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COMPUTING GRÖBNER BASES AND INVARIANTS OF THE SYMMETRIC ALGEBRA

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Abstract. We study algebraic invariants of the symmetric algebra $Sym_R(L)$ of the square-free monomial ideal $L = I_{n-1} + J_{n-1}$ in the polynomial ring $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$, where I_{n-1} (resp. J_{n-1}) is generated by all square-free monomials of degree $n-1$ in the variables $X_1,\dots,X_n$ (resp. $Y_1,\dots,Y_n$). In particular, the dimension and the depth of $Sym_R(L)$ are investigated by techniques of Gröbner bases and theory of s-sequences.

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Introduction

Let R be a commutative noetherian ring and M be a finitely generated R-module $M = Rf_1 + \cdots + Rf_q$. If (a_{ij}), $i = 1,\dots,q$, $j = 1,\dots,p$, is the matrix associated to a free presentation of M, then $Sym_R(M) = R[T_1,\dots,T_q]/J$, where J is generated by the linear forms $g_j = \sum_{i=1}^{q} a_{ij} T_i$ for $j = 1,\dots,p$.

In [3], in order to study the symmetric algebra $Sym_R(M)$, it is introduced the concept of s-sequence for the generators $f_1,\dots,f_q$ of M. We say that $f_1,\dots,f_q$ is an s-sequence for M if there exists a monomial order $\prec$ for the monomials in the variables T_i with $T_1 \prec T_2 \prec \cdots \prec T_q$ such that $in_\prec(J) = (\mathcal{I}_1 T_1,\dots,\mathcal{I}_q T_q)$, with $\mathcal{I}_i = (f_1,\dots,f_{i-1}) :_R f_i$ ideals of R.

The ideals I_i are called the annihilator ideals of the sequence $f_1,\dots,f_q$. If M is generated by an s-sequence, the standard algebraic invariants of M can be expressed only by the ideals $\mathcal{I}_i$ and in more cases the dimension can be computed in terms of the annihilators ideal of the sequence. The crucial point is that we can easily calculate the invariants, starting by the structure of the initial ideal $in_\prec(J)$ of J, stated that it is $\underline{T}$-linear, being $\underline{T} = \{T_1,\dots,T_q\}$ the variables that correspond in the presentation of symmetric algebra of M to the generators of M. A natural question arises: if $in_\prec(J)$ is not linearly generated in the variables $T_1,\dots,T_q$ and we write $in_\prec(J) = J_L + J^*$, where J^* is the not linear part of $in_\prec(J)$, what is the maximum degree (with respect the variables $T_1,\dots,T_q$) of the monomial generators of J^*?

We are interested to check when J^* is generated by monomials of minimum degree (quadratic). Moreover, we are interested to describe explicitly a Gröbner basis of J, given the importance of the initial ideal in more open problems about the invariants of $Sym_R(M)$ ([2], [4], [5], [8]).

Now, let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring on a field K in two sets of variables and L be a mixed products ideal, as defined in [7]. In [6] the authors selected the mixed products ideals that are generated by an s-sequence in order to compute the value or a bound for standard invariants of the symmetric algebra $Sym_R(L)$. The problem is open for ideals L that are not generated by an s-sequence. In this paper we examine a first class of mixed products ideals not generated by an s-sequence. More precisely, we consider the ideal $L = I_{n-1} + J_{n-1}$ of $K[X_1, \dots, X_n; Y_1, \dots, Y_n]$, with I_{n-1} (resp. J_{n-1}) the monomial ideal of R generated by all square-free monomials of degree $n-1$ in the variables $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$).

The aim is to compute a Gröbner basis of the relations ideal J of the symmetric algebra $Sym_R(L)$ and to study some invariants of $Sym_R(L)$. More precisely, in section 1 we give the structure of a Gröbner basis of J with respect to the lexicographic order $\prec$. It should be noted that we are able to compute the generators of $in_\prec(J)$ not linear in the variables $T_1 \dots, T_{2n}$ and to establish the degree. For $n = m = 3$, we obtain the only case in which the not linear part of $in_\prec(J)$ is of degree two in the variables T_i. In section 2, we compute the dimension and the depth of $Sym_R(L)$. For the computation of the dimension, we inspire to the techniques used in the theory of the s-sequences. More precisely, we consider the linear part J_L of $in_\prec(J)$ and we apply the results given in [3] for computing dimension in terms of the annihilator ideals of the monomial sequence generating L. Then we obtain that $Sym_R(L)$ is a Cohen-Macaulay algebra.

1. Gröbner bases of relation ideals

In [3] the notion of s-sequence is introduced for finitely generated modules in a noetherian ring R.

For every $i = 1, \dots, q$, we set $M_{i-1} = Rf_1 + \dots + Rf_{i-1}$ and $\mathcal{I}_i = M_{i-1} :_R f_i$ be the colon ideal. We set $\mathcal{I}_0 = (0)$. Since $M_i/M_{i-1} \simeq R/\mathcal{I}_i$, so $\mathcal{I}_i$ is the annihilator of the cyclic module $R/\mathcal{I}_i$. $\mathcal{I}_i$ is called *annihilator ideal* of the sequence $f_1, \dots, f_q$.

Let (a_{ij}), for $i = 1, \dots, q$, $j = 1, \dots, p$, be the relation matrix of M. The symmetric algebra $Sym_R(M)$ has a presentation $R[T_1, \dots, T_q]/J$, with $J = (g_1, \dots, g_p)$ where $g_j = \sum_{i=1}^{q} a_{ij} T_i$ for $j = 1, \dots, p$.

We consider $S = R[T_1, \dots, T_q]$ as a graded ring by assigning to each variable T_i degree 1 and to the elements of R degree 0.

Let $\prec$ be a monomial order on the monomials of S in the variables T_i such that $T_1 \prec T_2 \prec \dots \prec T_q$. With respect to this term order, if $f = \sum a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha =$

$T_1^{\alpha_1} \cdots T_q^{\alpha_q}$ and $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$, we put $in_{\prec}(f) = a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$.

So we define the monomial ideal $in_{\prec}(J) = (\{in_{\prec}(f) | f \in J\})$. In general we have $(\mathcal{I}_1 T_1, \mathcal{I}_2 T_2, \dots, \mathcal{I}_q T_q) \subseteq in_{\prec}(J)$ and the two ideals coincide in degree 1.

The sequence $f_1, \dots, f_q$ is an *s-sequence* for M if

$$(\mathcal{I}_1 T_1, \mathcal{I}_2 T_2, \dots, \mathcal{I}_q T_q) = in_{\prec}(J).$$

If R is a polynomial ring over a field and $f_1, \dots, f_q$ are monomials of R, then we have a criterion to be s-sequences. Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i \neq j$, where $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j. J is generated by $g_{ij} = f_{ij} T_j - f_{ji} T_i$ for $1 \leq i < j \leq q$. The monomial sequence $f_1, \dots, f_q$ is an s-sequence if and only if g_{ij}, for $1 \leq i < j \leq q$, is a Gröbner basis for J for a term order that agrees with the order of the variables in $S = R[T_1, \dots, T_q]$. Note that the annihilator ideals of the monomial sequence $f_1, \dots, f_q$ are the ideals $\mathcal{I}_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$ for $i = 1, \dots, q$ ([3]).

Now we consider the polynomial ring $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ over a field K in two sets of variables and the class of monomial *ideals of mixed products* of $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$:

$$L = I_k J_r + I_s J_t,$$

where I_k (resp. J_r) is the monomial ideal of R generated by all the square-free monomials of degree k (resp. r) in the variables $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_m$).

In [6] the authors investigate in which cases these monomial ideals are generated by an s-sequence. The system of generators of L is an s-sequence only in the following cases:
1) $L = I_{n-1} J_m$, 2) $L = I_1 J_m$, 3) $L = I_{n-1} J_m + I_n J_{m-1}$, 4) $L = J_m + I_n J_1$.

Set $I_0 = J_0 = R$, then we have the following case for $r = 0$ and $s = 0$ $L = I_k + J_k$ with $1 \leq k \leq \inf\{n, m\}$. We study this class of square-free monomial ideals for $k = n-1$ and $n = m$, then $R = K[X_1, \dots, X_n; Y_1, \dots, Y_n]$ and $L = I_{n-1} + J_{n-1}$.

Since the property to be an s-sequence may depend on the order of the sequence, in the sequel we will suppose $L = (f_1, f_2, \dots, f_{2n})$ where $f_1 \prec f_2 \prec \cdots \prec f_{2n}$ with respect to the monomial order $\prec_{lex}$ on $X_1, \dots, X_n, Y_1, \dots, Y_n$ and $X_1 \prec X_2 \prec \cdots \prec X_n \prec Y_1 \prec Y_2 \prec \cdots \prec Y_n$.

J is generated by $g_{ij} = f_{ij} T_j - f_{ji} T_i$ for $1 \leq i < j \leq 2n$. The monomial sequence $f_1, \dots, f_{2n}$ is an s-sequence if and only if g_{ij} for $1 \leq i < j \leq 2n$ is a Gröbner basis for J in $K[X_1, \dots, X_n; Y_1, \dots, Y_n; T_1, \dots, T_{2n}]$, with $X_1 \prec X_2 \prec \cdots \prec X_n \prec Y_1 \prec Y_2 \prec \cdots \prec Y_n \prec T_1 \prec T_2 \prec \cdots \prec T_{2n}$.

Theorem 1 ([6]). *Let $R = K[X_1, \dots, X_n]$ be the polynomial ring over a field K and I_k with $2 \leq k \leq n$. The ideal I_k is generated by an s-sequence if and only if $k = n - 1$.*

Theorem 2. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$. L is not generated by an s-sequence for any $n \neq 2$.*

Proof. For $n = 2$, $L = I_1 + J_1 = (X_1, \dots, X_n, Y_1, \dots, Y_n)$ is generated by an s-sequence for any admissible term order, since L is generated by a regular sequence ([3]).

For $n > 2$, let $L = I_{n-1} + J_{n-1} = (f_1, f_2, \dots, f_n) + (f_{n+1}, f_{n+2}, \dots, f_{2n})$, where $f_1 \prec f_2 \prec \cdots \prec f_n$ and $f_{n+1} \prec f_{n+2} \prec \cdots \prec f_{2n}$ with respect to the monomial order $\prec_{lex}$. L is generated by an s-sequence $\Leftrightarrow$ $G = \{g_{ij} = f_{ij} T_j - f_{ji} T_i \mid\ 1 \le i < j \le 2n\}$ is a Gröbner basis for J $\Leftrightarrow$ $S(g_{ij}, g_{hl}) \overset{G}{\to} 0$ for all $i, j, h, l \in \{1, \dots, 2n\}$ and $g_{ij} \neq g_{hl}$. We consider a lexicographic Gröbner basis for J with respect to the order on the variables $T_1 \prec T_2 \prec \cdots \prec T_{2n}$.

The generators of L are the following: $f_1 = X_1 \cdots X_{n-1}$, $f_2 = X_1 \cdots X_{n-2} X_n$, $f_n = X_2 \cdots X_n$, $f_{n+1} = Y_1 \cdots Y_{n-1}$, $f_{n+2} = Y_1 \cdots Y_{n-2} Y_n$, ..., $f_{2n} = Y_2 \cdots Y_n$.

One has $S(g_{1n}, g_{2,n+1}) = \frac{f_{1n} f_{n+1,2}}{[f_{1n}, f_{2,n+1}]} T_2 T_n - \frac{f_{2,n+1} f_{n1}}{[f_{1n}, f_{2,n+1}]} T_1 T_{n+1} = Y_1 \cdots Y_{n-1} T_2 T_n - X_2 \cdots X_{n-2} X_n^2 T_1 T_{n+1}$. By the structure of the generators of L there is no $g_{st} \in G$ whose initial term with respect to the admissible order on the variables $T_1 \prec T_2 \prec \cdots \prec T_{2n}$ divides the terms of $S(g_{1n}, g_{2,n+1})$. It follows that it is not possible to get a standard expression of $S(g_{1n}, g_{2,n+1})$ with respect G with remainder 0. Hence G is not a Gröbner basis for J. It follows that L can not be generated by an s-sequence ([3], Lemma 1.2). In fact, from the theory of Gröbner bases, suppose that $f_1, \dots, f_{2n}$ is a monomial s-sequence with respect to some admissible term order $\prec$, then $f_1, \dots, f_{2n}$ is an s-sequence for any other admissible term order, and as a consequence it is an s-sequence for the lexicographic order, that is a contradiction. □

The main result of this section gives the structure of the Gröbner basis of the relation ideal J of the symmetric algebra $Sym_R(L)$.

Theorem 3. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$. A Gröbner basis for J is the set*

$$BG(J) = \{g_{ij} = f_{ij} T_j - f_{ji} T_i, 1 \le i < j \le 2n\} \cup \mathcal{S},$$

where $\mathcal{S} = \bigcup_{t=1}^{n-2} H_t$ with $H_1 = \bigcup_{k_1=3}^{n} H_{1k_1}$ where $H_{1k_1} = \{S(g_{1k_1}, g_{ij}) | i < i < k_1,\ \ j = n+1, \dots, 2n\}$ and $H_t = \bigcup_{k_t=t+2}^{n} H_{tk_t}$ where $H_{tk_t} = \{S(g_{1k_t}, h) | h \in H_{t-1k_{t-1}},\ \ k_{t-1} < k_t\}$.

Proof. Let $L = I_{n-1} + J_{n-1} = (f_1, \dots, f_n) + (f_{n+1} \dots, f_{2n})$ and $G = \{g_{ij} = f_{ij} T_j - f_{ji} T_i \mid\ 1 \le i < j \le 2n\}$, where $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i \neq j$ and $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j. By the Theorem 2 L is not generated by an s-sequence, then $G \subsetneq BG(J)$ and there are S-polynomials $S(g_{ij}, g_{kl})$ that have not a standard expression with respect G with remainder 0. Let $g_{ij}, g_{kl} \in G$, it is known that $S(g_{ij}, g_{kl}) = \frac{f_{ij} f_{lk}}{[f_{ij}, f_{kl}]} T_j T_k - \frac{f_{kl} f_{ji}}{[f_{ij}, f_{kl}]} T_i T_l$. Knowing the structure of the monomials $f_1, \ldots, f_{2n}$, we are able to compute the S-polynomials of G that do not reduce to 0 modulo G and using Buchberger algorithm we construct the elements of $BG(J) \setminus G$. Then for $j = n+1, \ldots, 2n$ we compute $S(g_{1k_1}, g_{ij})$ with $i < k_1$ and $k_1 = 3, \ldots, n$:

$$S(g_{13}, g_{2j}) = f_j T_2 T_3 - \delta_1 X_n^2 T_1 T_j$$

$$S(g_{14}, g_{2j}) = f_j T_2 T_4 - \delta_2 X_n^2 T_1 T_j$$

$$S(g_{15}, g_{2j}) = f_j T_2 T_5 - \delta_4 X_n^2 T_1 T_j$$

....

$$S(g_{1n}, g_{n-1,n+1}) = f_j T_{n-1} T_n - \delta_{\binom{n-1}{n-3}} X_n^2 T_1 T_j,$$

where $\delta_1, \delta_2, \ldots, \delta_{\binom{n-1}{n-3}}$ are the generators of the Veronese ideal of degree $n-3$ in the variables $X_1, X_2, \ldots, X_{n-1}$. Being $f_1 = X_1 \cdots X_{n-1}$, $f_2 = X_1 \cdots X_{n-2} X_n$, $f_n = X_2 \cdots X_n$, $f_{n+1} = Y_1 \cdots Y_{n-1}$, $f_{n+2} = Y_1 \cdots Y_{n-2} Y_n$, ..., $f_{2n} = Y_2 \cdots Y_n$, it follows that there is no $g_{st} \in G$ whose initial term divides the terms of these S-polynomials. Then $S(g_{1k_1}, g_{ij}) \in BG(J)$ for $j = n+1, \ldots, 2n$, $i < k_1$ and $k_1 = 3, \ldots, n$. Set $H_{1k_1} = \{S(g_{1k_1}, g_{ij}) \mid \ j = n+1, \ldots, 2n,\ i < k_1\ \}$ for $k_1 = 3, \ldots, n$. Now we continue to compute the S-polynomials $S(g_{1k_2}, S(g_{1k_1}, g_{ij}))$ with $i < k_1 < k_2$ and $k_2 = 4, \ldots, n$, $j = n+1, \ldots, 2n$:

$$S(g_{14}, S(g_{13}, g_{2,n+1})) = f_{n+1} T_2 T_3 T_4 - \gamma_1 X_n^3 T_1^2 T_{n+1}$$

$$S(g_{14}, S(g_{13}, g_{2,n+2})) = f_{n+2} T_2 T_3 T_4 - \gamma_2 X_n^3 T_1^2 T_{n+2}$$

....

$$S(g_{14}, S(g_{13}, g_{2,2n})) = f_{2n} T_2 T_3 T_4 - \gamma_{\binom{n-1}{n-4}} X_n^3 T_1^2 T_{2n}$$

and so on up to $k_2 = n$:

$$S(g_{1n}, S(g_{1,n-1}, g_{n-2,n+1})) = f_{n+1} T_{n-2} T_{n-1} T_n - \gamma_1 X_n^3 T_1^2 T_{n+1}$$

$$S(g_{1n}, S(g_{1,n-1}, g_{n-2,n+2})) = f_{n+2} T_{n-2} T_{n-1} T_n - \gamma_2 X_n^3 T_1^2 T_{n+2}$$

....

$$S(g_{1n},S(g_{1,n-1},g_{n-2,2n})) = f_{2n}T_{n-2}T_{n-1}T_n - \gamma_{\binom{n-1}{n-4}}X_n^3T_1^2T_{2n},$$

where $\gamma_1,\gamma_2,\ldots,\gamma_{\binom{n-1}{n-4}}$ are the generators of the Veronese ideal of degree $n-4$ in the variables $X_1,X_2,\ldots,X_{n-1}$. The terms of these S-polynomials are not divided by the initial term of any $g_{st}\in G$, then $S(g_{1k_2},S(g_{1k_1},g_{ij}))\in BG(J)$ for $i<k_1<k_2$ and $k_2=4,\ldots,n$, $j=n+1,\ldots,2n$. Set $H_{2k_2}=\{S(g_{1k_2},h_{k_1})|h_{k_1}\in H_{1k_1}\}$. Continuing the computation of the S-polynomials one obtains:

$H_{3k_3}=\{S(g_{1k_3},h_{k_2})|h_{k_2}\in H_{2k_2}\}$ for $k_2<k_3$, $k_3=5,\ldots,n$

....

$H_{n-3k_{n-3}}=\{S(g_{1k_{n-3}},h_{k_{n-4}})|h_{k_{n-4}}\in H_{n-4k_{n-4}}\}$ for $k_{n-4}<k_{n-3}$, $k_{n-3}=n-1,n$

$H_{n-2k_{n-2}}=\{S(g_{1k_{n-2}},h_{k_{n-3}})|h_{k_{n-3}}\in H_{n-3k_{n-3}}\}$ for $k_{n-3}<k_{n-2}$, $k_{n-2}=n$.

Set $\mathcal{S}=\bigcup_{t=1}^{n-2}H_t$ with $H_1=\bigcup_{k_1=3}^{n}H_{1k_1}$ where $H_{1k_1}=\{S(g_{1k_1},g_{ij})|i<i<k_1,\ \ j=n+1,\ldots,2n\}$ and $H_t=\bigcup_{k_t=t+2}^{n}H_{tk_t}$ where $H_{tk_t}=\{S(g_{1k_t},h)|h\in H_{t-1k_{t-1}},\ \ k_{t-1}<k_t\}$. Then elements of $\mathcal{S}$ do not reduce to 0 modulo G. Moreover by construction no term of an element h of $\mathcal{S}$ is divisible by the initial term of an element of $\mathcal{S}\setminus\{h\}$. Set $B=G\cup\mathcal{S}$, it follows that $BG(J)\supseteq B$.

In order to show that $BG(J)=B$ we must prove that for all $g,h\in B$ the S-polynomial $S(g,h)$ reduces to 0 modulo B.

Let's start to prove that all the S-polynomials $S(g_{ij},g_{hl})$, with $i,j,h,l\in\{1,\ldots,2n\}$, has a standard expression with respect to B with remainder 0. We have:

$$S(g_{ij},g_{hl}) = \frac{f_{ij}f_{lh}}{[f_{ij},f_{hl}]}T_jT_h - \frac{f_{hl}f_{ji}}{[f_{ij},f_{hl}]}T_iT_l \ . \ (*)$$

Let's find a standard expression of $S(g_{ij},g_{hl})$, for all $i,j,h,l\in\{1,\ldots,n-1\}$.

If $[in_\prec(g_{ij}),in_\prec(g_{hl})]=1$, then $S(g_{ij},g_{hl})=f_{lh}g_{ij}T_h-f_{ji}g_{hl}T_i$.

If $[in_\prec(g_{ij}),in_\prec(g_{hl})]\neq 1$, we apply $(*)$ to obtain a standard expression for the S-polynomials $S(g_{ij},g_{hl})$. It results:

$S(g_{ij},g_{il})=-[f_{ji},f_{li}]g_{jl}T_i$
$S(g_{ij},g_{lj})=[f_{ji},f_{jl}]g_{il}T_j$
$S(g_{ij},g_{lk})=[f_{ji},f_{kl}](\frac{f_{kl}}{[f_{ji},f_{kl}]}g_{il}T_j-\frac{f_{lk}}{[f_{ij},f_{lk}]}g_{jk}T_i)$ or
$S(g_{ij},g_{lk})=[f_{ji},f_{kl}](\frac{f_{ji}}{[f_{ji},f_{kl}]}g_{il}T_k-\frac{f_{ij}}{[f_{ij},f_{lk}]}g_{jk}T_l)$.

Hence all the S-polynomials $S(g_{ij},g_{hl})$ reduce to 0 with respect to B. It remains to prove that the elements of $B\setminus G$ reduce to 0 with respect to B. If the elements of B have initial terms coprime, then they reduce to 0 with respect to B. Otherwise we observe that by the structure of the elements of $\mathcal{S}$ it follows that the initial terms of the elements of $\mathcal{S}$ are $\gamma_{i_k} X_n^t T_1^s T_j$ with γ_{i_k} a generator of the Veronese ideal of degree k in the variables $X_1, X_2, \ldots, X_{n-1}$ for $n+1 \leq j \leq 2n, 0 \leq k \leq n-3, 0 \leq t \leq n-1, 0 \leq s \leq n-3, k+t = n-1, s = t-1$.

Let $f, g \in B$, let d be the greater common divisor of $\mathrm{in}_{\prec}(f)$ and $\mathrm{in}_{\prec}(g)$, c be the greater common divisor of the no initial terms of f and g. In order to prove that $S(f,g)$ reduces to 0 with respect to B for all $f, g \in B$, we start to consider the elements of B of the form $g_{ij} \in G$ and $g_{1m,st} = S(g_{1m}, g_{st}) \in H_{1k_1}$. We have the following cases:

– if T_j is a variable of d, then $S(g_{ij}, g_{1m,st})$ reduces to 0 by the elements of B g_{is} and g_{1m} for $i < s$;

– if T_j is a variable of d and T_i is a variable of c, then $S(g_{ij}, g_{1j,si})$ reduces to 0 by the element $g_{1s} \in B$;

– if no variable $T_1, \ldots, T_{2n}$ is in d and c, then $S(g_{ij}, g_{1m,st})$ reduces to 0 by the elements of B g_{1i} and $g_{11m,stj} = S(g_{1j}, S(g_{1t}, g_{sm})) \in H_{2k_2}$ for $s < t < j$.

For the elements of the form $g_{1j,lk} = S(g_{1j}, g_{lk}) \in G$ and $g_{1m,st} = S(g_{1m}, g_{st}) \in H_{1k_1}$, we have the following cases:

– if T_1, T_j are variables of d and T_l is a variable of c, then $S(g_{1j,lk}, g_{1j,tl})$ reduces to 0 by $g_{kt} \in B$ for $k < t$;

– if T_1 is a variable of d and T_l, T_k are variables of c, then $S(g_{1j,lk}, g_{1j,lk})$ reduces to 0 by $g_{jt} \in B$ for $j < t$;

– if T_k is a variable of c then $S(g_{1j,lk}, g_{1t,mk})$ reduces to 0 by $g_{jt}, g_{lm} \in B$ for $j < t, l < m$.

The same argument is applied for the S-polynomials of the elements of all H_{jk_j} for $j > 1$. The assertion follows. □

Corollary 1. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$.*
The presentation ideal of $Sym_R(L)$ admits a lexicographic Gröbner basis of degree $\leq n-1$ in the variables $T_1, \ldots, T_{2n}$.

Corollary 2. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$. Then we have:*

1) The presentation ideals of $Sym(I_{n-1})$ and $Sym(J_{n-1})$ have a linear Gröbner basis respectively in the variables $U_1,\dots,U_n$ and $V_1,\dots,V_n$, which correspond to the generators of I_{n-1} and J_{n-1} respectively.
2) The presentation ideal of $Sym(I_{n-1}+J_{n-1})$ has a Gröbner basis not linear in the variables $T_1,\dots,T_{2n}$, which correspond to the generators of $L = I_{n-1}+J_{n-1}$.

Proof. 1) Since I_{n-1} and J_{n-1} are generated by an s-sequence, there exist a monomial order $\prec_1$ in the variables $U_1,\dots,U_n$ with $U_1 \prec_1 \cdots \prec_1 U_n$ and a monomial order $\prec_2$ in the variables $V_1,\dots,V_n$ with $V_1 \prec_2 \cdots \prec_2 V_n$ such that the presentation ideals of $Sym(I_{n-1})$ and $Sym(J_{n-1})$ have a linear Gröbner basis respectively in the variables $U_1,\dots,U_n$ and $V_1,\dots,V_n$.

2) It follows by Corollary 1. □

2. Studying standard invariants

In this section, we shall compute the dimension and the depth of the symmetric algebra $Sym_R(L)$, with L the mixed product ideal $L = I_{n-1}+J_{n-1}$.

In order to apply the theory of s-sequences, at the beginning we prove the main result concerning the annihilator ideals of the monomial sequence that generates L.

Proposition 1. *Let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$ be the polynomial ring over a field K and $L = I_{n-1}+J_{n-1} = (f_1,\dots,f_{2n})$. Then the annihilator ideals of the sequence $f_1,\dots,f_{2n}$ are:*

$$\mathcal{I}_1 = (0), \mathcal{I}_i = \begin{cases} (X_{n-i+1}) & \text{for } i = 2,\dots,n \\ I_{n-1} & \text{for } i = n+1 \\ (I_{n-1}, Y_{2n-i+1}) & \text{for } i = n+2,\dots,2n \end{cases}$$

Proof. Set $f_{ij} = \frac{f_i}{[f_i,f_j]}$ for $i<j$ and $i,j = 1,\dots,2n$. Then the annihilator ideals of the monomial sequence $f_1,\dots,f_{2n}$ are $\mathcal{I}_i = (f_{1i}, f_{2i},\dots,f_{i-1,i})$ for $i = 1,\dots,2n$. For $i=1$ we have $\mathcal{I}_1 = (0)$ and by the structure of these monomials it follows $\mathcal{I}_2 = (f_{12}) = (X_{n-1})$, $\mathcal{I}_3 = (f_{13}, f_{23}) = (X_{n-2})$, ... , $\mathcal{I}_{n-1} = (f_{1,n-1},\dots,f_{n-2,n-1}) = (X_2)$, $\mathcal{I}_n = (f_{1n}, f_{2n},\dots,f_{n-1,n}) = (X_1)$, $\mathcal{I}_{n+1} = (f_{1,n+1}, f_{2,n+1},\dots,f_{n,n+1}) = (f_1,\dots,f_n) = I_{n-1}$, $\mathcal{I}_{n+2} = (f_{1,n+2},\dots,f_{n+1,n+2}) = (f_1,\dots,f_n,Y_{n-1}) = (I_{n-1},Y_{n-1})$, ..., $\mathcal{I}_{2n} = (f_{1,2n},\dots,f_{2n-1,2n}) = (f_1,\dots,f_n,Y_1) = (I_{n-1},Y_1)$. Hence the assertion follows. □

Proposition 2. *Let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$ be the polynomial ring over a field K and $L = I_{n-1}+J_{n-1}$ for $n \geq 3$. Then*

$$in_\prec(J) = (\mathcal{I}_2T_2,\dots,\mathcal{I}_{2n}T_{2n}) + (I'_k X_n^t T_1^s T_j)$$

where $n+1 \le j \le 2n$, $0 \le k \le n-3$, $0 \le t \le n-1$, $0 \le s \le n-2$ *such that* $k+t=n-1$, $s=t-1$, *and* I'_k *is the square-free Veronese ideal generated by all the monomials of degree* k *in the variables* $X_1,\dots,X_{n-1}$.

Proof. It is known that the initial ideal of J is given by $\mathrm{in}_\prec(J) = (\{\mathrm{in}_\prec(f) | f \in BG(J)\})$. One has $BG(J) = \{g_{ij} = f_{ij}T_j - f_{ji}T_i, 1 \le i < j \le 2n\} \cup \mathcal{S}$ as in Theorem 3. By the structure of the monomials $f_1,\dots,f_{2n}$ that generate L we deduce the linear forms $g_{ij} = f_{ij}T_j - f_{ji}T_i, 1 \le i < j \le 2n$ and then we compute $\mathrm{in}_\prec(g_{ij}) = f_{ij}T_j$ for $1 \le i < j \le 2n$:

$\{X_{n-1}T_2,\dots,X_2T_{n-1},X_1T_n,Y_{n-1}T_{n+2},\dots,Y_1T_{2n},I_{n-1}T_j,\ j=n+1,\dots,2n\}$.

It remains to compute the initial terms of the element of $\mathcal{S}$. By the structure of the S-polynomials of $\mathcal{S}$ (see Theorem 3) we obtain that the initial term of these S-polynomials are the elements of the set $\{I'_k X_n^t T_1^s T_j\}$, where I'_k is the ideal generated by the square-free monomials of degree k in the variables $X_1,\dots,X_{n-1}$ and $j=n+1,\dots,2n$, $k+t=n-1$, $s=t-1$, $k=0,\dots,n-3$, $t=2,\dots,n-1$, $s=1,\dots,n-2$. Then the initial ideal of J is:

$$\mathrm{in}_\prec(J) = (X_{n-1}T_2,\dots,X_1T_n,I_{n-1}T_{n+1},(I_{n-1},Y_{n-1})T_{n+2},\dots,(I_{n-1},Y_1)T_{2n}) + (I'_k X_n^t T_1^s T_j).$$

Using Proposition 1 we can write $\mathrm{in}_\prec(J) = (\mathcal{J}_2T_2,\dots,\mathcal{J}_{2n}T_{2n}) + (\{I'_k X_n^t T_1^s T_j | n+1 \le j \le 2n, 0 \le k \le n-3, 0 \le t \le n-1, 0 \le s \le n-2, k+t=n-1, s=t-1\})$. □

Now, we recall some general results about an ideal $I = (H_1T_1,\dots,H_tT_t) \subset R[T_1,\dots,T_t]$, where R is a noetherian ring and $H_1,\dots,H_t$ are ideals of R. We say that I is linear in the variables $T_1,\dots,T_t$, or $\underline{T}$-linear. For $\underline{T}$-linear ideals we have:

Proposition 3 ([3], Lemma 2.3)**.** *Let* $I = (H_1T_1,\dots,H_tT_t)$ *be a* $\underline{T}$*-linear ideal of* $R[T_1,\dots,T_t]$*. Then*

$$I = \bigcap_{1 \le r \le t} (H_{i_1}+\ldots+H_{i_r},T_1,\dots,\widehat{T}_{i_1}\dots,\widehat{T}_{i_r},\dots,T_t),$$

with $1 \le i_1 \le \ldots \le i_r \le t$.

Proposition 4 ([3], Prop. 2.4)**.** *Let* $I = (H_1T_1,\dots,H_tT_t)$ *be a* $\underline{T}$*-linear ideal of* $R[T_1,\dots,T_t]$*. Then*

$$d = dim(R[T_1,\dots,T_t]/I) = max_{1 \le r \le t}\{dim(R/(H_{i_1}+\ldots+H_{i_r}))+r\}$$

with $1 \le i_1 \le \ldots \le i_r \le t$.

In order to apply the previous results, let $\mathrm{in}_\prec(J) = (\mathcal{J}_2T_2,\dots,\mathcal{J}_{2n}T_{2n}) + (I'_k X_n^t T_1^s T_j)$ as in Proposition 2. We set $J_L = (\mathcal{J}_2T_2,\dots,\mathcal{J}_{2n}T_{2n})$ the *linear part* of $\mathrm{in}_\prec(J)$ and $J^* = (I'_k X_n^t T_1^s T_j)$, where $n+1 \le j \le 2n$, $0 \le k \le n-3$, $0 \le t \le n-1$,

$0 \leq s \leq n-2$ such that $k+t=n-1$, $s=t-1$, and I'_k is the square-free Veronese ideal generated by all the monomials of degree k in the variables $X_1, \ldots, X_{n-1}$.

Proposition 5. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K, $L = I_{n-1} + J_{n-1}$ and $J_L \subset R[T_1, \ldots, T_{2n}]$. Then:*

$$dim(R[T_1, \ldots, T_{2n}]/J_L) = 2n+2.$$

Proof. Let $J_L = (\mathcal{I}_1 T_1, \ldots, \mathcal{I}_{2n} T_{2n})$, $\dim(R[T_1, \ldots, T_{2n}]/(\mathcal{I}_1 T_1, \ldots, \mathcal{I}_{2n} T_{2n})) = \max_{1 \leq r \leq 2n}\{\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r, \ \ 1 \leq i_1 \leq \ldots \leq i_r \leq 2n\}$ by Proposition 3. For $r = 1, \ldots, 2n$ one computes:

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r \leq 2n+1$ for $r < n$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(n-1)] + n = 2n+1$ for $r = n$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(n-1)] + n + 1 = 2n+2$ for $r = n+1$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = (2n-n) + n + 2 = 2n+2$ for $r = n+2$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(n+1)] + n + 3 = 2n+2$ for $r = n+3$,

....

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(2n-2)] + 2n = 2n+2$ for $r = 2n$.

Hence $\max_{1 \leq r \leq 2n}\{\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r\} = 2n+2$. □

Using the Proposition 5 we state the following

Theorem 4. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K, $L = I_{n-1} + J_{n-1}$ and $J \subset R[T_1, \ldots, T_{2n}]$ be the relation ideal of $Sym_R(L)$. Then:*
1) $dim(R[T_1, \ldots, T_{2n}]/in_{\prec}(J)) = 2n+1$
2) $depth(R[T_1, \ldots, T_{2n}]/in_{\prec}(J)) = 2n+1$
3) $pd(R[T_1, \ldots, T_{2n}]/in_{\prec}(J)) = 2n-1$.

Proof. 1) Let $\mathrm{in}_{\prec}(J) = J_L + J^*$ as in Proposition 2. By the structure of the ideals, $\mathrm{ht}(J_L + J^*) = \mathrm{ht}(J_L) + 1$. Hence one has

$\dim(R[T_1, \ldots, T_{2n}]/\mathrm{in}_{\prec}(J)) = \dim(R[T_1, \ldots, T_{2n}]/(J_L + J^*)) = \dim(R[T_1, \ldots, T_{2n}]) - \mathrm{ht}(J_L) - 1 = \dim(R[T_1, \ldots, T_{2n}]/J_L) - 1 = (2n+2) - 1 = 2n+1$ (by Proposition 5).

2) $\mathrm{ht}(\mathrm{in}_{\prec}(J)) = \mathrm{ht}(J_L) + 1 = \dim(R[T_1,\dots,T_{2n}]) - \dim(R[T_1,\dots,T_{2n}]/J_L) + 1 = 4n - (2n+2) + 1 = 2n-1$. Hence $\mathrm{depth}(\mathrm{in}_{\prec}(J)) \le \mathrm{ht}(\mathrm{in}_{\prec}(J)) = 2n-1$.

By Proposition 2 one has $\mathrm{depth}(\mathrm{in}_{\prec}(J)) \ge 2n-1$ being $\{X_{n-1}T_2,\dots,X_1T_n, Y_{n-1}T_{n+2},\dots,Y_1T_{2n}, I_k' X_n^t T_1^s T_j\}$ a regular sequence of $2n-1$ elements of $\mathrm{in}_{\prec}(J)$, where $k+t = n-1$, $s = t-1$ and $n+1 \le j \le 2n$. One has $2n-1 \le \mathrm{depth}(\mathrm{in}_{\prec}(J)) \le 2n-1$, hence the equality holds. If follows $\mathrm{depth}(\mathrm{in}_{\prec}(J)) = \mathrm{ht}(\mathrm{in}_{\prec}(J))$, then $\mathrm{in}_{\prec}(J)$ is Cohen-Macaulay that is equivalent to say that $R[T_1,\dots,T_{2n}]/\mathrm{in}_{\prec}(J)$ is Cohen-Macaulay.

Hence $\mathrm{depth}(R[T_1,\dots,T_{2n}]/\mathrm{in}_{\prec}(J)) = \dim(R[T_1,\dots,T_{2n}]/\mathrm{in}_{\prec}(J)) = 2n+1$.

3) $\mathrm{pd}(R[T_1,\dots,T_{2n}]/\mathrm{in}_{\prec}(J)) = \dim(R[T_1,\dots,T_{2n}]) - \mathrm{depth}(R[T_1,\dots,T_{2n}]/\mathrm{in}_{\prec}(J)) = 4n - (2n+1) = 2n-1$. □

The following result holds for the the symmetric algebra $Sym_R(L)$.

Theorem 5. *Let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$, $L = I_{n-1} + J_{n-1} \subset R$, $Sym_R(L) = S/J$ with $S = R[T_1,\dots,T_{2n}]$.*

Then:

1) dim($Sym_R(L)$) $= 2n+1$

2) depth($Sym_R(L)$) $= 2n+1$.

3) pd($Sym_R(L)$) $= 2n-1$.

Proof. 1) $\dim(Sym_R(L)) = \dim(S/\mathrm{in}_{\prec}(J)) = 2n+1$.

2) One has $\mathrm{depth}(Sym_R(L)) \ge \mathrm{depth}(\mathrm{in}_{\prec}(J)) = 2n+1$. On the other hand $\mathrm{depth}(Sym_R(L)) \le \dim(Sym_R(L)) = 2n+1$. The thesis follows.

3) $\mathrm{pd}(Sym_R(L)) = \mathrm{pd}(S/J) = \dim(S) - \mathrm{depth}(S/J) = 4n - (2n+1) = 2n-1$. □

Corollary 3. *Let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$, $L = I_{n-1} + J_{n-1} \subset R$. Then $Sym_R(L)$ is Cohen-Macaulay.*

Example 1. $R = K[X_1,X_2,X_3;Y_1,Y_2,Y_3]$

$L = I_2 + J_2 = (X_1X_2, X_1X_3, X_2X_3, Y_1Y_2, Y_1Y_3, Y_2Y_3)$

Set $f_1 = X_1X_2$, $f_2 = X_1X_3$, $f_3 = X_2X_3$, $f_4 = Y_1Y_2$,
$f_5 = Y_1Y_3$, $f_6 = Y_2Y_3$, where $f_1 \prec \cdots \prec f_6$ with respect to the lex order and $X_1 \prec X_2 \prec X_3 \prec Y_1 \prec Y_2 \prec Y_3$.
$G = \{X_2T_2 - X_3T_1, X_1T_3 - X_3T_1, X_1X_2T_4 - Y_1Y_2T_1, X_1X_2T_5 - Y_1Y_3T_1, X_1X_2T_6 - Y_2Y_3T_1, X_1T_3 - X_2T_2, X_1X_3T_4 - Y_1Y_2T_2, X_1X_3T_5 - Y_1Y_3T_2, X_1X_3T_6 - Y_2Y_3T_2, X_2X_3T_4 - Y_1Y_2T_3, X_2X_3T_5 - Y_1Y_3T_3, X_2X_3T_6 - Y_2Y_3T_3, Y_2T_5 - Y_3T_4, Y_1T_6 - Y_3T_4, Y_1T_6 - Y_2T_5\}$ is a set of generators for J.

$$BG(J) = G \cup \{S(g_{13}, g_{24}), S(g_{13}, g_{25}), S(g_{13}, g_{26})\},$$

where

$$S(g_{13}, g_{24}) = Y_1Y_2T_2T_3 - X_3^2T_1T_4;$$

$$S(g_{13}, g_{25}) = Y_1Y_2T_2T_3 - X_3^2T_1T_5;$$

$$S(g_{13}, g_{26}) = Y_1Y_2T_2T_3 - X_3^2T_1T_6.$$

The annihilator ideals are

$$\mathcal{I}_2 = (X_2);$$

$$\mathcal{I}_3 = (X_1);$$

$$\mathcal{I}_4 = (X_1X_2, X_1X_3, X_2X_3) = I_2;$$

$$\mathcal{I}_5 = (X_1X_2, X_1X_3, X_2X_3, Y_2) = (I_2, Y_2);$$

$$\mathcal{I}_6 = (X_1X_2, X_1X_3, X_2X_3, Y_1) = (I_2, Y_1).$$

Moreover $\text{in}_{\prec}(J) = J_L + J^*$, with $J_L = ((X_2)T_2, (X_1)T_3, (X_1X_2, X_1X_3, X_2X_3)T_4, (X_1X_2, X_1X_3, X_2X_3, Y_2)T_5, (X_1X_2, X_1X_3, X_2X_3, Y_1)T_6)$ and $J^* = (X_3^2T_1T_4, X_3^2T_1T_5, X_3^2T_1T_6)$.

By direct calculations, one obtains:

$$\dim(Sym_R(L)) = \text{depth}(Sym_R(L)) = 7$$

$$\text{pd}(Sym_R(L)) = 5.$$

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