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PARTIAL SUMS OF GENERALIZED STRUVE FUNCTIONS

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Abstract. Let $(f_{v,d,c})_n(z) = z + \sum_{k=1}^{n} b_k z^{k+1}$ be the sequence of partial sums of generalized and normalized Struve functions $f_{v,d,c}(z) = z + \sum_{k=1}^{\infty} b_k z^{k+1}$ where $b_k = \frac{(-c/4)^k}{(3/2)_k (F)_k}$ and $F := v + (d+2)/2 \neq 0, -1, -2, \ldots$. The purpose of the present paper is to determine lower bounds for $\Re\left\{\frac{f_{v,d,c}(z)}{(f_{v,d,c})_n(z)}\right\}$, $\Re\left\{\frac{(f_{v,d,c})_n(z)}{f_{v,d,c}(z)}\right\}$, $\Re\left\{\frac{f'_{v,d,c}(z)}{(f_{v,d,c})'_n(z)}\right\}$ and $\Re\left\{\frac{(f_{v,d,c})'_n(z)}{f'_{v,d,c}(z)}\right\}$. Furthermore, we give lower bounds for $\Re\left\{\frac{\Lambda[f_{v,d,c}](z)}{(\Lambda[f_{v,d,c}])_n(z)}\right\}$ and $\Re\left\{\frac{(\Lambda[f_{v,d,c}])_n(z)}{\Lambda[f_{v,d,c}](z)}\right\}$, where $\Lambda[f_{v,d,c}]$ is the Alexander transform of $f_{v,d,c}$.

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1. INTRODUCTION AND PRELIMINARY RESULTS

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and satisfy the usual normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ contains all functions which are univalent in $\mathcal{U}$. Also let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex and close-to-convex of order α in $\mathcal{U}$ $(0 \leq \alpha < 1)$.

The Alexander transform $\Lambda[f] : \mathcal{U} \longrightarrow \mathbb{C}$ of f is defined by [1],

$$\Lambda[f](z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{k=2}^{\infty} \frac{a_k}{k} z^k. \tag{1.2}$$

We consider the following second-order linear inhomogeneous differential equation (see, for details [6]):

$$z^2 w''(z) + dzw'(z) + \left[cz^2 - v^2 + (1-d)v\right] w(z) = \frac{4(z/2)^{v+1}}{\sqrt{\pi}\Gamma(v+d/2)} \tag{1.3}$$

where $c, d, v \in \mathbb{C}$.

A particular solution of the differential equation (1.3), which is denoted by $w_{v,d,c}(z)$, is called the generalized Struve function of the first kind of order v. In fact we have the following series representation for the function $w_{v,d,c}(z)$:

$$w_{v,d,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(k+3/2)\Gamma(v+k+\frac{d+2}{2})} \left(\frac{z}{2}\right)^{2k+v+1} \quad (z \in \mathbb{C}), \tag{1.4}$$

where $\Gamma(z)$ stands for Euler gamma function. The series in (1.4) permits us to study the Struve and the modified Struve functions in a unified manner. Each of these particular cases of the function $w_{v,d,c}(z)$ is worthy of mention here.

• For $d = c = 1$ in (1.4), we get the Struve function $H_v(z)$ defined by (see [11] and [6]):

$$H_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(v+k+3/2)} \left(\frac{z}{2}\right)^{2k+v+1} \quad (z \in \mathbb{C}). \tag{1.5}$$

• For $d = -c = 1$ in (1.4), we get the modified Struve function $L_v(z)$ defined by (see [11] and [6]):

$$L_v(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+3/2)\Gamma(v+k+3/2)} \left(\frac{z}{2}\right)^{2k+v+1} \quad (z \in \mathbb{C}). \tag{1.6}$$

We now consider the function $f_{v,d,c}(z)$ defined, in terms of the generalized Struve function $w_{v,d,c}(z)$, by (see [6]):

$$\begin{aligned} f_{v,d,c}(z) &= 2^v \sqrt{\pi}\Gamma(v+\frac{d+2}{2}) z^{\frac{-v+1}{2}} w_{v,d,c}(\sqrt{z}) \\ &= z + \sum_{k=1}^{\infty} \frac{\sqrt{\pi}(-c)^k \Gamma(v+\frac{d+2}{2})}{2^{2k+1}\Gamma(k+3/2)\Gamma(v+k+\frac{d+2}{2})} z^{k+1} \quad (z \in \mathcal{U}) \end{aligned} \tag{1.7}$$

According to Weierstrass M-test the series in (1.7) converges uniformly for $z \in \mathcal{U}$. By taking

$$\sum_{k=0}^{\infty} \frac{\sqrt{\pi}(-c)^k \Gamma(v+\frac{d+2}{2})}{2^{2k+1}\Gamma(k+3/2)\Gamma(v+k+\frac{d+2}{2})} = \sum_{k=0}^{\infty} M_k,$$

we see that from the Ratio Test the series $\sum_{k=0}^{\infty} M_k$ is convergent. That means the function $f_{v,d,c}(z)$ is analytic for $z \in \mathcal{U}$. Moreover, $f_{v,d,c}(z)$ satisfies the normalization condition $f_{v,d,c}(0) = f'_{v,d,c}(0) - 1 = 0$. So that, $f_{v,d,c} \in \mathcal{A}$.

By using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_k = \Gamma(\lambda+k)/\Gamma(\lambda) = \lambda(\lambda+1)...(\lambda+k-1)$, we obtain the following series representation for the function $f_{v,d,c}(z)$ given by (1.7):

$$f_{v,d,c}(z) = z + \sum_{k=1}^{\infty} b_k z^{k+1} \tag{1.8}$$

where $b_k = \frac{(-c/4)^k}{(3/2)_k (F)_k}$ and $F := v + (d+2)/2 \neq 0, -1, -2, ...$

For further results on this relative $f_{v,d,c}(z)$ of the generalized Struve function $w_{v,d,c}(z)$, we refer the reader to the recent papers (see, for example, [6, 12, 13]).

In this work, we will examine the ratio of a function of the form (1.8) to its sequence of partial sums $\left(f_{v,d,c}\right)_n(z) = \sum\limits_{k=0}^{n} b_k z^{k+1}$ when the parameters c, d, v satisfy appropriate conditions. We will determine lower bounds for $\Re\left\{\frac{f_{v,d,c}(z)}{\left(f_{v,d,c}\right)_n(z)}\right\}$, $\Re\left\{\frac{\left(f_{v,d,c}\right)_n(z)}{f_{v,d,c}(z)}\right\}$, $\Re\left\{\frac{f'_{v,d,c}(z)}{\left(f_{v,d,c}\right)'_n(z)}\right\}$, $\Re\left\{\frac{\left(f_{v,d,c}\right)'_n(z)}{f'_{v,d,c}(z)}\right\}$, $\Re\left\{\frac{\Lambda[f_{v,d,c}](z)}{\left(\Lambda[f_{v,d,c}]\right)_n(z)}\right\}$ and $\Re\left\{\frac{\left(\Lambda[f_{v,d,c}]\right)_n(z)}{\Lambda[f_{v,d,c}](z)}\right\}$, where $\Lambda[f_{v,d,c}]$ is the Alexander transform of $f_{v,d,c}$.

For various interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) refered to the works of Brickman et al. [2], Lin and Owa [3], Orhan and Güneş [4], Orhan and Yağmur [5], Owa et.al [7], Sheil-Small [8], Silverman [9], Silvia [10].

Lemma 1. *If the parameters $d, v \in \mathbb{R}$, $c \in \mathbb{C}$ and $F := v + (d+2)/2 \neq$ $0, -1, -2, ...$ then the function*

$$f_{v,d,c} : \mathcal{U} \longrightarrow \mathbb{C}$$

given by (1.8) satisfies the following inequalities:

(i) If $F > \frac{|c|}{6}$ then

$$\left|f_{v,d,c}(z)\right| \leq \frac{6F}{6F-|c|} \quad (z \in \mathcal{U}),$$

(ii) If $F > \frac{|c|}{4}$ then

$$\left|f'_{v,d,c}(z)\right| \leq \frac{12F+|c|}{3(4F-|c|)} \quad (z \in \mathcal{U}),$$

(iii) If $F > \frac{|c|}{6}$ then

$$\left|\Lambda[f_{v,d,c}](z)\right| \leq \frac{12F-|c|}{12F-2|c|} \quad (z \in \mathcal{U}).$$

Proof. (i) If we use the well-known triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the inequalities $(3/2)_k \geq (3/2)^k$, $(F)_k \geq F^k$, $(k \in \mathbb{N})$ we obtain

$$\begin{aligned} \left| f_{v,d,c}(z) \right| &= \left| z + \sum_{k=1}^{\infty} \frac{(-c/4)^k}{(3/2)_k\,(F)_k} z^{k+1} \right| \leq 1 + \sum_{k=1}^{\infty} \left(\frac{|-c/4|}{\frac{3}{2}F} \right)^k \\ &= 1 + \frac{|c|}{6F} \sum_{k=1}^{\infty} \left(\frac{|c|}{6F} \right)^{k-1} = \frac{6F}{6F - |c|}, \quad \left(F > \frac{|c|}{6} \right). \end{aligned}$$

(ii) Suppose that $F > \frac{|c|}{4}$, by using well-known triangle inequality and the inequalities: $\left(\frac{3}{2}\right)_k \geq \frac{3(k+1)}{4}$, $(F)_k \geq F^k$, $(k \in \mathbb{N})$, we get

$$\begin{aligned} \left| f'_{v,d,c}(z) \right| &= \left| 1 + \sum_{k=1}^{\infty} \frac{(k+1)(-c/4)^k}{(3/2)_k\,(F)_k} z^k \right| \leq 1 + \frac{4}{3} \frac{|c|}{4F} \sum_{k=1}^{\infty} \left(\frac{|c|}{4F} \right)^{k-1} \\ &= \frac{12F + |c|}{3\,(4F - |c|)}, \quad \left(F > \frac{|c|}{4} \right) \end{aligned} \tag{1.9}$$

(iii) In order to prove the part (iii) of Lemma 1, we make use of the well-known triangle inequality and the inequalities

$$(k+1)\,(3/2)_k \geq 2\,(3/2)^k\,, (F)_k \geq F^k, \ (k \in \mathbb{N})\,.$$

We thus find

$$\begin{aligned} \left| \Lambda[f_{v,d,c}](z) \right| &= \left| z + \sum_{k=1}^{\infty} \frac{(-c/4)^k}{(k+1)\,(3/2)_k\,(F)_k} z^{k+1} \right| \leq 1 + \sum_{k=1}^{\infty} \frac{|c|^k}{2\,(3/2)^k\,(4F)^k} \\ &= 1 + \frac{1}{2} \frac{|c|}{6F} \sum_{k=1}^{\infty} \left(\frac{|c|}{6F} \right)^{k-1} \\ &= \frac{12F - |c|}{12F - 2|c|}\,, \quad \left(F > \frac{|c|}{6} \right). \end{aligned}$$

□

2. Main results

Theorem 1. *If the parameters $d, v \in \mathbb{R}$, $c \in \mathbb{C}$ and , $F = v + (d+2)/2 \neq 0, -1, -2, ...$ are so constrained that $F > \frac{|c|}{3}$, then*

$$\Re \left\{ \frac{f_{v,d,c}(z)}{\left(f_{v,d,c}\right)_n(z)} \right\} \geq \frac{6F - 2|c|}{6F - |c|} \quad (z \in \mathcal{U})\,, \tag{2.1}$$

and

$$\Re\left\{\frac{\left(f_{v,d,c}\right)_n(z)}{f_{v,d,c}(z)}\right\} \geq \frac{6F-|c|}{6F} \quad (z\in\mathcal{U}). \tag{2.2}$$

Proof. Firstly, we know that for $z\in\mathcal{U}$ the image domains of the functions $z\longmapsto \left(\frac{f_{v,d,c}}{\left(f_{v,d,c}\right)_n}\right)(z)$ and $z\longmapsto\left(\frac{\left(f_{v,d,c}\right)_n}{f_{v,d,c}}\right)(z)$ do not contain the origin. Because

$$\frac{f_{v,d,c}(z)}{\left(f_{v,d,c}\right)_n(z)}=\frac{1+\sum_{k=1}^{\infty}b_kz^k}{1+\sum_{k=1}^{n}b_kz^k}=1+...,$$

and so

$$\frac{\left(f_{v,d,c}\right)_n(z)}{f_{v,d,c}(z)}=\frac{1+\sum_{k=1}^{n}b_kz^k}{1+\sum_{k=1}^{\infty}b_kz^k}=1+....$$

It means that the functions $f_{v,d,c}$ and $\left(f_{v,d,c}\right)_n$ does not vanish for $z\in\mathcal{U}$.

Now, we consider from part (i) of Lemma 1 that

$$1+\sum_{k=1}^{\infty}|b_k|\leq\frac{6F}{6F-|c|},$$

which is equivalent to

$$\frac{6F-|c|}{|c|}\sum_{k=1}^{\infty}|b_k|\leq 1,$$

where $b_k=\frac{(-c/4)^k}{(3/2)_k(F)_k}$.

We may write

$$\begin{aligned}&\frac{6F-|c|}{|c|}\left[\frac{f_{v,d,c}(z)}{\left(f_{v,d,c}\right)_n(z)}-\frac{6F-2|c|}{6F-|c|}\right]\\ =&\ \frac{1+\sum_{k=1}^{n}b_kz^k+\frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}b_kz^k}{1+\sum_{k=1}^{n}b_kz^k}\\ :&=\frac{1+w(z)}{1-w(z)}.\end{aligned}$$

So that,

$$w(z)=\frac{\frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}b_kz^k}{2+2\sum_{k=1}^{n}b_kz^k+\frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}b_kz^k}$$

and

$$|w(z)|\leq\frac{\frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}|b_k|}{2-2\sum_{k=1}^{n}|b_k|-\frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}|b_k|}.$$

Now $|w(z)| \leq 1$ if and only if

$$2\frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}|b_k| \leq 2-2\sum_{k=1}^{n}|b_k|,$$

which is equivalent to

$$\sum_{k=1}^{n}|b_k| + \frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}|b_k| \leq 1. \tag{2.3}$$

It suffices to show that the left hand side of (2.3) is bounded above by $\frac{6F-|c|}{|c|}\sum_{k=1}^{\infty}|b_k|$, which is equivalent to

$$\frac{6F-2|c|}{|c|}\sum_{k=1}^{n}|b_k| \geq 0.$$

To prove the result (2.2), we write

$$\begin{aligned}
&\frac{6F}{|c|}\left[\frac{\left(f_{\upsilon,d,c}\right)_n(z)}{f_{\upsilon,d,c}(z)} - \frac{6F-|c|}{6F}\right] \\
&= \frac{1+\sum_{k=1}^{n} b_k z^k - \frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty} b_k z^k}{1+\sum_{k=1}^{\infty} b_k z^k} \\
&: \quad = \frac{1+w(z)}{1-w(z)}
\end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{6F}{|c|}\sum_{k=n+1}^{\infty}|b_k|}{2-2\sum_{k=1}^{n}|b_k| - \frac{6F-2|c|}{|c|}\sum_{k=n+1}^{\infty}|b_k|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{k=1}^{n}|b_k| + \frac{6F-|c|}{|c|}\sum_{k=n+1}^{\infty}|b_k| \leq 1 \tag{2.4}$$

Since the left hand side of (2.4) is bounded above by $\frac{6F-|c|}{|c|}\sum_{k=1}^{\infty}|b_k|$, the proof is completed. □

Theorem 2. *If the parameters* $d, \upsilon \in \mathbb{R}$, $c \in \mathbb{C}$ *and* , $F = \upsilon + (d+2)/2 \neq 0, -1, -2, \ldots$ *are so constrained that* $F > \frac{7|c|}{12}$ *then*

$$\Re\left\{\frac{f'_{\upsilon,d,c}(z)}{\left(f_{\upsilon,d,c}\right)'_n(z)}\right\} \geq \frac{12F-7|c|}{3(4F-|c|)} \quad (z \in \mathcal{U}), \tag{2.5}$$

and

$$\Re\left\{\frac{\left(f_{v,d,c}\right)_n'(z)}{f_{v,d,c}'(z)}\right\} \geq \frac{12F-3|c|}{12F+|c|} \quad (z \in \mathcal{U}). \tag{2.6}$$

Proof. Similar to in the proof of Theorem 1 we know that for $z \in \mathcal{U}$ the image domains of the functions $z \longmapsto \left(\frac{f_{v,d,c}'}{\left(f_{v,d,c}\right)_n'}\right)(z)$ and $z \longmapsto \left(\frac{\left(f_{v,d,c}\right)_n'}{f_{v,d,c}'}\right)(z)$ do not contain the origin. Because

$$\frac{f_{v,d,c}'(z)}{\left(f_{v,d,c}\right)_n'(z)} = \frac{1+\sum_{k=1}^{\infty}(k+1)b_k z^k}{1+\sum_{k=1}^{n}(k+1)b_k z^k} = 1+\ldots,$$

and

$$\frac{\left(f_{v,d,c}\right)_n'(z)}{f_{v,d,c}'(z)} = \frac{1+\sum_{k=1}^{n}(k+1)b_k z^k}{1+\sum_{k=1}^{\infty}(k+1)b_k z^k} = 1+\ldots.$$

So that the functions $f_{v,d,c}'$ and $\left(f_{v,d,c}\right)_n'$ does not vanish for $z \in \mathcal{U}$.

By using part (ii) of Lemma 1 we observe that

$$1+\sum_{k=1}^{\infty}(k+1)|b_k| \leq \frac{12F+|c|}{3(4F-|c|)},$$

which is equivalent to

$$\frac{3(4F-|c|)}{4|c|}\sum_{k=1}^{\infty}(k+1)|b_k| \leq 1,$$

where $b_k = \frac{(-c/4)^k}{(3/2)_k (F)_k}$.

Now, we write

$$\begin{aligned}
&\frac{3(4F-|c|)}{4|c|}\left[\frac{f_{v,d,c}'(z)}{\left(f_{v,d,c}\right)_n'(z)} - \frac{12F-7|c|}{3(4F-|c|)}\right] \\
= \; &\frac{1+\sum_{k=1}^{n}(k+1)b_k z^k + \frac{3(4F-|c|)}{4|c|}\sum_{k=n+1}^{\infty}(k+1)b_k z^k}{1+\sum_{k=1}^{n}(k+1)b_k z^k} \\
: \; = \; &\frac{1+w(z)}{1-w(z)},
\end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{3(4F-|c|)}{4|c|}\sum_{k=n+1}^{\infty}(k+1)|b_k|}{2-2\sum_{k=1}^{n}(k+1)|b_k| - \frac{3(4F-|c|)}{4|c|}\sum_{k=n+1}^{\infty}(k+1)|b_k|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{k=1}^{n}(k+1)|b_k| + \frac{3(4F-|c|)}{4|c|}\sum_{k=n+1}^{\infty}(k+1)|b_k| \leq 1. \tag{2.7}$$

It suffices to show that the left hand side of (2.7) is bounded above by $\frac{3(4F-|c|)}{4|c|}\sum_{k=1}^{\infty}(k+1)|b_k|$, which is equivalent to

$$\frac{12F-7|c|}{4|c|}\sum_{k=1}^{n}(k+1)|b_k| \geq 0.$$

To prove the result (2.6), we write

$$\begin{aligned} & \frac{12F+|c|}{4|c|}\left[\frac{\left(f_{v,d,c}\right)_n'(z)}{f_{v,d,c}'(z)} - \frac{3(4F-|c|)}{12F+|c|}\right] \\ : &= \frac{1+w(z)}{1-w(z)}, \end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{12F+|c|}{4|c|}\sum_{k=n+1}^{\infty}(k+1)|b_k|}{2-2\sum_{k=1}^{n}(k+1)|b_k| - \frac{12F-7|c|}{4|c|}\sum_{k=n+1}^{\infty}(k+1)|b_k|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{k=1}^{n}(k+1)|b_k| + \frac{3(4F-|c|)}{4|c|}\sum_{k=n+1}^{\infty}(k+1)|b_k| \leq 1. \tag{2.8}$$

Since the left hand side of (2.8) is bounded above by $\frac{3(4F-|c|)}{4|c|}\sum_{k=1}^{\infty}(k+1)|b_k|$, the proof is completed. □

Theorem 3. *If the parameters $d, v \in \mathbb{R}$, $c \in \mathbb{C}$ and , $F = v + (d+2)/2 \neq 0, -1, -2, \ldots$ are so constrained that $F > \frac{|c|}{4}$, then*

$$\Re\left\{\frac{\Lambda[f_{v,d,c}](z)}{\left(\Lambda[f_{v,d,c}]\right)_n(z)}\right\} \geq \frac{12F-3|c|}{12F-2|c|} \quad (z \in \mathcal{U}), \tag{2.9}$$

and

$$\Re\left\{\frac{\left(\Lambda[f_{v,d,c}]\right)_n(z)}{\Lambda[f_{v,d,c}](z)}\right\} \geq \frac{12F-2|c|}{12F-|c|} \quad (z \in \mathcal{U}), \tag{2.10}$$

where $\Lambda[f_{v,d,c}]$ is the Alexander transform of $f_{v,d,c}$.

Proof. We prove only (2.9), which is similar in spirit to the proof of Theorem 1. The proof of (2.10) follows the pattern of that in (2.2).

We consider from part (iii) of Lemma 1 that

$$1+\sum_{k=1}^{\infty}\frac{|b_k|}{k+1}\leq\frac{12F-|c|}{12F-2|c|},$$

which is equivalent to

$$\frac{12F-2|c|}{|c|}\sum_{k=1}^{\infty}\frac{|b_k|}{k+1}\leq 1,$$

where $b_k=\frac{(-c/4)^k}{(3/2)_k(F)_k}$.

We may write

$$\begin{aligned}
&\frac{12F-2|c|}{|c|}\left[\frac{\Lambda[f_{v,d,c}](z)}{\left(\Lambda[f_{v,d,c}]\right)_n(z)}-\frac{12F-3|c|}{12F-2|c|}\right]\\
&=\frac{1+\sum_{k=1}^{n}\frac{b_k}{k+1}z^k+\frac{12F-2|c|}{|c|}\sum_{k=n+1}^{\infty}\frac{b_k}{k+1}z^k}{1+\sum_{k=1}^{n}\frac{b_k}{k+1}z^k}\\
&:=\frac{1+w(z)}{1-w(z)},
\end{aligned}$$

where

$$|w(z)|\leq\frac{\frac{12F-2|c|}{|c|}\sum_{k=n+1}^{\infty}\frac{b_k}{k+1}z^k}{2-2\sum_{k=1}^{n}\frac{|b_k|}{k+1}-\frac{12F-2|c|}{|c|}\sum_{k=n+1}^{\infty}\frac{|b_k|}{k+1}}\leq 1.$$

The last inequality is equivalent to

$$\sum_{k=1}^{n}\frac{|b_k|}{k+1}+\frac{12F-2|c|}{|c|}\sum_{k=n+1}^{\infty}\frac{|b_k|}{k+1}\leq 1. \tag{2.11}$$

It suffices to show that the left hand side of (2.11) is bounded above by $\frac{12F-2|c|}{|c|}\sum_{k=1}^{\infty}\frac{|b_k|}{k+1}$, which is equivalent to

$$\frac{12F-2|c|}{|c|}\sum_{k=1}^{n}\frac{|b_k|}{k+1}\geq 0.$$

□

2.1. *Struve functions*

Choosing $d=c=1$, in (1.3) or (1.4), we obtain the Struve function $H_v(z)$ of the first kind of order v defined by (1.5). Let $\mathcal{H}_v:\mathcal{U}\longrightarrow\mathbb{C}$ be defined by

$$\mathcal{H}_v(z)=f_{v,1,1}(z)=2^v\sqrt{\pi}\,\Gamma(p+3/2)z^{\frac{-v+1}{2}}H_v(\sqrt{z}).$$

We observe that

$$\begin{aligned}\mathcal{H}_{-1/2}(z) &= \sqrt{z}\sin\sqrt{z}, \quad \mathcal{H}_{1/2}(z) = 2(1-\cos\sqrt{z}),\\ \mathcal{H}_{3/2}(z) &= 4\left(1+\frac{2}{z}\right) - 8\left(\frac{\sin\sqrt{z}}{\sqrt{z}} + \frac{\cos\sqrt{z}}{z}\right).\end{aligned}$$

In particular, the results of Theorems 1-3 become:

Corollary 1. *The following assertions hold true:*
(i) If $v > -\frac{7}{6}$, then

$$\Re\left\{\frac{\mathcal{H}_v(z)}{(\mathcal{H}_v)_n(z)}\right\} \geq \frac{6v+7}{6v+8} \quad (z \in \mathcal{U}), \tag{2.12}$$

and

$$\Re\left\{\frac{(\mathcal{H}_v)_n(z)}{\mathcal{H}_v(z)}\right\} \geq \frac{6v+8}{6v+9} \quad (z \in \mathcal{U}). \tag{2.13}$$

(ii) If $v > \frac{-11}{12}$ then

$$\Re\left\{\frac{\mathcal{H}_v'(z)}{\left(\mathcal{H}_v'\right)_n(z)}\right\} \geq \frac{12v+11}{12v+15} \quad (z \in \mathcal{U}), \tag{2.14}$$

and

$$\Re\left\{\frac{\left(\mathcal{H}_v'\right)_n(z)}{\mathcal{H}_v'(z)}\right\} \geq \frac{12v+15}{12v+19} \quad (z \in \mathcal{U}). \tag{2.15}$$

(iii) If $v > -\frac{5}{4}$, then

$$\Re\left\{\frac{\Lambda[\mathcal{H}_v](z)}{(\Lambda[\mathcal{H}_v])_n(z)}\right\} \geq \frac{12v+15}{12v+16} \quad (z \in \mathcal{U}), \tag{2.16}$$

and

$$\Re\left\{\frac{(\Lambda[\mathcal{H}_v])_n(z)}{\Lambda[\mathcal{H}_v](z)}\right\} \geq \frac{12v+16}{12v+17} \quad (z \in \mathcal{U}). \tag{2.17}$$

Remark 1. For $v = 1/2$ we have $\mathcal{H}_{1/2}(z) = 2(1-\cos\sqrt{z})$ and $\left(\mathcal{H}_{1/2}\right)_0(z) = z$. From (2.12) and (2.13) we obtain

$$\Re\left\{\frac{1-\cos\sqrt{z}}{z}\right\} \geq \frac{5}{11} \approx 0.45455 \quad (z \in \mathcal{U}), \tag{2.18}$$

and

$$\Re\left\{\frac{z}{1-\cos\sqrt{z}}\right\} \geq \frac{11}{6} \approx 1.8333 \quad (z \in \mathcal{U}). \tag{2.19}$$

Furthermore we have $\mathcal{H}'_{1/2}(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$ and $\left(\mathcal{H}'_{1/2}\right)_1(z) = 1 - \frac{1}{6}z$, so from (2.14) and (2.15) we obtain

$$\Re\left\{\frac{\sin\sqrt{z}}{\sqrt{z}\left(1-\frac{1}{6}z\right)}\right\} \geq \frac{17}{21} \approx 0,809 \ (z \in \mathcal{U}), \tag{2.20}$$

and

$$\Re\left\{\frac{\sqrt{z}\left(1-\frac{1}{6}z\right)}{\sin\sqrt{z}}\right\} \geq \frac{21}{25} = 0,84 \ \ (z \in \mathcal{U}). \tag{2.21}$$

2.2. *Modified Struve functions*

Taking $d = 1$ and $c = -1$, in (1.3) or (1.4), we get the modified Struve function $L_v(z)$ of the first kind of order v defined by (1.6). Let the function $\mathcal{L}_v : \mathcal{U} \longrightarrow \mathbb{C}$ be defined by

$$\mathcal{L}_v(z) = f_{v,1,-1}(z) = 2^v\sqrt{\pi}\,\Gamma(v+3/2)z^{\frac{-v+1}{2}}L_v(\sqrt{z}).$$

The properties of the function $\mathcal{L}_v$ are the same like for the function $\mathcal{H}_v$, because in this case we have $|c| = 1$. More precisely, we have the following results.

Corollary 2. *The following assertions are true:*
(i) If $v > -\frac{7}{6}$, then

$$\Re\left\{\frac{\mathcal{L}_v(z)}{(\mathcal{L}_v)_n(z)}\right\} \geq \frac{6v+7}{6v+8} \quad (z \in \mathcal{U}), \tag{2.22}$$

and

$$\Re\left\{\frac{(\mathcal{L}_v)_n(z)}{\mathcal{L}_v(z)}\right\} \geq \frac{6v+8}{6v+9} \quad (z \in \mathcal{U}). \tag{2.23}$$

(ii) If $v > \frac{-11}{12}$ then

$$\Re\left\{\frac{\mathcal{L}'_v(z)}{\left(\mathcal{L}'_v\right)_n(z)}\right\} \geq \frac{12v+11}{12v+15} \quad (z \in \mathcal{U}), \tag{2.24}$$

and

$$\Re\left\{\frac{\left(\mathcal{L}'_v\right)_n(z)}{\mathcal{L}'_v(z)}\right\} \geq \frac{12v+15}{12v+19} \quad (z \in \mathcal{U}). \tag{2.25}$$

(iii) If $v > -\frac{5}{4}$, then

$$\Re\left\{\frac{\Lambda[\mathcal{L}_v](z)}{(\Lambda[\mathcal{L}_v])_n(z)}\right\} \geq \frac{12v+15}{12v+16} \quad (z \in \mathcal{U}), \tag{2.26}$$

and

$$\Re\left\{\frac{(\Lambda[\mathcal{L}_v])_n(z)}{\Lambda[\mathcal{L}_v](z)}\right\} \geq \frac{12v+16}{12v+17} \quad (z \in \mathcal{U}). \tag{2.27}$$

Remark 2. If we take $\upsilon = 1/2$ we have $\mathcal{L}_{1/2}(z) = 2(\cosh\sqrt{z} - 1)$ and for $n = 0$, we get $\left(\mathcal{L}_{1/2}\right)_0(z) = z$, so, from (2.22) and (2.23) we obtain

$$\Re\left\{\frac{\cosh\sqrt{z}-1}{z}\right\} \geq \frac{5}{11} \approx 0.45455 \quad (z \in \mathcal{U}), \tag{2.28}$$

and

$$\Re\left\{\frac{z}{\cosh\sqrt{z}-1}\right\} \geq \frac{11}{6} \approx 1.8333 \quad (z \in \mathcal{U}). \tag{2.29}$$

Furthermore we have $\mathcal{L}'_{1/2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}$ and $\left(\mathcal{L}'_{1/2}\right)_1(z) = 1 + \frac{1}{6}z$, so from (2.24) and (2.25) we get

$$\Re\left\{\frac{\sinh\sqrt{z}}{\sqrt{z}\left(1+\frac{1}{6}z\right)}\right\} \geq \frac{17}{21} \approx 0,809 \quad (z \in \mathcal{U}), \tag{2.30}$$

and

$$\Re\left\{\frac{\sqrt{z}\left(1+\frac{1}{6}z\right)}{\sinh\sqrt{z}}\right\} \geq \frac{21}{25} = 0,84 \quad (z \in \mathcal{U}). \tag{2.31}$$

3. Illustrative examples and image domains

In this section, we present four illustrative examples along with the geometrical descriptions of the image domains of the unit disk by the ratio of Struve (modified Struve) function to its sequence of partial sums or the ratio of its sequence of partial sums to the function which we considered in our remarks in Section 2.

Example 1. The image domain of the unit disk under the function $f_1(z) = \frac{1-\cos\sqrt{z}}{z}$, $(z \in \mathcal{U})$ is shown in Figure 1.

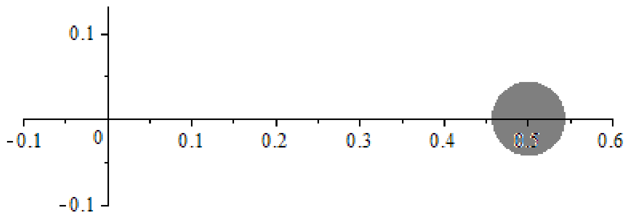

Figure 1.

Example 2. The image domain of the unit disk under the function $f_2(z) = \frac{z}{1-\cos\sqrt{z}}$, $(z \in \mathcal{U})$ is shown in Figure 2.

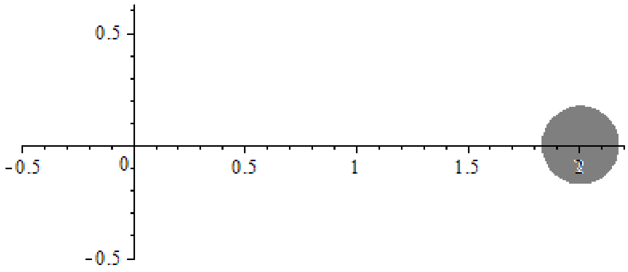

FIGURE 2.

Example 3. The image domain of the unit disk under the function $f_3(z) = \frac{\cosh\sqrt{z}-1}{z}$, $(z \in \mathcal{U})$ is shown in Figure 3.

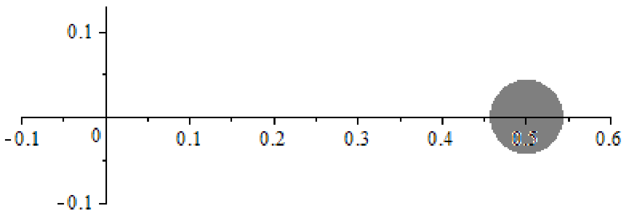

FIGURE 3.

Example 4. The image domain of the unit disk under the function $f_4(z) = \frac{z}{\cosh\sqrt{z}-1}$, $(z \in \mathcal{U})$ is shown in Figure 4.

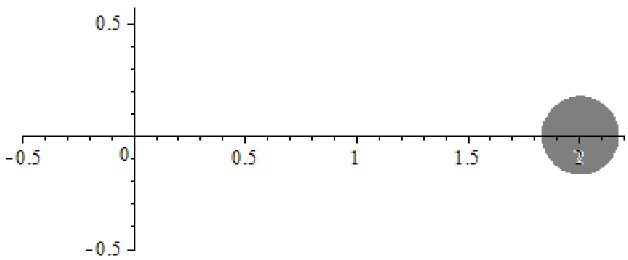

FIGURE 4.

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