\lambda-\lim_n(\lambda_n-\lambda_{t_n})^{-1}\sum_{j=t_n+1}^{n}x_j=L. \tag{2.9}$$

Proof. Let us suppose that $t>1$. After some calculations we get:

$$(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n+1}^{t_n}x_j=T_n+\lambda_{t_n}(\lambda_{t_n}-\lambda_n)^{-1}(T_{t_n}-T_n)+$$

$$(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n-\lambda_n+1}^{t_n}x_j-(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=t_n-\lambda_{t_n}+1}^{t_n}x_j,$$

respectively

$$(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n+1}^{t_n}x_j=T_n+\lambda_{t_n}(\lambda_{t_n}-\lambda_n)^{-1}(T_{t_n}-T_n)+$$

$$(\lambda_{t_n}-\lambda_n)^{-1}\left(\sum_{j=n-\lambda_n+1}^{t_n}x_j-\sum_{j=t_n-\lambda_{t_n}+1}^{t_n}x_j\right). \tag{2.10}$$

From definition of the sequence $\lambda=(\lambda_n)$, we obtain

$$st_\lambda-\limsup_n\sum_{j=n-\lambda_n+1}^{t_n}x_j=st_\lambda-\limsup_n\sum_{j=t_n-\lambda_{t_n}+1}^{t_n}x_j. \tag{2.11}$$

Really, let us suppose that $st_\lambda - \lim_n \sup \sum_{j=n-\lambda_n+1}^{t_n} x_j = L$, and for every $\epsilon > 0$, we get:

$$\frac{\left|\left\{k \in I_{t_n} : \left|\sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j - L\right| \geq \epsilon\right\}\right|}{\lambda_{t_n}}$$
$$\leq \frac{\left|\left\{k \in I_n : \left|\sum_{j=k-\lambda_k+1}^{t_k} x_j - L\right| \geq \epsilon\right\}\right|}{\lambda_n}$$
$$+ \frac{\left|\left\{k \in I_n : \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j \neq \sum_{j=k-\lambda_k+1}^{t_k} x_j\right\}\right|}{\lambda_n}.$$

The first summand in the right side of the inequality tends statistically to zero as $n \to \infty$ and second summands tends to zero, too (from relations (2.7)). And this means that $st_\lambda - \lim_n \sup \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j = L$.

Since by (2.1)

$$st_\lambda - \limsup_n \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1} < \infty, st_\lambda - \limsup_n (\lambda_{t_n} - \lambda_n)^{-1} < \infty, \tag{2.12}$$

now relation (2.8) follows from (2.10), (2.11), (2.12), Lemma 2 and statistical convergence of T_n.

Case where $0 < t < 1$. In this case we have:

$$(\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} x_j = T_n + \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) +$$

$$(\lambda_n - \lambda_{t_n})^{-1} \sum_{j=n-\lambda_n+1}^{n} x_j - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=t_n-\lambda_{t_n}+1}^{n} x_j.$$

Following Lemma 2 and the conclusions like in the previous case, we get that relation (2.9) is valid. □

In what follows we will prove Theorem 1.

Proof of Theorem 2. Necessity. Let us suppose that $st_\lambda - \lim_k x_k = L$, and $st_\lambda - \lim_k T_k = L$. For every $t > 1$ following Lemma 2, we get relation (2.2) and in case where $0 < t < 1$, again applying Lemma 2 we obtain relation (2.3).

Sufficient: Assume that $st_\lambda - \lim_n T_n = L$, and conditions (2.1), (2.2) and (2.3) are satisfied. In what follows we will prove that $st_\lambda - \lim_n x_n = L$. Or equivalently, $st_\lambda - \lim_n (T_n - x_n) = 0$.

First we consider the case where $t > 1$. We will start from this estimation:

$$x_n - T_n = \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} (x_j - x_n).$$

For any $\epsilon > 0$, we obtain:

$$\{k \in I_n : x_n - T_n \geq \epsilon\} \subset \left\{k \in I_n : \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) \geq \frac{\epsilon}{2}\right\} \cup$$
$$\left\{k \in I_n : (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} (x_j - x_n) \leq -\frac{\epsilon}{2}\right\}.$$

From relation (2.2), it follows that for every $\gamma > 0$, exists a $t > 1$ such that

$$\limsup_n \frac{1}{\lambda_n} \left| \left\{k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \leq -\epsilon \right\} \right| \leq \gamma.$$

By Lemma 2 and relation (2.12) we get:

$$\limsup_n \frac{1}{\lambda_n} \left| \left\{k \in I_n : |\lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n)| \geq \frac{\epsilon}{2}\right\} \right| = 0.$$

Combining last three relations we have:

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \geq \epsilon\}| \leq \gamma,$$

and γ is arbitrary, we conclude that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \geq \epsilon\}| = 0. \tag{2.13}$$

Now we consider case where $0 < t < 1$. From above we get that:

$$x_n - T_n = \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) + (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} (x_n - x_j).$$

For any $\epsilon > 0$,

$$\{k \in I_n : x_n - T_n \leq -\epsilon\} \subset \left\{k \in I_n : \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) \leq -\frac{\epsilon}{2}\right\} \cup$$
$$\left\{k \in I_n : (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} (x_k - x_j) \leq -\frac{\epsilon}{2}\right\}.$$

For same reasons as in the case where $t > 1$, by Lemma 2, we have that for every $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \leq -\epsilon\}| = 0. \tag{2.14}$$

Finally from relations (2.13) and (2.14) we get:

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_n - T_n| \geq \epsilon\}| = 0.$$

□

In the next result we will consider the case where $x = (x_n)$ is a sequence of complex numbers.

Theorem 3. *Let (λ_n) be a sequence of complex numbers defined above,which satisfied relation (2.1) and let us consider that $st_\lambda - \lim T_k = L$. Then (x_k) is $st_\lambda-$ statistically convergent to the same number L if and only if the following to conditions holds: for every $\epsilon > 0$,*

$$\inf_{t>1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \geq \epsilon \right\} \right| = 0 \tag{2.15}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} (x_k - x_j) \geq \epsilon \right\} \right| = 0. \tag{2.16}$$

Remark 2. Let us suppose that $st_\lambda - \lim_k x_k = L$, $st_\lambda - \lim_k T_k = L$ and relation (2.1) satisfies. Then for every $t > 1$, relation (2.15) holds, and in case where $0 < t < 1$, relation (2.16) is valid.

Proof of Theorem 3. We omit it, because it is similar to that of Theorem 1. □

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