GENERALIZED TERRACED MATRICES

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Abstract. We know that every terraced matrix has the factorization $R_b = D_b C$, where $C$ is the Cesàro matrix and $D_b = \text{diag}\{ (n+1)b_n \}$. In the present paper, we define the generalized terraced matrix by using the generalized Cesàro matrix in the expression above, and some properties of this matrix are given.

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1. INTRODUCTION

In [13], A.G. Siskakis gives the spectrum of the Cesàro matrix on $H^p$ by using the integral representation of the Cesàro operator.

Let $H(\mathbb{D})$ denotes the space of complex valued analytic functions on the unit disk $\mathbb{D}$, for $1 \leq p < \infty$, $H^p$ denotes the standard Hardy space on $\mathbb{D}$, and $\ell^p$ denotes the standard space of $p$-summable complex-valued sequences on the set of non-negative integers.

Suppose that $1 < p < \infty$ and $(b) = \{b_n\}_{n=0}^\infty$ is in $\ell^p$. Then the sequences

$$C(b) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} b_k \right\}^\infty_{n=0}$$

have $\ell^p$-norms satisfying

$$\|C(b)\|_p \leq \frac{p}{p-1} \|(b)\|_p$$

and the constant in this inequality is the best possible [4, 6, 7, 10]. Thus $C$ is a bounded linear operator on $\ell^p$ for $1 < p < \infty$ with its norm equal to $p/(p-1)$.

If $f(z) = \sum_{k=0}^\infty b_k z^k$ is in $H^p$, let

$$C(f)(z) = \sum_{n=0}^\infty \left( \frac{1}{n+1} \sum_{k=0}^{n} b_k \right) z^n.$$
By computing Taylor series, we see that $C$ has the following integral representation: for $f \in H^p$,

$$C(f)(z) = \frac{1}{z} \int_0^z \frac{f(t)}{1-t} dt$$  \hspace{1cm} (1.1)$$

In [19], Scott W. Young generalized Cesàro operator, by considering more general analytic functions instead of the function $1/(1-t)$ in equality (1.1), as follows.

**Definition 1.** Let $g$ be analytic on the unit disk. The operator $C_g : H^2 \rightarrow H^2$ defined by

$$C_g(f) := \frac{1}{z} \int_0^z f(t) g(t) dt$$  \hspace{1cm} (1.2)$$

is called the generalized Cesàro operator with symbol $g$.

**Definition 2.** Let $I$ be an arc of the unit circle $\mathbb{T}$, and let $\varphi : \mathbb{T} \rightarrow \mathbb{C}$. Then, let $\varphi_I = \frac{1}{|I|} \int_I |\varphi|$, where $|I|$ denotes the arclength of $I$. $\varphi$ is said to be of bounded mean oscillation if

$$\|\varphi\|_* = \sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |\varphi - \varphi_I| < \infty.$$  

We denote the set of all functions of bounded mean oscillation by $BMO$. If we endow $BMO$ with the norm $\|\varphi\|_{BMO} = \|\varphi\|_* + |\varphi(0)|$, then $BMO$ is a Banach space (see [5]).

We say that $g \in BMOA$ if $g \in H^2$ and $g(e^{i\theta}) \in BMO$.

**Definition 3.** Let $I$ be an arc of $\mathbb{T}$. We say that a function $\varphi : \mathbb{T} \rightarrow \mathbb{C}$ is of vanishing mean oscillation if

$$\lim_{\delta \to 0} \sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |\varphi - \varphi_I| = 0.$$  

We denote the set of all functions of vanishing mean oscillation by $VMO$. $VMO$ is a closed subspace of $BMO$.

As with $BMOA$, we define $VMOA$ as the set of $g \in H^2$ such that $g(e^{i\theta}) \in VMO$. $VMOA$ is a closed subspace of $BMOA$ (see [5]).

**Definition 4.** A vector $x$ is a cyclic vector for a bounded operator $T$ on a Hilbert space $H$ if the set $\{p(T)x : p$ is polynomial$\}$ is dense in $H$. If $T$ has a cyclic vector, then $T$ is called a cyclic operator.

We denote the spectrum of the linear operator $T$ by $\sigma(T)$. That is,

$$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ not invertible} \}.$$  

Let $G(z) = \int_0^z g(w) dw$. Pommerenke [12] showed that $C_g$ is bounded on the Hilbert space $H^2$ if and only if $G \in BMOA$. Aleman and Siskakis [2] extended
Pommerenke’s result to the Hardy spaces $H^p$ for all $p$, $1 \leq p < \infty$, and showed that $C_g$ is compact on $H^p$ if and only if $G \in VMOA$.

Continuity of the Cesàro operator $C$ on the Hilbert space $H^2(\mathbb{D})$ is due to Hardy, Littlewood and Polya [7], and to Siskakis for the general Hardy and the unweighted Bergman space cases, [13,14,16]. In [15], Siskakis considered a class of generalized Cesàro operators associated with semigroups of weighted composition operators on $H^2(\mathbb{D})$, $1 \leq p < \infty$, characterized compactness within this class and identified the spectrum of the operators $C_g|_{H^p}$ for $g(z) = \frac{1+z}{1-z}$. He also raised question of the extent to which these operators were hyponormal or subnormal on $H^2(\mathbb{D})$. Brown, Halmos and Shields [3] and Kriete and Trutt [9] investigated these properties for the classical Cesàro operator. In [1] Albrecht, Miller and Neumann showed that $C_{(1+z)/(1-z)}$ is hyponormal on $H^2(\mathbb{D})$.

The matrix representation of $C_g$ in the standard basis $\{z^n\}_{n=1}^{\infty}$ of $H^2$ follows

$$C_g = \begin{pmatrix}
a_0 \\
a_1 & a_0 \\
\frac{a_2}{2} & \frac{a_1}{2} & \frac{a_0}{2} \\
\frac{a_3}{3} & \frac{a_2}{3} & \frac{a_1}{3} & \frac{a_0}{3} \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \quad (1.3)$$

where, $a_j$ are Taylor coefficients of $g(z)$, i.e. $\sum_{j=0}^{\infty} a_j z^j = g(z) \in H(\mathbb{D})$.

Given a sequence $\{b_n\}$ of scalars, the terraced matrix $R_b$ is the lower triangular matrix with constant row-segments

$$R_b = \begin{pmatrix}
b_0 \\
b_1 & b_1 \\
b_2 & b_2 & b_2 \\
b_3 & b_3 & b_3 & b_3 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \quad (1.4)$$

The Cesàro matrix is $R_{1/(n+1)}$ and more generally, if we take $b_n = n^{-\zeta}$ we get the $z\text{-Cesàro matrix } C_z$.

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The example is a mathematical document discussing the matrix representation of Cesàro operators and terraced matrices in the context of Hardy spaces and related to composition operators. The text includes formulas and definitions that are typical in advanced mathematical analysis, particularly in the study of operator theory and complex analysis.
If $D$ is the diagonal matrix $diag \{d_n\}$, then $DR_{\{bn\}} = R_{\{bn\}}$. Hence every terraced matrix has the factorization $R_b = D_bC$, where $D_b = diag \{(n + 1)b_n\}$; while if every $b_n \neq 0$, $C = D_bR_b$, where $D_b = diag \{1/(n + 1)b_n\}_{n=0}^\infty$.

In the present paper, we define the generalized terraced matrix by using the generalized Cesàro matrix and we show that the Cesàro matrix $C$, obtained when $b_n = 1/(n + 1)$ and $g(z) = 1/(1 - z)$ are taken in the generalized terraced matrix, is essentially the only generalized terraced matrix that is a Hausdorff matrix. That is, any generalized terraced matrix that is not a scalar multiple of $C$ is not a Hausdorff matrix. And we prove that every generalized terraced matrix commutes with an infinite matrix $B$, then $B$ is a scalar multiple of unit matrix. Also, we prove necessary and sufficient conditions related to normality and self-adjointedness of generalized terraced matrix.

**Definition 5.** Let $\{b_n\}$ be a scalar sequence and $g(z) = \sum_{k=0}^\infty a_kz^k \in H(\mathbb{D})$. The matrix

$$R^g_b = \begin{pmatrix} a_0b_0 & \cdots & \cdots & \cdots \\ a_1b_1 & a_0b_1 & \cdots & \cdots \\ a_2b_2 & a_1b_2 & a_0b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(1.5)

is called the generalized terraced matrix with symbol $g$ on $H^2$.

The relation $R^g_b = D_bC_g$ is valid similar to the terraced matrix, where $D_b = diag \{(n + 1)b_n\}_{n=0}^\infty$. We recall that $C = C_g$ for $g(z) = 1/(1 - z)$, since $g(z) = \sum_{k=0}^\infty z^k$, which fixes then $a_n = 1$ for all $n \in \mathbb{N}$. Thus, from (1.5) we get

$$R^g_b = \begin{pmatrix} b_0 & b_1 & \cdots \\ b_1 & b_2 & \cdots \\ b_2 & b_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = R_b$$

On the other hand $R^g_{\{1/(n+1)\}} = C_g$. Therefore this definition could be regarded as a two-way generalization of both terraced and Cesàro operators.
From (1.5) we can write
\[
(R^g_b)_{nj} = \begin{cases} 
  a_{n-j}b_n , & n \geq j \\
  0 , & n < j 
\end{cases} \tag{1.6}
\]
and
\[
[(R^g_b)^*]_{nj} = \begin{cases} 
  \overline{a_j}b_j , & j \geq n \\
  0 , & j < n 
\end{cases} \tag{1.7}
\]

2. Results

**Theorem 1.** Let \( g(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( a_0 \neq 0 \neq a_1 \). If \( R^g_b \) commutes with \( C_g \), then \( R^g_b \) is a scalar multiple of \( C_g \).

**Proof.** We get by direct calculation
\[
[R^g_b C_g]_{nj} = \begin{cases} 
  b_n \sum_{k=0}^{n-j} a_k a_{n-k-j} / k + j + 1 , & n \geq j \\
  0 , & n < j 
\end{cases}
\]
and
\[
[C_g R^g_b]_{nj} = \begin{cases} 
  1 \sum_{k=0}^{n-j} a_k a_{n-k-j} b_{k+j} / n + 2 , & n \geq j \\
  0 , & n < j 
\end{cases}
\]
If \( R^g_b C_g = C_g R^g_b \), then equating the entries on the first subdiagonal,
\[
[R^g_b C_g]_{n+1,n} = [C_g R^g_b]_{n+1,n}.
\]
this gives
\[
a_0 a_1 b_{n+1} \left( \frac{1}{n+1} + \frac{1}{n+2} \right) = \frac{a_0 a_1}{n+2} (b_n + b_{n+1})
\]
for all nonnegative integers \( n \). From last equation we have
\[
b_{n+1} = \frac{n+1}{n+2} b_n \tag{2.1}
\]
From (2.1), we can prove by using strong induction that for every \( n \),
\[
b_n = \frac{1}{n+1} b_0
\]
Hence, we have \( R^g_b = R^g_{b_0} = b_0 R^g_{\frac{1}{n+1}} = b_0 C_g . \)

**Remark 1.** Proposition 2.1 of [11] is a special case of Theorem 1 with the case \( g(t) = 1/(1-t) \).

**Theorem 2.** If an infinite matrix \( B \) commutes with all generalized terraced matrices, then \( B \) is a scalar multiple of the identity matrix.
Proposition 3.6. □

Since normality is defined to be $g.0/\ a$ $Hence, the proof could be completed by Proposition 2.3 in [11].

Theorem 3. Let $b_n \neq 0$ for each $n \in \mathbb{Z}^+$. The matrix $R_b^g$ is normal if and only if $g(z) = c$ for some $c \in \mathbb{C}$.

Proof. We calculate $\left[(R_b^g)^* (R_b^g)\right]_{00}$ and $\left[(R_b^g) (R_b^g)^*\right]_{00}$ by matrix multiplication. We get

$$\left[(R_b^g)^* (R_b^g)\right]_{00} = \sum_{k=0}^{\infty} \left[(R_b^g)^*\right]_{0k} \left[(R_b^g)\right]_{k0} = \sum_{k=0}^{\infty} |a_k|^2 |b_k|^2$$

and

$$\left[(R_b^g) (R_b^g)^*\right]_{00} = \sum_{k=0}^{\infty} \left[(R_b^g)^*\right]_{0k} \left[(R_b^g)\right]_{k0} = a_0 b_0 \overline{a_0 b_0} = |a_0|^2 |b_0|^2.$$

Since normality is defined to be $(R_b^g)^* (R_b^g) = (R_b^g) (R_b^g)^*$, we require that $\left[(R_b^g)^* (R_b^g)\right]_{00} = \left[(R_b^g) (R_b^g)^*\right]_{00}$. This implies that

$$\sum_{k=1}^{\infty} |a_k|^2 |b_k|^2 = 0.$$ 

Hence, $\sum_{k=1}^{\infty} |a_k|^2 |b_k|^2 = 0$. Since $b_k \neq 0$ for every $k \geq 1$, then $a_k = 0$ for every $k \geq 1$. Thus, $g(z) = \sum_{k=0}^{\infty} a_k z^k = a_0$. The converse direction is trivial since $g(z) = a_0$ implies that $R_b^g = \text{diag} \{a_0 b_k\}_{k=1}^{\infty}$.

Corollary 1. Let $b_n \neq 0$ for each $n \in \mathbb{Z}^+$ and $b_0 \in \mathbb{R}$. $R_b^g$ is self-adjoint if and only if $g(z) = c$ for some $c \in \mathbb{C}$.

Proof. From (1.6) and (1.7)

$$a_0 b_0 = \overline{a_0 b_0}, \ a_1 b_1 = a_2 b_2 = a_3 b_3 = \cdots = 0$$

Since, $\forall n \in N, b_n \neq 0$, then

$$a_0 = \overline{a_0}, \ a_1 = a_2 = a_3 = \cdots = 0$$

Hence $a_0 \in \mathbb{R}$ and $g(z) = a_0 \in \mathbb{R}$. The other direction is obvious.

Theorem 4. Let $\forall n \in N, b_n > 0$ real number and $\{b_n\}$ be a strictly decreasing sequence. $(R_b^g)^*$ is cyclic for all $\int_0^z g(w)\ dw \in BMOA$.

Proof. If $g(0) = 0$, then the result follows from [17], Theorem 2. If $g(0) \neq 0$, then the diagonal entries in (1.7) are distinct. Therefore, it is cyclic. See, for example, [8], Proposition 3.6.
Theorem 5. Let \( g (\beta) := g (\beta z) \) with \( |\beta| = 1 \), then \( R^g_b \) is unitarily equivalent to \( R^g_b \).

Proof. Define the map \( U_\beta : H^2 \to H^2 \) by \( U_\beta \left( f \right)(z) = f (\beta z) \). It is easy to see that \( U_\beta \) is unitary with \( U_\beta^* = U_\beta \). Now, to show the unitary equivalence, we must prove that \( U_\beta^* R^g_b U_\beta = R^g_b \). The matrix representation of \( U_\beta \) in the basis \( \{ z^{n-1} \}_{n=1}^{\infty} \) is the diagonal matrix \( \text{diag} \{ \beta^n \} \). Moreover, we know that \( (U_\beta)^* = U_\overline{\beta} = (U_\beta)^{-1} \).

Thus we have \( U_\beta^* R^g_b U_\beta = R^g_b \) using these matrix representations and consequently \( R^g_b \) is unitarily equivalent to \( R^g_b \).

Corollary 2. Let \( D \) be a unit disk in the complex plane. If \( \beta \in \partial (D) \) and \( b_n > 0 \) \( \forall n \in \mathbb{N} \), then \( \sigma \left( R^{1/(1-\beta z)}_b \right) = \sigma (R_b) = \{ z : |z - L| \leq L \} \cup S \), where \( L = \lim_{n \to \infty} (n + 1) b_n \) and \( 0 \leq L < +\infty \), \( S = \{ b_n : n = 0, 1, 2, \ldots \} \).

Proof. This is immediate from the unitary equivalence and [18].

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