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# FIXED POINT THEOREMS IN BIMETRIC SPACE ENDOWED WITH BINARY RELATION AND APPLICATIONS

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*Abstract.* In this paper, we obtained some fixed point results for continuous mappings satisfying a generalized contractive condition in the setting of two metrics space endowed with a binary relation. Our theorems generalize and extend several known results in the literature. As application, we establish an existence theorem for the solution of a nonlinear first order differential equation.

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## 1. INTRODUCTION

The Banach contraction mapping is one of the pivotal results of analysis. It is very popular tool for solving existence and uniqueness problems in many different fields of mathematics. Due to its importance and applications potential, the Banach Contraction Mapping Principle has been investigated heavily by many authors. Consequently, a number of generalizations of this celebrated principle have appeared in the literature (see [1–19]).

Recently, Samet and Turinici [17] introduced the notion of contractive mapping in a metric space endowed with amorphous binary relation. They showed a theorem subsumes many known results in the literature. For further study about contractive mappings in a metric space endowed with binary relation, we refer the reader to [3–7], [13], [17] and [19].

The purpose of this paper is to establish some fixed point results in the setting of two metrics space; called bimetric space; endowed with a binary relation. Our results generalized and extended many existing fixed point theorems, for generalized contractive and quasi-contractive mappings, in a metric space endowed with binary relation. Also, an application to the study the existence of solution to a nonlinear first order differential equation has been given.

## 2. PRELIMINARIES

We first recall Maia's fixed point theorem:

**Theorem 1** ([14])**.** *Let $(X,d,\delta)$ be a bimetric space. Assume that for $T : X \to X$, the following conditions are satisfied:*

(i) *$d(x,y) \leq \delta(x,y)$ for all $x,y \in X$;*
(ii) *$X$ is complete with respect to $d$;*
(iii) *$T$ is continuous with respect to $d$;*
(iv) *there exists a constant $\alpha \in [0,1)$ such that*

$$\delta(Tx,Ty) \leq \alpha\delta(x,y).$$

*Then $T$ has a unique fixed point in $X$.*

In the sequel, let $(X,d,\delta)$ be a bimetric space, and $T : X \to X$ be a mapping. Denote

$$Fix(T) := \{x^* \in X : x^* = Tx^*\}.$$

Let $\mathcal{R}$ be a binary relation on $X$, and let $\mathcal{S}$ be the *symmetric binary relation* defined by

$$x,y \in X, \quad x\mathcal{S}y \Longleftrightarrow x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

From $x_0 \in X$, we define the sequence $\{x_n\}$ by

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

**Definition 1.** Let $(X,\delta)$ be metric space and $n \in \mathbb{N} \cup \{0\}$. For $A \subset X$ we denote by $D(A) := \sup\{\delta(a,b) : a,b \in A\}$. For each $x_0 \in X$, the orbit sets of $T$ at $x_0$ are defined as following

$$O_n(x_0) = \{x_0, x_1, \ldots, x_n\} \quad \text{and} \quad O_\infty(x_0) = \{x_0, x_1, x_2, \ldots\}.$$

We say that $(X,\delta)$ is *$T$-orbitally complete* iff every $\delta$-Cauchy sequence from $O_\infty(x)$ for some $x \in X$ converges in $X$.

**Definition 2** ([17])**.** A subset $D$ of $X$ is called $\mathcal{R}$-directed if for every $x,y \in D$, there exists $z \in X$ such that $x\mathcal{R}z$ and $y\mathcal{R}z$.

**Definition 3.** A mapping $T : X \to X$ is called $\mathcal{R}$-preserving mapping if

$$x,y \in X, \quad x\mathcal{R}y \Longrightarrow Tx\mathcal{R}Ty.$$

Next, we define the set $\Phi$ of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying:

(I) $\varphi$ is nondecreasing;
(II) $\displaystyle\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$, where $\varphi^n$ is the $n$-th iterate of $\varphi$.

*Remark* 1. Let $\varphi \in \Phi$. We have $\varphi(t) < t$ for all $t > 0$.

*Remark* 2. Let $\varphi \in \Phi$. We have $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

**Definition 4.** Assume that for $T : X \to X$, there exists $\varphi \in \Phi$ such that

$$\delta(Tx,Ty) \le \varphi(M_{\mathcal{S}}(x,y)), \qquad \text{for all } x,y \in X \text{ with } x\mathcal{S}y.$$

A mapping $T$ is called a *generalized contractive* with respect to $\delta$, if

$$M_{\mathcal{S}}(x,y) = \max\{\delta(x,y), \frac{1}{2}[\delta(x,Tx)+\delta(y,Ty)], \frac{1}{2}[\delta(x,Ty)+\delta(y,Tx)]\}).$$

A mapping $T$ is called a *generalized quasi-contractive* with respect to $\delta$, if

$$M_{\mathcal{S}}(x,y) = \max\{\delta(x,y),\delta(x,Tx),\delta(y,Ty),\delta(x,Ty),\delta(y,Tx)\}).$$

## 3. Main results

To begin with, we have the following result.

**Theorem 2.** *Let $(X,d,\delta)$ be a bimetric space, and $\mathcal{S}$ be a binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(A1) *$d(x,y) \le \delta(x,y)$ for all $x,y \in X$;*
(A2) *$(X,d)$ is $T$-orbitally complete;*
(A3) *$T$ is continuous with respect to $d$;*
(A4) *$T$ is $\mathcal{S}$-preserving;*
(A5) *there exists $x_0 \in X$ with $x_0\mathcal{S}Tx_0$;*
(A6) *$T$ is a generalized contractive with respect to $\delta$.*

*Then $T$ has a fixed point $x^*$ in $X$. Moreover, if in addition, $Fix(T)$ is $\mathcal{S}$-directed, then $x^*$ is the unique fixed point of $T$ in $X$.*

*Proof.* From (A5), there exists $x_0 \in X$ with $x_0\mathcal{S}Tx_0$, and from (A4), $T$ is $\mathcal{S}$-preserving, we get

$$x_n\mathcal{S}Tx_n \text{ for all } n \ge 0. \tag{3.1}$$

If $x_n = Tx_n$, then $x_n$ is a fixed point of $T$. Suppose that $x_n \ne Tx_n$ for all $n$. Since (3.1) is satisfied for all $n \ge 1$, by applying the contraction condition (A6), we have

$$\delta(x_n,Tx_n) \le \varphi(\max\Big\{\delta(x_{n-1},x_n), \frac{1}{2}[\delta(x_{n-1},x_n)+\delta(x_n,x_{n+1})], \frac{1}{2}[\delta(x_{n-1},x_{n+1})+\delta(x_n,x_n)]\Big\}))$$

Hence, we get

$$\delta(x_n,x_{n+1}) \le \varphi(\max\{\delta(x_{n-1},x_n),\delta(x_n,x_{n+1})\}).$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence in $(X,\delta)$. If for some $n \ge 1$, we have $\delta(x_{n-1},x_n) \le \delta(x_n,x_{n+1})$, then from Remark 1, we get

$$\delta(x_n,x_{n+1}) \le \varphi(\delta(x_n,x_{n+1})) < \delta(x_n,x_{n+1}),$$

which is a contradiction. Thus, $\delta(x_{n-1},x_n) > \delta(x_n,x_{n+1})$, and

$$\delta(x_n,x_{n+1}) \le \varphi(\delta(x_{n-1},x_n)), \qquad \text{for all } n \ge 1.$$

Hence, by induction, we obtain

$$\delta(x_n, x_{n+1}) \leq \varphi^n(\delta(x_0, x_1)), \quad \text{for all } n \geq 1. \tag{3.2}$$

Fix $\varepsilon > 0$ and let $n_\varepsilon$ be a positive integer such that

$$\sum_{n \geq n_\varepsilon} \varphi^n(\delta(x_1, x_0)) < \varepsilon.$$

Using the triangular inequality and (3.2), for some $m > n > n_\varepsilon$, we get

$$\delta(x_n, x_m) \leq \sum_{k=n}^{m-1} \delta(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi^k(\delta(x_1, x_0)) \leq \sum_{n \geq n_\varepsilon} \varphi^n(\delta(x_1, x_0)) < \varepsilon.$$

Consequently, the sequence $\{x_n\}$ is $\delta$-Cauchy, so by (A1), $\{x_n\}$ is $d$-Cauchy too. Since from (A2), we have metric space $(X, d)$ is $T$-orbitally complete, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{3.3}$$

From (A3), we have that $T$ is continuous with respect to $d$, and, so it follows that $x^* = \lim_{n\to\infty} T x_n = T(\lim_{n\to\infty} x_n) = T x^*$, that is, $x^*$ is a fixed point of $T$.

Next, suppose that $Fix(T)$ is $\mathcal{S}$-directed, and we will show that $x^*$ is the unique fixed point of $T$ in $X$. Suppose that $y^* \in Fix(T)$ is another fixed point of $T$. Then, there exists $z_0 \in X$ such that $z_0 \mathcal{S} x^*$ and $z_0 \S y^*$. Define the sequence $\{z_n\}$ in $X$ by $z_{n+1} = T z_n$ for all $n \geq 0$. Since $T$ is $\mathcal{S}$-preserving, for all $n \geq 0$, we have $z_n \mathcal{S} x^*$ and $z_n \mathcal{S} y^*$. Applying (A6), for all $n \geq 0$, we get

$$\begin{aligned}
\delta(z_{n+1}, x^*) &= \delta(T z_n, T x^*) \\
&\leq \varphi(M(z_n, x^*)) \\
&= \varphi(\max\{\delta(z_n, x^*), \tfrac{1}{2}[\delta(z_n, z_{n+1}) + \delta(x^*, x^*)], \tfrac{1}{2}[\delta(z_n, x^*) + \delta(x^*, z_{n+1})]\}) \\
&\leq \varphi(\max\{\delta(z_n, x^*), \tfrac{1}{2}[\delta(z_n, x^*) + \delta(x^*, z_{n+1})], \tfrac{1}{2}[\delta(z_n, x^*) + \delta(x^*, z_{n+1})]\}) \\
&\leq \varphi(\max\{\delta(z_n, x^*), \delta(x^*, z_{n+1})\}).
\end{aligned}$$

Now, we will show that $\lim_{n\to\infty} \delta(z_n, x^*) = 0$. Without the loss of generality, suppose that $\delta(z_n, x^*) > 0$ for all $n$. Assume that for some $n$, we have $\delta(z_n, x^*) \leq \delta(x^*, z_{n+1})$. Hence,

$$\delta(z_{n+1}, x^*) \leq \varphi(\delta(x^*, z_{n+1})) < \delta(x^*, z_{n+1}),$$

which is a contradiction. Then, for all $n \geq 0$, we have $\delta(z_n, x^*) > \delta(x^*, z_{n+1})$. Consequently, for all $n$, we obtain

$$\delta(z_{n+1}, x^*) \leq \varphi(\delta(x^*, z_n)).$$

By induction, for all $n$, we get

$$\delta(z_n, x^*) \le \varphi^n(\delta(x^*, z_0)).$$

Which implies that $\lim_{n\to\infty} \delta(z_n, x^*) = 0$. Similarly, we can prove that

$$\lim_{n\to\infty} \delta(z_n, y^*) = 0.$$

Hence, $x^* = y^*$. □

To prove our next result, we need the following lemmas which will be used in the sequel.

**Lemma 1.** *Let $n \in \mathbb{N}$, $(X,\delta)$ be a metric space, and $\mathcal{R}$ a transitive binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(a1) *there exists $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$;*
(a2) *$T$ is $\mathcal{R}$-preserving mapping;*
(a3) *$T$ is generalized quasi-contractive with respect to $\delta$.*

*Then, $\delta(x_i, x_j) \le \varphi(D(O_n(x_0)))$ for all $i, j \in \{1, \dots, n\}$.*

*Proof.* From (a1) there exists $x_0 \in X$ such that $x_0 \mathcal{R} x_1$. Hence, by (a2) we get $x_k \mathcal{R} x_{k+1}$ for all $k$. Since, $\mathcal{R}$ is transitive, then

$$x_{i-1} \mathcal{R} x_{j-1} \text{ for all } 1 \le i < j \le n. \tag{3.4}$$

Without loss of generality, we can suppose that $1 \le i < j \le n$ for which we have $x_{i-1}, x_i, x_{j-1}, x_j \in O_n(x_0)$. Now, using (a3) and (3.4), we get

$$\delta(Tx_{i-1}, Tx_{j-1}) \le \varphi\big(\max\{\delta(x_{i-1}, x_{j-1}), \delta(x_{i-1}, x_i), \delta(x_{j-1}, x_j), \delta(x_{i-1}, x_j), \delta(x_i, x_{j-1})\}\big),$$

which implies that

$$\delta(x_i, x_j) \le \varphi(D(O_n(x_0))).$$ □

*Remark* 3. From the previous lemma and Remark 1, we have

$$\delta(x_i, x_j) \le \varphi(D(O_n(x_0))) < D(O_n(x_0)) \text{ for all } 1 \le i < j \le n,$$

which implies the existence of an integer $k \le n$ such that $\delta(x_0, T^k x_0) = D(O_n(x_0))$.

**Lemma 2.** *Let $(X,\delta)$ be a metric space, and $\mathcal{R}$ a transitive binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(b1) *there exists $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$;*
(b2) *$T$ is $\mathcal{R}$-preserving mapping;*
(b3) *$T$ is generalized quasi-contractive with respect to $\delta$.*

*Then,*

$$D(O_\infty(x_0)) \le \sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)).$$

*Proof.* At first, we note that we have

$$D(O_1(x_0)) \le D(O_2(x_0)) \le \dots,$$

which implies that $D(O_\infty(x)) = \sup\{D(O_n(x)) : n \in \mathbb{N}\}$. By (b1), there exists $x_0 \in X$ such that $x_0 \mathcal{R} x_1$, then (3.4) holds. Let $n \in \mathbb{N}$ so by Lemma 1, there exists $T^k x_0 \in O_n(x_0)$ for $1 \le k \le n$, such that $\delta(x_0, T^k x_0) = D(O_n(x_0))$. Using the triangular inequality, (3.4) and (b3), we get

$$\begin{aligned} D(O_n(x_0)) = \delta(x_0, T^k x_0) &\le \delta(x_0, Tx_0) + \delta(Tx_0, T^k x_0) \\ &\le \delta(x_0, Tx_0) + \varphi(D(O_n(x_0))), \end{aligned}$$

that is,

$$(id - \varphi)(D(O_n(x_0))) \le \delta(x_0, Tx_0)$$

where $id : [0, +\infty) \to [0, +\infty)$ is the identity function. It follows from the monotony of $\varphi$ that $\sum_{\ell=0}^{k} \varphi^\ell$ is monotone too for all $k \in \mathbb{N}$. Thus, by applying $\sum_{\ell=0}^{k} \varphi^\ell$ to both side of the previous inequality, we get

$$\sum_{\ell=0}^{k} \varphi^\ell((id - \varphi)(D(O_n(x_0)))) \le \sum_{\ell=0}^{k} \varphi^\ell(\delta(x_0, Tx_0))$$

which implies

$$D(O_n(x_0)) - \varphi^{k+1}(D(O_n(x_0))) \le \sum_{\ell=0}^{k} \varphi^\ell(\delta(x_0, Tx_0))$$

Since $\sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)) < \infty$, then by letting $k \to \infty$ in the above inequality, we get

$$D(O_n(x_0)) \le \sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)), \quad \text{for all } n \ge 1.$$

Hence, we get

$$D(O_\infty(x_0)) \le \sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)).$$ □

Now, we are ready to state our second main result.

**Theorem 3.** *Let $(X, d, \delta)$ be a bimetric space, and $\mathcal{R}$ a transitive binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(B1) *$d(x, y) \le \delta(x, y)$ for all $x, y \in X$;*
(B2) *$(X, d)$ is $T$-orbitally complete;*
(B3) *$T$ is continuous with respect to $d$;*
(B4) *$T$ is $\mathcal{R}$-preserving;*

(B5) *there exists* $x_0 \in X$ *with* $x_0 \mathcal{R} T x_0$;

(B6) $T$ *is a generalized quasi-contractive with respect to* $\delta$.

*Then* $T$ *has a fixed point* $x^*$ *in* $X$. *Moreover, if in addition,* $\mathcal{R}$ *is symmetric and* $Fix(T)$ *is* $\mathcal{R}$*-directed, then* $x^*$ *is the unique fixed point of* $T$ *in* $X$.

*Proof.* Let $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$. Define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$, for all $n \geq 0$. Since $T$ is an $\mathcal{R}$-preserving, then $x_n \mathcal{R} x_{n+1}$ for all $n$. Let $n$ and $m$ $(n < m)$ be any positive integers. From (B6) and Lemma 1 it follows

$$\delta(T^n x_0, T^m x_0) = \delta(T T^{n-1} x_0, T^{m-n+1} T^{n-1} x_0) \leq \varphi(D(O_{m-n+1}(T^{n-1} x_0))).$$

From Remark 1, there exists an integer $k_1$, $1 \leq k_1 \leq m-n+1$, such that

$$D(O_{m-n+1}(T^{n-1} x_0)) = \delta(T^{n-1} x, T^k T^{n-1} x_0).$$

Using Lemma 1 again, we get

$$\begin{aligned} \delta(T^{n-1} x_0, T^k T^{n-1} x_0) &= \delta(T T^{n-2} x_0, T^{k_1+1} T^{n-2} x_0) \\ &\leq \varphi(D(O_{k_1+1}(T^{n-2} x_0))) \\ &\leq \varphi(D(O_{m-n+2}(T^{n-2} x_0))) \end{aligned}$$

Combining, the above inequalities, we get

$$\delta(T^n x_0, T^m x_0) \leq \varphi(D(O_{m-n+1}(T^{n-1} x_0))) \leq \varphi^2(D(O_{m-n+2}(T^{n-2} x_0))).$$

Continue this process, we obtain

$$\delta(T^n x_0, T^m x_0) \leq \varphi(D(O_{m-n+1}(T^{n-1} x_0))) \leq \cdots \leq \varphi^n(D(O_m(x_0))).$$

By Lemma 2, we get

$$\delta(T^n x_0, T^m x_0) \leq \varphi^n(\sum_{\ell=0}^{\infty} \varphi^{\ell}(\delta(x_0, T x_0))).$$

Since $\lim_{n\to\infty} \varphi^n(s) = 0$ for all $s \geq 0$, it follows that the sequence $\{T^n x_0\}$ is a $\delta$-Cauchy sequence. Therefore, by (B1) the sequence $\{T^n x_0\}$ is a $d$-Cauchy sequence too. Now, since the metric space $(X, d)$ is $T$-orbitally complete, we deduce that the sequence $\{T^n x_0\}$ converges to some $x^*$ in $X$. From (B3), we have that $T$ is continuous with respect to $d$, and, so it follows that $x^* = \lim_{n\to\infty} T x_n = T(\lim_{n\to\infty} x_n) = T x^*$, that is, $x^*$ is a fixed point of $T$.

Now, suppose that $Fix(T)$ is $\mathcal{R}$-directed, and we claim that the fixed point is unique. Let $x^*$ and $y^*$ two fixed points of $T$. Suppose that $x^* \neq y^*$. Since $Fix(T)$ is $\mathcal{R}$-directed, then there exists $z \in X$ such that $z \mathcal{R} x^*$ and $z \mathcal{R} y^*$, which implies by the transitivity of $\mathcal{R}$, that $x^* \mathcal{R} y^*$. Then, we apply the contraction condition (B6), we get

$$\begin{aligned} &\delta(x^*, y^*) = \delta(T x^*, T y^*) \leq \\ &\varphi(\max\{\delta(x^*, y^*), \delta(x^*, T x^*), \delta(y^*, T y^*), \delta(x^*, T y^*), \delta(y^*, T x^*)\}) \end{aligned}$$

which implies, by Remark 1, that

$$\delta(x^*, y^*) \leq \varphi(\delta(x^*, y^*)) < \delta(x^*, y^*). \tag{3.5}$$

which is a contradiction. Consequently, $x^* = y^*$, and our claim holds. □

The following results are an immediate consequences of Theorems 2 and 3.

**Corollary 1.** *Theorem 1 is a particular case of Theorem 2.*

**Corollary 2.** *Let $(X, \leq)$ be a partially ordered set and $(X, d, \delta)$ be a bimetric space. Assume that for $T : X \to X$, the following conditions are satisfied:*

(C1) *$d(x, y) \leq \delta(x, y)$ for all $x, y \in X$;*
(C2) *$X$ is complete with respect to $d$;*
(C3) *$T$ is continuous with respect to $d$;*
(C4) *$T$ is monotone nondecreasing mapping;*
(C5) *there exists $x_0 \in X$ with $x_0 \leq Tx_0$;*
(C6) *there exists $\varphi \in \Phi$ such that*

$$\delta(Tx, Ty) \leq \varphi(M_\delta(x, y)) \text{ for all } x, y \in X \text{ with } x \leq y.$$

*Then $T$ has a unique fixed point in $X$.*

**Corollary 3.** *Let $(X, \leq)$ be a partially ordered set and $(X, d)$ be a complete metric space. Assume that for $T : X \to X$, the following conditions are satisfied:*

(D1) *$T$ is continuous;*
(D2) *$T$ is monotone nondecreasing mapping;*
(D3) *there exists $x_0 \in X$ with $x_0 \leq Tx_0$;*
(D4) *there exists a constant $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha M_d(x, y), \quad \text{for all } x, y \in X \text{ with } x \leq y.$$

*Then $T$ has a unique fixed point in $X$.*

**Corollary 4.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be continuous mapping. Suppose there exists $\varphi \in \Phi$ such that*

$$d(Tx, Ty) \leq \varphi\left(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}\right), \ \forall x, y \in X.$$

*Then $T$ has a unique fixed point in $X$.*

## 4. Application to Cauchy problem

In this section, we study the Cauchy problem for a class of nonlinear differential equations, using the results obtained in the previous section.

**Theorem 4.** *Let $(X, \leq)$ is a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a* *nondecreasing and continuous mappings with respect to $d$ and consider the following power series*

$$\sum_{n=0}^{\infty} \lambda^n d(T^n x, T^n y). \tag{4.1}$$

*Suppose there exists $\lambda \in (0,1)$, for which the power series (4.1) converges for all $x, y \in X$. For such $\lambda$ we define a function $\delta$ by*

$$\delta(x, y) = \sum_{n=0}^{\infty} \lambda^n d(T^n x, T^n y). \tag{4.2}$$

*Suppose that also that*

$$\delta(x, y) \leq \frac{1}{1-\lambda^2} d(x, y) \quad \textit{for all } x \leq y. \tag{4.3}$$

*Then,*

(i) *$\delta$ is a metric satisfies $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$;*
(ii) *$T$ has a unique fixed point in $X$.*

*Proof.* The function $\delta$ is a metric. Indeed, If $x = y$, then $\delta(x, x) = 0$. On the other hand, if $\delta(x, y) = 0$, so $\sum_{n=0}^{\infty} \lambda^n d(T^n, x, T^n y) = 0$ which implies that $x = y$ since $d(T^n, x, T^n y) \geq 0$ for all $n$.
Next, it is easy to prove that $\delta(x, y) = \delta(y, x)$ and that $\delta$ satisfies also the triangle inequality.

Now, we shall prove (ii). From hypothesis, there exists some $\lambda \in (0,1)$ such that

$$\begin{aligned} \delta(x, y) &= \sum_{n=0}^{\infty} \lambda^n d(T^n x, T^n y) \\ &= d(x, y) + \lambda \sum_{0}^{\infty} \lambda^n d(T^{n+1} x, T^{n+1} y) \\ &= d(x, y) + \lambda \delta(T x, T y) \end{aligned} \tag{4.4}$$

On the other hand, from condition (4.3) we have

$$\delta(x, y) \leq \frac{1}{1-\lambda^2} d(x, y) \quad \text{for all } x \leq y.$$

Therefore, from (4.4) we get

$$\delta(T x, T y) \leq \lambda \delta(x, y) \quad \text{for all } \ x \leq y.$$

All conditions of Theorem 2 are satisfied, then we conclude (ii). □

**Theorem 5.** *Let $(X,\leq)$ is a partially ordered set and suppose that there exist a metric $d$ on $X$ such that $(X,d)$ is a complete metric space. Let $T:X\to X$ be a self mappings. Suppose that the condition (4.3) holds, further suppose that there exists a constant $L\geq 1$ and $k\geq 1$ such that*

$$d(T^k x,T^k y)\leq Ld(x,y) \quad \textit{for all } x\leq y. \tag{4.5}$$

*Then the power series*

$$S=\sum_{n=0}^{\infty}\lambda^n d(T^n x,T^n y), \tag{4.6}$$

*converges in $X$ for all $x,y\in X$ with $x\leq y$.*

*Proof.* From hypothesis, there exists some $L\geq 1$ and $k\leq 1$ such that:

$$d(T^k x,T^k y)\leq Ld(x,y) \quad \text{for all } x\leq y.$$

By induction we get

$$d(T^{nk}x,T^{nk}y)\leq L^n d(x,y) \quad \text{for all } n\geq 1 \text{ and } x\leq y. \tag{4.7}$$

Hence, we get

$$\begin{aligned}
S&=\sum_{n=0}^{\infty}\lambda^{nk}d(T^{nk}x,T^{nk}y)+\lambda^{nk+1}d(T^{nk+1}x,T^{nk+1}y)+\dots\\
&\qquad+\lambda^{nk+(k-1)}d(T^{nk+(k-1)}x,T^{nk+(k-1)}y)\\
&\leq\sum_{n=0}^{\infty}\lambda^{nk}L^n d(x,y)+\lambda^{nk+1}L^{n+1}d(Tx,Ty)+\dots\\
&\qquad+\lambda^{nk+(k-1)}L^{n+(k-1)}d(T^{(k-1)}x,T^{(k-1)}y)\\
&\leq\sum_{p=0}^{k-1}\lambda^p L^{2p}\sum_{n=0}^{\infty}(\lambda^k L)^n d(x,y)
\end{aligned}$$

for all $x\leq y$. We can choose $0<\lambda^k<\frac{1}{L}$ to obtain the convergence of the power series (4.6) for all $x,y\in X$ with $x\leq y$. □

Now, consider the nonlinear differential equation

$$\begin{cases} x'(t) &= f(t,x(t)), \quad t\in[a,b]\\ x(t_0) &= x_0, \end{cases} \tag{4.8}$$

where $a,b,t_0\in\mathbb{R}$ and $f:[a,b]\times\mathbb{R}\to\mathbb{R}$. Let $X=C([a,b],\mathbb{R})$ denotes the space of all continuous $\mathbb{R}$-valued functions on $[a,b]$. We endow this space with the metric $d$ given by

$$d(u,v)=\sup_{t\in[a,b]}|u(t)-v(t)|, \quad \text{for all } u,v\in X.$$

It is well known that $(X,d)$ is a complete metric space. We define an order relation $\leq$ on $X$ by

$$u,v \in X, \quad u \leq v \Longleftrightarrow u(t) \leq v(t), \text{ for all } t \in [a,b].$$

Consider the mapping $T : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(s,x(s)))ds; \quad t \in [a,b],$$

for all $x \in C([a,b],\mathbb{R})$. Clearly, $x^* \in C([a,b],\mathbb{R})$ is a solution of (4.8) if and only if $x^*$ is a fixed point of $T$.
Furthermore, we consider the following assumptions:

(H1) $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is continuous;
(H2) $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is nondecreasing with respect to the second variable;
(H3) $|f(t,x(t)) - f(t,y(t))| \leq L|x(t) - y(t)|$ for all $x(t) \leq y(t)$ and $t \in [a,b]$.

It is worth noting that condition (H3) is weaker compared to those used by Maia for studying Cauchy problem in [14], that is, $f$ is $L$-Lipschitzien function on the whole space.

The next result shows that $T$ satisfies the conditions of Corollary 2.
Let $d_r(x,y) = \sup_{a\leq t\leq r}|x(t)-y(t)|$ where $r \in [a,b]$. From condition (H3) it follows

$$d_r(T^n x, T^n y) \leq \frac{L^n(b-a)^n}{n!} d_r(x,y) \quad \text{for all } x \leq y, \quad n \geq 0. \tag{4.9}$$

Indeed, obviously (4.9) holds for $n = 0$. Suppose that (4.9) holds for some integer $n$, then using (H2) and (H3), we get

$$\begin{aligned} d_r(T^{n+1}x, T^{n+1}y) &= \sup_{a\leq t\leq r}\left|\int_{t_0}^{t}(f(s,T^n x(s)) - f(s,T^n y(s)))\,ds\right| \\ &\leq \int_a^r L d_r(T^n x(s), T^n y(s))\,ds \\ &\leq \frac{L^{n+1}}{n!}\int_a^r (s-a)^n\,ds\; d_r(x,y) = \frac{L^{n+1}(r-a)^{n+1}}{(n+1)!} d_r(x,y) \end{aligned}$$

This implies that (4.9) holds. Next, by taking $r = b$ in (4.9), we obtain

$$d(T^n x, T^n y) \leq \frac{L^n(b-a)^n}{n!} d(x,y) \quad \text{for all } x \leq y \text{ and } n \geq 0. \tag{4.10}$$

**Theorem 6.** *Suppose that (H1)–(H3) hold. Then (4.8) has at least one solution* $x^* \in C([a,b],\mathbb{R})$.

*Proof.* We shall show the existence of $\lambda \in (0,1)$ such that (4.3) holds. First, we have

$$d(T^n x, T^n y) \leq \frac{L^n(b-a)^n}{n!} d(x,y) \quad \text{for all } x \leq y, \quad n \geq 0.$$

Thus, we get

$$\delta(x,y) \leq \exp(L(b-a))d(x,y) \quad \text{for all } x \leq y.$$

If we take $\lambda = \sqrt{1-\exp(L(a-b))}$, then the condition (4.3) holds.
Since, we have

$$d(T^n x, T^n y) \leq \exp(L(b-a))d(x,y) \quad \text{for all } x \leq y, \quad n \geq 0,$$

then the condition (4.5) holds too. We deduce by using Corollary 2 that $T$ has a unique fixed point $x^* \in C([a,b],\mathbb{R})$. □

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