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REGIONS OF VARIABILITY FOR JANOWSKI FUNCTIONS

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Abstract. Let $A \in \mathbb{C}$, $B \in [-1, 0)$. Then $P[A, B]$ denotes the class of analytic functions p in the open unit disk with $p(0) = 1$ such that

$$p(z) = \frac{1 + Aw(z)}{1 + Aw(z)},$$

where $w(0) = 0$ and $|w(z)| < 1$. In this article we find the regions of variability $V(z_0, \lambda)$ for $\int_0^{z_0} p(\rho)\, d\rho$ when p ranges over the class $P_\lambda[A, B]$ defined as

$$P_\lambda[A, B] = \left\{p \in P[A, B] : p'(0) = (A - B)\lambda\right\}$$

for any fixed $z_0 \in E$ and $\lambda \in \overline{E}$. As a consequence, the regions of variability are also illustrated graphically for different sets of parameters.

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1. INTRODUCTION

Let A denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk $E = \{z : |z| < 1\}$. Also Consider A as a topological vector space endowed with the topology of uniform convergence over a compact subsets of E. Also let $\mathcal{B}$ denote the class of analytic functions w in E such that $|w(z)| < 1$ and $w(0) = 0$. A function f is said to be subordinate to a function g written as $f \prec g$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular if g is univalent in E, then $f(0) = g(0)$ and $f(E) \subset g(E)$. Let $P[A, B]$ denote the class of analytic functions p such that $p(0) = 1$ and

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \; w \in B, z \in E, \tag{1.2}$$

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where $A \in \mathbb{C}$, $B \in [-1,0)$ with $A \neq B$. Note that $P[1,-1] = P$.

Let $p \in P[A,B]$. Then using the Herglotz representation, there exists a unique positive unit measure μ in $(-\pi,\pi]$ such that

$$p(z) = \int_{-\pi}^{\pi} \frac{1+Ae^{it}z}{1+Be^{it}z} d\mu(t), z \in E. \tag{1.3}$$

It can easily be seen from (1.2) that

$$w_p(z) = \frac{p(z)-1}{A-Bp(z)}, z \in E, \tag{1.4}$$

and conversely, we have

$$p'(0) = (A-B)\, w_p'(0) \tag{1.5}$$

Let $p \in P[A,B]$. Then by using classical Schwarz lemma, that is $\left|w_p'(0)\right| \leq 1$ (see [1],) we obtain

$$p'(0) = (A-B)\lambda$$

for some $\lambda \in \overline{E}$. By using (1.4) one can compute

$$\frac{w_p''(0)}{2} = \frac{p''(0)}{2(A-B)} + B\lambda^2.$$

Now, if we let

$$g(z) = \begin{cases} \dfrac{\frac{w_p(z)}{z}-\lambda}{1-\overline{\lambda}\frac{w_p(z)}{z}}, & |\lambda| < 1, \\ 0 & |\lambda| = 1. \end{cases}$$

This implies that

$$g'(0) = \begin{cases} \frac{1}{1-|\lambda|^2}\left(\frac{w_p(z)}{z}\right)' |_{z=0} = \frac{1}{1-|\lambda|^2}\frac{w_p''(z)}{2}, & |\lambda| < 1, \\ 0 & |\lambda| = 1. \end{cases}$$

By Schwarz lemma for $|\lambda| < 1$, we see that $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$. Equality is attained in both cases if $g(z) = e^{i\alpha}z$ for some $\alpha \in \mathbb{R}$. Now the condition $|g'(0)| \leq 1$ shows that there exists an $a \in \overline{E}$ such that $g'(0) = a$. Thus, we have

$$p''(0) = 2(A-B)\left[\left(1-|\lambda|^2\right)a - B\lambda^2\right].$$

Consequently for $\lambda \in \overline{E}$ and $z_0 \in E$, we have $V(z_0,\lambda)$ for $\int_0^{z_0} p(\rho)\,d\rho$ when p ranges over the class $P_\lambda[A,B]$ defined as

$$P_\lambda[A,B] = \left\{p \in P[A,B] : p'(0) = (A-B)\lambda\right\}$$

and

$$V_\lambda(z_0,A,B) = \left\{\int_0^{z_0} p(\rho)\,d\rho, \qquad p \in P_\lambda[A,B]\right\}.$$

For related study we refer to [2–9] and the references therein. The aim of this paper is to investigate explicitly the region of variability $V_\lambda(z_0,A,B)$ for the class $P_\lambda[A,B]$.

2. Basic properties of $V_\lambda(z_0, A, B)$

Proposition 1. *(i)* $V_\lambda(z_0, A, B)$ *is a compact subset of* $\mathbb{C}$.

(ii) $V_\lambda(z_0, A, B)$ *is convex subset of* $\mathbb{C}$.

(iii) If $|\lambda| = 1$ *or* $z_0 = 0$, *then* $V_\lambda(z_0, A, B) = \left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}$ *and if* $|\lambda| < 1$ *and* $z_0 \neq 0$, *then* $\left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}$ *is the interior point of the set* $V_\lambda(z_0, A, B)$.

Proof. *(i)* Since $P_\lambda[A, B]$ is a compact subset of $\mathbb{C}$, therefore $V_\lambda(z_0, A, B)$ is also compact.

(ii) Let $p_1, p_2 \in P_\lambda[A, B]$. Then

$$p(z) = (1-t)\, p_1(z) + t p_2(z),\, t \in [0,1]$$

is also in $P_\lambda[A, B]$, therefore $V_\lambda(z_0, A, B)$ is convex.

(iii) Since if $|\lambda| = |w_f'(0)| = 1$, then from Schwarz lemma, we obtain $w_f(z) = \lambda z$, which yields $p(z) = \frac{1+\lambda A z}{1+\lambda B z}$. This implies that

$$p(z) = 1 + \left(\frac{A-B}{B}\right)\left\{1 - \frac{1}{1 + B\lambda z}\right\}.$$

Therefore

$$V_\lambda(z_0, A, B) = \int_0^{z_0} p(\rho)\, d\rho = \left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}.$$

This also trivially holds true when $z_0 = 0$. For $\lambda \in E$ and $\alpha \in \overline{E}$, set

$$\delta(z, \lambda) = \frac{z + \lambda}{1 + \overline{\lambda} z},$$

$$\int_0^z H_{a,\lambda}(\rho)\, d\rho = \int_0^z \frac{1 + A\rho\delta(a\rho, \lambda)}{1 + B\rho\delta(a\rho, \lambda)} d\rho,\ z \in E. \tag{2.1}$$

Then $\int_0^z H_{a,\lambda}(\rho)\, d\rho$ is in $P_\lambda[A, B]$ and $w_f(z) = z\delta(az, \lambda)$. For fixed $\lambda \in E$ and $z_0 \in E \backslash \{0\}$ the function

$$E \ni a \mapsto \int_0^{z_0} H_{a,\lambda}(\rho)\, d\rho = \int_0^{z_0} \frac{1 + (\overline{\lambda} a + A\lambda)\rho + A a \rho^2}{1 + (\overline{\lambda} a + B\lambda)\rho + B a \rho^2} d\rho$$

is a non-constant analytic function of $a \in E$, and therefore is an open mapping. Hence $\int_0^{z_0} H_{0,\lambda}(\rho)\, d\rho = \left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}$ is an interior point of $\left\{\int_0^{z_0} H_{a,\lambda}(\rho)\, d\rho : a \in E\right\} \subset V_\lambda[z_0, A, B]$. $\square$

Keeping in view the above proposition, it is sufficient to find $V_\lambda[z_0, A, B]$ for $0 \le \lambda < 1$ and $z_0 \in E \backslash \{0\}$.

3. Main results

In this section, we state and prove some results which are needed in the proof of our main theorems.

Proposition 2. *For $p \in P_\lambda[A,B]$, we have*

$$|p(z) - q(z,\lambda)| \leq r(z,\lambda), \; z \in E, \lambda \in \overline{E}, \tag{3.1}$$

where

$$q(z,\lambda) = \frac{(1+\lambda A z)\left(1+B\overline{\lambda}\overline{z}\right) - |z|^2(\lambda + B\overline{z})\left(\overline{\lambda} + Az\right)}{1 - B^2|z|^4 + 2B(1-|z|^2)\Re(\lambda z) + |\lambda|^2|z|^2(B^2-1)}, \tag{3.2}$$

$$r(z,\lambda) = \frac{|A-B|\,(1-|\lambda|^2)|z|^2}{1 - B^2|z|^4 + 2B(1-|z|^2)\Re(\lambda z) + |\lambda|^2|z|^2(B^2-1)}.$$

The inequality is sharp for $z_0 \in E\backslash\{0\}$ if and only if $f(z) = H_{e^{i\theta},\lambda}(z)$ for some $\theta \in \mathbb{R}$.

Proof. Let $p \in P_\lambda[A,B]$. Then there exists $w_p \in \mathcal{B}$ such that

$$\left| \frac{\frac{w_p(z)}{z} - \lambda}{1 - \overline{\lambda}\frac{w_p(z)}{z}} \right| \leq |z|. \tag{3.3}$$

From (1.4) this can be written equivalently as

$$\left| \frac{p(z) - b(z,\lambda)}{p(z) + c(z,\lambda)} \right| \leq |z|\,|\tau(z,\lambda)|, \tag{3.4}$$

where

$$b(z,\lambda) = \frac{1+\lambda A z}{1+\lambda B z}, \quad c(z,\lambda) = -\frac{\overline{\lambda} + Az}{Bz + \overline{\lambda}}, \quad \tau(z,\lambda) = \frac{\overline{\lambda} + Bz}{1+\lambda B z}. \tag{3.5}$$

Simple computations show that the inequality (3.4) can be written as

$$\left| p(z) - \frac{b(z,\lambda) + |z|^2\,|\tau(z,\lambda)|^2\,c(z,\lambda)}{1 - |z|^2\,|\tau(z,\lambda)|^2} \right| \leq \frac{|z|\,|\tau(z,\lambda)|\,|b(z,\lambda) + c(z,\lambda)|}{1 - |z|^2\,|\tau(z,\lambda)|^2}. \tag{3.6}$$

Now we have

$$1 - |z|^2\,|\tau(z,\lambda)|^2 = \frac{1 - B^2|z|^4 + 2B(1-|z|^2)\Re(\lambda z) + |\lambda|^2|z|^2(B^2-1)}{|1+\lambda z B|^2},$$

$$b(z,\lambda) + c(z,\lambda) = \frac{(B-A)(1-|\lambda|^2)z}{(1+\lambda B z)(Bz + \overline{\lambda})},$$

$$b(z,\lambda) + |z|^2\,|\tau(z,\lambda)|^2\,c(z,\lambda)$$

$$= \frac{(1+\lambda Az)\left(1+B\overline{\lambda}\overline{z}\right)}{|1+\lambda zB|^2} - \frac{|z|^2(\lambda+B\overline{z})\left(\overline{\lambda}+Az\right)}{|1+\lambda zB|^2}.$$

By simple calculations, we obtain

$$q(z,\lambda) = \frac{b(z,\lambda)+|z|^2|\tau(z,\lambda)|^2 c(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2},$$

$$r(z,\lambda) = \frac{|z||\tau(z,\lambda)||b(z,\lambda)+c(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}.$$

All these relations together with (3.6) give (3.1). Equality occurs in (3.1) when $p = H_{i\theta,\lambda}(z)$, for some $z \in E$. Conversely if equality occurs in (3.1) for some $z \in E\backslash\{0\}$, then equality must hold in (3.3). Thus by Schwarz lemma there exists $\theta \in \mathbb{R}$ such that $w_p(z) = z\delta(e^{i\theta}z,\lambda)$ for all $z \in E$. This implies $P = H_{i\theta,\lambda}$. □

Geometrically the above lemma means that the functional p lies in the closed disk centred at $q(z,\lambda)$ with radius $r(z,\lambda)$.

For $\lambda = 0$, we have the following result

Corollary 1. *For $p \in P_0[A,B]$, we have*

$$\left| p(z) - \frac{1-AB|z|^4}{1-B^2|z|^4} \right| \leq \frac{(A-B)|z|^2}{1-B^2|z|^4},\ z \in E\backslash\{0\}.$$

Equality is attained if and only if $p = H_{i\theta,0}$.

Corollary 2. *Let $\gamma : z(t), 0 \leq t \leq 1$, be a C^1–curve in E with $z(0) = 0$ and $z(1) = z_0$. Then*

$$V_\lambda(z_0,A,B) \subset \{w \in \mathbb{C} : |w - Q(\lambda,\gamma)| \leq R(\lambda,\gamma)\},$$

where

$$Q(\lambda,\gamma) = \int_0^1 q(z(t),\lambda)z'(t)dt,\quad R(\lambda,\gamma) = \int_0^1 r(z(t),\lambda)z'(t)dt.$$

Proof. Since p is in $P_\lambda[A,B]$, therefore by using Proposition 2, we get

$$\begin{aligned} \left|\int_0^1 p(z(t))z'(t)dt - Q(\lambda,\gamma)\right| &= \left|\int_0^1 p(z(t))z'(t)dt - \int_0^1 q(z(t),\lambda)z'(t)dt\right| \\ &= \left|\int_0^1 (p(z(t)) - q(z(t),\lambda))z'(t)dt\right| \\ &\leq \int_0^1 r(z(t),\lambda)|z'(t)|dt = R(\lambda,\gamma). \end{aligned}$$

This implies the required result. □

For our next result we need the following lemma:

Lemma 1. *For $\theta \in \mathbb{R}$ and $|\lambda| < 1$, the function*

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta^2}{\left(1 + (\overline{\lambda}e^{i\theta} + B\lambda)\zeta + Be^{i\theta}\zeta^2\right)^2} d\zeta, \quad z \in E,$$

has zeros of order 3 at the origin and no zero elsewhere in E. Moreover, there exists a starlike normalized univalent function s in E such that $G(z) = 3^{-1}e^{i\theta}s^3(z)$.

The above lemma is due to Ponnusamy et al. [4].

Proposition 3. *Let $\theta \in (-\pi, \pi]$, $z_0 \in E \backslash \{0\}$. Then, $\int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\, d\rho \in \partial V_\lambda(z_0, A, B)$. Moreover, $\int_0^{z_0} p(\rho)\, d\rho = \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\, d\rho$ implies $p = H_{e^{i\theta},\lambda}$ for some $p \in P_\lambda[A, B]$ and $\theta \in (-\pi, \pi]$.*

Proof. It follows from (2.1) that

$$\begin{aligned} H_{a,\lambda}(z) &= \frac{1 + Az\delta(az,\lambda)}{1 + Bz\delta(az,\lambda)} \\ &= \frac{1 + (\overline{\lambda}a + A\lambda)z + Aaz^2}{1 + (\overline{\lambda}a + B\lambda)z + Baz^2}. \end{aligned}$$

Thus from (3.5), it follows that

$$H_{a,\lambda}(z) - b(z,\lambda) = \frac{(A-B)(1-|\lambda|^2)az^2}{\left(1 + (\overline{\lambda}a + B\lambda)z + Baz^2\right)(1 + \lambda Bz)},$$

$$H_{a,\lambda}(z) + c(z,\lambda) = \frac{(B-A)(1-|\lambda|^2)z}{\left(1 + (\overline{\lambda}a + B\lambda)z + Baz^2\right)(Bz + \overline{\lambda})}.$$

Therefore

$$\begin{aligned} &H_{a,\lambda}(z) - q(z,\lambda) \\ &= H_{a,\lambda}(z) - \frac{b(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 c(z,\lambda)}{1 - |z|^2|\tau(z,\lambda)|^2} \\ &= \frac{1}{1 - |z|^2|\tau(z,\lambda)|^2}\left[H_{a,\lambda}(z) - b(z,\lambda) - |z|^2|\tau(z,\lambda)|^2\left(H_{a,\lambda}(z) + c(z,\lambda)\right)\right] \\ &= \frac{(A-B)(1-|\lambda|^2)\left[az(1 + B\overline{\lambda z}) + |z|^2(B\overline{z} + \lambda)\right]}{\left(\begin{array}{c} 1 - B^2|z|^4 + 2B(1-|z|^2)\Re(\lambda z) \\ +|\lambda|^2|z|^2(B^2-1) \end{array}\right)\left(1 + (\overline{\lambda}a + B\lambda)z + Baz^2\right)}. \end{aligned}$$

Putting $a = e^{i\theta}$, we obtain

$$H_{e^{i\theta},\lambda}(z) - q(z,\lambda)$$

$$= \frac{r(z,\lambda)e^{i\theta}z^2}{|z|^2}\frac{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)\overline{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)}}{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)^2}$$

$$= \frac{r(z,\lambda)e^{i\theta}z^2}{|z|^2}\frac{\left|1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right|^2}{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)^2}.$$

Now using $G(z)$ defined in Lemma 1, it follows that

$$H_{e^{i\theta},\lambda}(z)-q(z,\lambda)=r(z,\lambda)\frac{G'(z)}{|G'(z)|}. \tag{3.7}$$

Using the argument of Lemma 1 that $G=3^{-1}e^{i\theta}s^3$, where s is starlike in E with $s(0)=s'(0)-1=0$, for any $z_0\in E\backslash\{0\}$ the linear segment joining 0 and $s(z_0)$ lies entirely in $s(E)$. Let γ_0 be the curve defined by

$$\gamma_0 : z(t)=s^{-1}(ts(z_0)),t\in[0,1].$$

Since $G(z(t))=2^{-1}e^{i\theta}(s(z(t)))^2=3^{-1}e^{i\theta}(ts(z_0))^3=t^3G(z_0)$. Differentiation w.r.t t gives us

$$G'(z(t))z'(t)=3t^2G(z_0),\quad t\in[0,1]. \tag{3.8}$$

This relation together with (3.7), leads to

$$\begin{aligned}\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho-Q(\lambda,\gamma_0)&=\int_0^1\left(H_{e^{i\theta},\lambda}(z(t))-q(z(t),\lambda)\right)z'(t)dt\\&=\int_0^1 r(z(t),\lambda)\frac{G'(z(t))z'(t)}{|G'(z(t))z'(t)|}|z'(t)|dt\\&=\frac{G(z_0)}{|G(z_0)|}\int_0^1 r(z(t),\lambda)|z'(t)|dt\\&=\frac{G(z_0)}{|G(z_0)|}R(\lambda,\gamma_0).\end{aligned} \tag{3.9}$$

This implies that $\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho\in\partial\overline{E}\,(Q(\lambda,\gamma_0),R(\lambda,\gamma_0))$, where $Q(\lambda,\gamma_0)$ and $R(\lambda,\gamma_0)$ are defined as in Corollary 2. Hence from Corollary 2, we have $\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho\in\partial V_\lambda(z_0,A,B)$. For uniqueness, we suppose that

$$\int_0^{z_0}p(\rho)\,d\rho=\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho$$

for some $\theta\in(-\pi,\pi]$ and $p\in P_\lambda[A,B]$. Let

$$h(t)=\frac{\overline{G(z_0)}}{|G(z_0)|}(p(z(t))-q(z(t),\lambda))z'(t),$$

where $\gamma_0 : z(t),\ 0 \le t \le 1$. Then the function h is continuous and

$$|h(t)| = \frac{\left|\overline{G(z_0)}\right|}{|G(z_0)|} \left|(p(z(t)) - q(z(t),\lambda))\right| \left|z'(t)\right|.$$

Now using Proposition 2, we get $|h(t)| \le r(z(t),\lambda)|z'(t)|$. Further from (3.9), we have

$$\begin{aligned}\int_0^1 \Re(h(t))dt &= \int_0^1 \Re(\left[\frac{\overline{G(z_0)}}{|G(z_0)|}(p(z(t)) - q(z(t),\lambda))\, z'(t)\right])dt \\ &= \Re\left[\frac{\overline{G(z_0)}}{|G(z_0)|}\int_0^{z_0}\{H_{e^{i\theta},\lambda}(\rho)\,d\rho - Q(z(t),\lambda)\}\right] \\ &= \int_0^1 \Re(r((z(t),\lambda)))|z'(t)|dt.\end{aligned}$$

Thus $h(t) = r(z(t),\lambda)|z'(t)|$, for all $t \in [0,1]$. From (3.7) and (3.8) we have $\int_0^{z_0} p(\rho)\,d\rho = \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho$ on γ_0. The identity theorem for analytic functions yields us $p = H_{e^{i\theta},\lambda}$, $z \in E$. □

Main Theorem

Theorem 1. *Let $\lambda \in E$ and $z_0 \in E\backslash\{0\}$. Then boundary $\partial V_\lambda(z_0, A, B)$ is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \mapsto \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho = \int_0^{z_0} \frac{1 + A\rho\delta(e^{i\theta}\rho,\lambda)}{1 + B\rho\delta(e^{i\theta}\rho,\lambda)}d\rho$$

If $\int_0^{z_0} p(\rho)\,d\rho = \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho$ for some $p \in P_\lambda[A,B]$ and $\theta \in (-\pi,\pi]$, then $p(z) = H_{e^{i\theta},\lambda}(z)$.

Proof. First we have to show that the curve

$$(-\pi,\pi] \ni \theta \mapsto \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho$$

is simple. Let us assume that

$$\int_0^{z_0} H_{e^{i\theta_1},\lambda}(\rho)\,d\rho = \int_0^{z_0} H_{e^{i\theta_2},\lambda}(\rho)\,d\rho$$

for some $\theta_1,\theta_2 \in (-\pi,\pi]$ with $\theta_1 \ne \theta_2$. Then the use of Proposition 3 yields us that $H_{e^{i\theta_1},\lambda}(z_0) = H_{e^{i\theta_2},\lambda}(z_0)$, which further gives the following relation

$$\tau\left(\frac{w_{H_{e^{i\theta_1},\lambda}}(z)}{z},\lambda\right) = \tau\left(\frac{w_{H_{e^{i\theta_2},\lambda}}(z)}{z},\lambda\right).$$

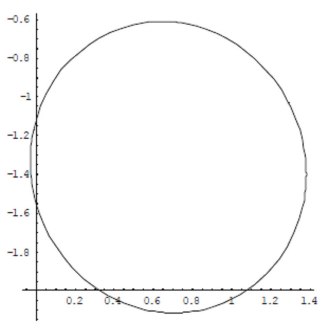
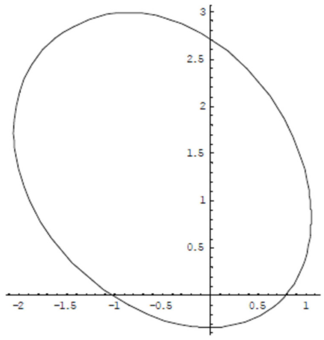
This implies that

$$\frac{B(ze^{i\theta_1}+\lambda)+\overline{\lambda}(1+\overline{\lambda}e^{i\theta_1}z)}{1+\overline{\lambda}e^{i\theta_1}z+\lambda B(ze^{i\theta_1}+\lambda)}=\frac{B(ze^{i\theta_2}+\lambda)+\overline{\lambda}(1+\overline{\lambda}e^{i\theta_2}z)}{1+\overline{\lambda}e^{i\theta_2}z+\lambda B(ze^{i\theta_2}+\lambda)}.$$

After some simplification, we obtain $ze^{i\theta_1}=ze^{i\theta_2}$, which leads us to a contradiction. Hence the curve is simple. Since $V_\lambda(z_0,A,B)$ is compact convex subset of $\mathbb{C}$ and has non-empty interior, therefore the boundary $\partial V_\lambda(z_0,A,B)$ is a simple closed curve. From Proposition 3 the curve $\partial V_\lambda(z_0,A,B)$ contains the curve $(-\pi,\pi]\ni\theta\mapsto\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho$. Since a simple closed curve cannot contain any simple closed curve other than itself. Thus $\partial V_\lambda(z_0,A,B)$ is given by $(-\pi,\pi]\ni\theta\mapsto\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho$. □

Geometric View of Theorem 1

By using Mathematica, we have the following views for different parameters.

Values of parameters	$z_0 = 0.335192 - 0.787333i$ $\lambda = 0.0737292 + 0.466706i$ $A = 1, \ B = -1$	$z_0 = -0.261209 + 0.926935i$ $\lambda = 0.0737292 + 0.466706i$ $\beta = -1.991244, \quad \gamma = 0.383292$ $A = 2(1-\beta)\cos\gamma e^{-i\gamma} - 1, \quad B = -1$
Geometric View of Theorem 1		

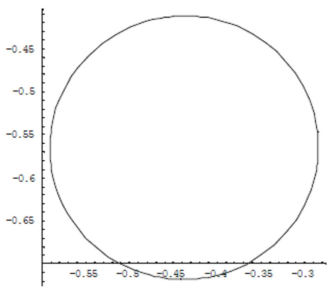
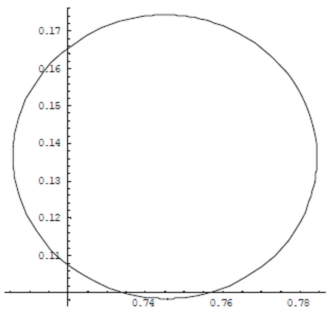
Values of parameters	$z_0 = -0.41227 - 0.521734i$ $\lambda = -0.0875648 + 0.0714166i$ $A = 0.9868233 + 0.00835453i$ $B = -0.50$	$z_0 = 0.771264 + 0.1512040i$ $\lambda = -0.391149 - 0.294747i$ $A = -0.2346400 - 0.180560i$ $B = -0.50$
Geometric View of Theorem 1		

FIGURE 1. For $A = 2(1-\beta)(\cos\gamma)e^{-i\gamma} - 1, \ B = -1$, where $\beta < 1, |\gamma| < \pi/2$, we have the known result proved by Ponnusamy and Vasudevarao [3] as special cases of our results.

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