MODULES THAT HAVE A WEAK SUPPLEMENT IN EVERY EXTENSION

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Abstract. We say that over an arbitrary ring a module $M$ has the property (WE) (respectively, (WEE)) if $M$ has a weak supplement (respectively, ample weak supplements) in every extension. In this paper, we provide various properties of modules with these properties. We show that a module $M$ has the property (WEE) iff every submodule of $M$ has the property (WE). A ring $R$ is left perfect iff every left $R$-module has the property (WE) iff every left $R$-module has the property (WEE). A ring $R$ is semilocal iff every left $R$-module has a weak supplement in every extension with small radical. We also study modules that have a weak supplement (respectively, ample weak supplements) in every coatomic extension, namely the property (WE) (respectively, (WEE)).

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1. INTRODUCTION

Throughout this paper, $R$ is an associative ring with identity and all modules are unital left $R$-modules, unless otherwise stated. Let $M$ be an $R$-module. The notation $U \leq M$ means that $U$ is a submodule of $M$. A submodule $U$ of $M$ is called small in $M$, denoted as $U \ll M$, if $M \neq U + L$ for every proper submodule $L$ of $M$. By $\text{Rad}(M)$ we denote the intersection of all maximal submodules of $M$, equivalently the sum of all small submodules of $M$ (see [14]). A module $M$ is called radical if $M$ has no maximal submodules, that is, $M = \text{Rad}(M)$.

As a proper generalization of direct summands of a module, the notion of supplement submodules is defined. For $U$, $V$ submodules of a module $M$, $V$ is called a supplement of $U$ in $M$ if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V \ll V$. Then, it is natural to introduce a generalization of supplement submodules by [14, Section 19.3.(2)]. A submodule $V$ of $M$ is called a weak supplement of $U$ in $M$ if $U + V = M$ and $U \cap V \ll M$. A module $M$ is called weakly supplemented if every submodule of $M$ has a weak supplement in $M$ (see [9], [14] and [17]). A submodule $U$ of $M$ has ample (weak) supplements in $M$.
if, whenever $M = U + L$, $L$ contains a (weak) supplement of $U$ in $M$. Under given definitions, we clearly have the following implication on submodules:

$$\text{direct summand} \implies \text{supplement} \implies \text{weak supplement}$$

Let $R$ be a ring and $M$ be an $R$-module. An $R$-module $N$ is called an extension of $M$ provided $M \subseteq N$. A module $M$ is said to be injective if it is a direct summand in its every extension $N$.

Modules that have a supplement (resp. ample supplements) in every extension, i.e. modules with the property $(E)$ (resp. $(EE)$), was first introduced by H. Zöschinger in [16], as a generalization of injective modules. The author determined in the same paper the structure of modules with these properties.

Adapting his concepts, we introduce the properties $(WE)$ and $(WEE)$ as a generalization of the properties $(E)$ and $(EE)$ in Section 2. We call a module that has the property $(WE)$ (resp. $(WEE)$) if it has a weak supplement (resp. ample weak supplements) in every extension. Moreover in this section, we show that a module $M$ has the property $(WEE)$ if and only if every submodule of $M$ has the property $(WE)$. This gives us that every submodule of a module with the property $(WEE)$ is weakly supplemented. We prove that the property $(WE)$ is inherited by direct summands. In Corollary 2, we obtain that if a ring $R$ is left hereditary, then every factor module of an $R$-module with the property $(WE)$ has the property $(WE)$. Thanks to Lemma 3.3 of Zöschinger’s paper [16], we directly say that over a complete local dedekind domain $R$, an $R$-module $M$ has the property $(WE)$ if and only if every left $R$-module has the property $(E)$. We also give new characterizations of left perfect rings via the modules with the properties $(WE)$ and $(WEE)$.

Let $R$ be a ring and $M$ be an $R$-module. R. Alizade et al. [1] say a submodule $U$ of $M$ cofinite in $M$ if the factor module $\frac{M}{U}$ is finitely generated. In [5], H. Çalışıcı and E. Türkmen called an extension $N$ of $M$ cofinite extension if $M$ is cofinite in $N$. Following [5], the authors studied modules that have a supplement (resp. ample supplements) in every cofinite extension, namely the property $(CE)$ (resp. $(CEE)$), as a generalization of the property $(E)$ (resp. $(EE)$). In addition, they showed in [5, Theorem 2.12] that a ring $R$ is semiperfect if and only if every left $R$-module has the property $(CE)$.

In [15], a module $M$ is said to be coatomic if $\text{Rad}(\frac{M}{K}) = \frac{M}{K}$ implies that $K = M$ for some submodule $K$ of $M$, that is, every radical factor module of $M$ is zero. $M$ is coatomic if and only if every proper submodule of $M$ is contained in a maximal submodule of $M$. Note that semisimple modules are coatomic.

Let $R$ be a ring and $M, N$ be $R$-modules. $N$ is called a coatomic extension of $M$ in case $M \subseteq N$ and $\frac{N}{M}$ is coatomic. In [11], B. N. Türkmen studied on modules that have a supplement (resp. ample supplements) in every coatomic extension and termed these modules $E^*$-modules (resp. $EE^*$-modules). Since finitely generated modules are coatomic, $E^*$-modules (resp. $EE^*$-modules) have the property $(CE)$ (resp. $(CEE)$).
In Section 3, we also call a module that has the property \((WE^*)\) (resp. \((WEE^*)\)) if it has a weak supplement (resp. ample weak supplements) in every coatomic extension. We prove that over a left \(V\)-ring \(R\), every left \(R\)-module with \((WE^*)\) is injective. In addition, we give also a characterization of semilocal rings via the modules that have a weak supplement in every extension with small radical. Finally, we give an example of modules that have a weak supplement in every extension with small radical but not have the property \((CEE)\).

2. MODULES WITH THE PROPERTIES \((WE)\) AND \((WEE)\)

It is shown in [16, Lemma 1.3.(a)] that direct summands of modules with the property \((E)\) have the property \((E)\). Now we give an analogue of this fact for the modules with the property \((WE)\).

**Proposition 1.** Let \(M\) be a module. If \(M\) has the property \((WE)\), then every direct summand of \(M\) has the property \((WE)\).

**Proof.** Let \(M_1\) be a direct summand of \(M\). Then there exists a submodule \(M_2\) of \(M\) such that \(M = M_1 \oplus M_2\). Let \(N\) be any extension of \(M_1\). Let \(N'\) be the external direct sum \(N \oplus M_2\) and \(\delta : M \to N'\) be the canonical embedding. Then \(M \cong \delta(M)\) has the property \((WE)\). Hence, there exists a submodule \(V\) of \(N'\) such that \(N' = \delta(M) + V\) and \(\delta(M) \cap V \ll N'\). By the projection \(\pi : N' \to N\), we have that \(M_1 + \pi(V) = N\). Also since \(\text{Ker}(\pi) \subseteq \delta(M)\), \(\pi(\delta(M) \cap V) = \pi(\delta(M)) \cap \pi(V) = M_1 \cap \pi(V) \ll N\). Hence \(\pi(V)\) is a weak supplement of \(M_1\) in \(N\). □

**Proposition 2.** A module \(M\) has the property \((WEE)\) if and only if every submodule of \(M\) has the property \((WE)\).

**Proof.** Suppose that every submodule of \(M\) has the property \((WE)\). For any extension \(N\) of \(M\), let \(N = M + K\) for some submodule \(K\) of \(N\). Since \(M \cap K\) has the property \((WE)\), there exists a submodule \(L\) of \(K\) such that \((M \cap K) + L = K\) and \((M \cap K) \cap L = M \cap L \ll K\). Note that \(N = M + K = M + ((M \cap K) + L) = M + L\). It follows that \(L\) is a weak supplement of \(M\) in \(N\).

Conversely, let \(M\) be a module with the property \((WEE)\) and \(M_1\) be any submodule of \(M\). For any extension \(N\) of \(M_1\), let \(F = \frac{M \oplus N}{H}\), where the submodule \(H\) is the set of all elements \((m',-m')\) of \(M \oplus N\) with \(m' \in M_1\) and let \(\gamma : M \to F\) via \(\gamma(m) = (m,0) + H\), \(\psi : N \to F\) via \(\psi(n) = (0,n) + H\) for all \(m \in M, n \in N\). For inclusion homomorphisms \(\iota_1 : M_1 \to N\) and \(\iota_2 : M_1 \to M\), we can draw the following pushout:

\[
\begin{array}{c}
M_1 \xrightarrow{\iota_1} N \\
\downarrow \iota_2 \\
M \xrightarrow{\gamma} F
\end{array}
\]
It is clear that $F = \text{Im}(\gamma) + \text{Im}(\psi)$. Since $\gamma$ is monomorphism, by assumption, $\text{Im}(\gamma)$ has the property (WEE). It means that $\text{Im}(\gamma)$ has a weak supplement $V$ in $F$ such that $V \leq \text{Im}(\psi)$, i.e. $F = \text{Im}(\gamma) + V$ and $\text{Im}(\gamma) \cap V \ll F$. Then we obtain that $N = \psi^{-1}(\text{Im}(\gamma)) + \psi^{-1}(V) = M_1 + \psi^{-1}(V)$ and $M_1 \cap \psi^{-1}(V) \ll N$. Hence $\psi^{-1}(V)$ is a weak supplement of $M_1$ in $N$.

**Corollary 1.** Every submodule of a module with the property (WEE) is weakly supplemented.

**Lemma 1.** Every simple submodule $S$ of a module $M$ is either a direct summand of $M$ or small in $M$.

**Proof.** Suppose that $S$ is not small in $M$, then there exists a proper submodule $K$ of $M$ such that $S + K = M$. Since $S$ is simple and $K \not= M$, $S \cap K = 0$. Thus $M = S \oplus K$.

Let $R$ be a ring and $M$ be an $R$-module. $M$ is called *local* if the sum of all proper submodules of $M$ is a proper submodule of $M$. $R$ is called a *local ring* if $R$ (or $RR$) is a local module.

**Proposition 3.** Local modules have the property (WE).

**Proof.** Let $S$ be a module and $N$ be any extension of $S$. If $S$ is small in $N$, $N$ is a weak supplement of $S$ in $N$. Suppose that $S$ is not small in $N$. Then there is a proper submodule $S'$ of $N$ such that $S + S' = N$. From Lemma 1, if $S$ is simple, $S'$ is a direct summand of $N$. If $S$ is local, $S' \cap S'$ is small in $S$. In both cases, $S'$ is a weak supplement of $S$ in $N$.

Let $M$ be a module and $U$ be a submodule of $M$. If the factor module $M/U$ has the property (WE), $M$ does not need to have the property (WE). For example, for the ring $R = \mathbb{Z}$, the $R$-module $M = \mathbb{Z}/2\mathbb{Z}$ has a weak supplement in every extension because it is simple. But $2\mathbb{Z}$ does not have a weak supplement in its extension $\mathbb{Z}$. Now we show that the statement mentioned above is true under a special condition.

**Proposition 4.** Let $M$ be a module and $U$ be a submodule of $M$. If $U \ll M$ and the factor module $M/U$ has the property (WE), $M$ has the property (WE).

**Proof.** Let $N$ be any extension of $M$. Since $M/U$ has the property (WE), there exists a submodule $V$ of $N$ such that $M/V = N/U$ and $M \cap V \ll N/U$. Note that $M + V = N$. Suppose that $M \cap V + S = N$ for a submodule $S$ of $N$. Then we obtain $M \cap U + S \cap U = N$. Since $M \cap U \ll N/U$, we have that $S \cap U = N/U$. By hypothesis, it follows that $N = S + U = S$. Hence $M \cap V \ll N$.

For a module $M$, we will denote by $\text{Soc}(M)$ the sum of all simple submodules of $M$. Note that $\text{Soc}(M)$ is the largest semisimple submodule of $M$. 
Remark 1. Let $M$ be a finitely generated semisimple module. Then $M$ is artinian. Since artinian modules have the property $(E)$, it has the property $(WE)$. Note that here the condition "finitely generated" is necessary. For example, consider the left $\mathbb{Z}$-module $M = \prod_{p \in \Omega} \mathbb{Z}/p\mathbb{Z}$, where $\Omega$ is the set of all prime numbers. Then, the semisimple module $\text{Soc}(M) = \bigoplus_{p \in \Omega} \mathbb{Z}/p\mathbb{Z}$. By [3, Lemma 2.9], there exists a submodule $N$ of $M$ such that $N_{\text{Soc}(M)} \cong \mathbb{Q}$. If $\text{Soc}(M)$ has a weak supplement $K$ in $N$, we have $N = \text{Soc}(M) \oplus K$ since $\text{Rad}(M) = 0$. Therefore, $K$ is injective and so $K = \text{Rad}(K) \subseteq \text{Rad}(M) = 0$, a contradiction.

In [7] a ring $R$ is said to be a left $V$-ring if every simple left $R$-module is injective. It is well known that a ring $R$ is a left $V$-ring if and only if $\text{Rad}(M) = 0$ for every left $R$-module $M$. A ring $R$ is called left hereditary if every left ideal of $R$ is projective. $R$ is a left hereditary ring if and only if every factor module of an injective left $R$-module is injective [14, Section 39.16].

The next example shows that every factor module of a module with the property $(WE)$ does not need to have the property $(WE)$. Firstly we need the following lemma.

Lemma 2. Let $R$ be a left $V$-ring. An $R$-module $M$ has the property $(WE)$ if and only if $M$ is injective.

Proof. Let $M$ has the property $(WE)$ and $N$ be any extension of $M$. Then $M$ has a weak supplement $V$ in $N$. We have $M + V = N$, $M \cap V \ll N$. Hence $M \cap V \subseteq \text{Rad}(N)$. Since $\text{Rad}(N) = 0$, we have $N = M \oplus V$.

Conversely, let $M$ be injective and $N$ be any extension of $M$. Then there exists a submodule $K$ of $N$ such that $N = M \oplus K$. Hence $K$ is a weak supplement of $M$ in $N$. □

Example 1. Let $R$ be the product of the family $\{F_i\}_{i \in I}$, where each $F_i$ is a field for an infinite index set $I$. The ring $R$ is a commutative Von Neumann regular but not hereditary [10, Example 2.15]. Then by [14, Section 23.5], $R$ is a left $V$-ring. $R$ is injective from [8, Corollary, 3.11.B]. By Lemma 2, the left $R$-module $RR$ has the property $(WE)$. Since $R$ is not hereditary, there is at least one factor module of $R$ which is not injective. This factor module does not have the property $(WE)$ by using Lemma 2.

Next we prove that under proper conditions a factor module of a module with the property $(WE)$ has the property $(WE)$.

Proposition 5. Let $K \subseteq M \subseteq L$ be modules with $L/K$ injective. If $M$ has the property $(WE)$, then $M/K$ has the property $(WE)$.

Proof. Let $N$ be any extension of $M/K$. Since $L/K$ is injective, by [10, Lemma 2.16] we have the following commutative diagram with exact rows:
Since $h$ is monomorphism and $M$ has the property $(WE)$, $M \cong Im(h)$ has a weak supplement $V$ in $P$, that is, $Im(h) + V = P$ and $Im(h) \cap V \ll P$. We claim that $g(V)$ is a weak supplement of $\frac{M}{K}$ in $N$. 

Corollary 2. If $R$ is a left hereditary ring and $M$ is an $R$-module with the property $(WE)$, then every factor module of $M$ has the property $(WE)$.

If a module $M$ has a supplement in its injective envelope, $M$ need not to have a weak supplement in every extension. For example, for the ring $R = \mathbb{Z}$, the $R$-module $M = 2\mathbb{Z}$ has a supplement in its injective envelope $\mathbb{Q}$. But $M = 2\mathbb{Z}$ does not have a weak supplement in its extension $\mathbb{Z}$. Now we prove that over a local Dedekind domain, a module $M$ has a supplement in its injective envelope if and only if $M$ has a weak supplement in every extension.

Lemma 3. Let $R$ be a local Dedekind domain and $M$ be an $R$-module. The following statements are equivalent:

1. $M$ has a supplement in its injective envelope.
2. $M$ has the property $(WE)$.
3. $M$ is an $E^*$-module.

Proof. It is clear by [16, Lemma 3.3]. □

Proposition 6. Let $R$ be a complete local Dedekind domain and $M$ be an $R$-module. $M$ has the property $(WE)$ if and only if $M$ has the property $(E)$.

Proof. Let $M$ has the property $(WE)$ and $N$ be any extension of $M$. Since $M$ has the property $(WE)$, there exists a submodule $X$ of $N$ such that $M + X = N$, $M \cap X \ll N$. By [16, Section 3, Corollary 5], there exists a supplement $V$ of $M$ in $N$ with $V \subset X$. Hence $M$ has the property $(E)$. □

Proposition 7. Let $R$ be a non-local Dedekind domain and $M$ be a semisimple $R$-module. Then, the following three statements are equivalent:

1. $M$ has the property $(WE)$.
2. $M$ has the property $(E)$.
3. $M$ is of the form $K \oplus \prod_p A_p$, where $K$ is injective and $A_p$ is a bounded $p$-primary module for every prime element $p \in R$. 

\begin{center}
\begin{tikzcd}
0 \arrow{r} & K \arrow{r}{\sigma} \arrow{d}{id} & M \arrow{r}{h} \arrow{d}{f} & M/K \arrow{r} \arrow{d}{g} & 0 \\
0 \arrow{r} & K \arrow{r}{p} & P \arrow{r}{g} & N \arrow{r} & 0
\end{tikzcd}
\end{center}
**Proof.** (1) $\iff$ (2) It follows from [12, Proposition 2.1].
(2) $\iff$ (3) By [16, Theorem 5.6].

It is known from [14, Section 43.9] that a ring $R$ is left perfect if and only if every left $R$-module has the property $(E)$. The next theorem gives new characterizations of left perfect rings via their modules which have the property $(WE)$.

**Theorem 1.** For a ring $R$ the following statements are equivalent:

1. $R$ is left perfect.
2. Every left $R$-module is weakly supplemented.
3. Every left $R$-module has the property $(WE)$.
4. $R^{(b)}$ is weakly supplemented.
5. $R^{(b)}$ has the property $(WEE)$.
6. Every left $R$-module has the property $(WEE)$.

**Proof.** (1) $\iff$ (2) $\iff$ (4) is clear from [4, Theorem 1]. (3) $\Rightarrow$ (6) and (5) $\Rightarrow$ (4) follow from Proposition 2. (1) $\Rightarrow$ (3) follows from [14, Section 43.9]. (6) $\Rightarrow$ (5) is clear.

The following definitions are given in the paper [6], and we recall them for the convenience of the reader:

By a **valuation ring** (also called a **chain ring**) we mean a commutative ring $R$ whose ideals are totally ordered by inclusion. Equivalently, if $a, b \in R$, then either $a \in Rb$ or $b \in Ra$. A valuation ring that is a domain will be called a **valuation domain**. A valuation ring $R$ is called **maximal** if $R$ is linearly compact, i.e., every family of cosets $\{a_i + L_i | i \in I\}$ with the finite intersection property has a non-empty intersection. Since linearly compact modules have ample supplements in every extension, a maximal ring $R$ has the property $(WEE)$.

The following example shows that a ring with the property $(WEE)$ need not be left perfect, in general.

**Example 2.** Let $R$ be the localization ring $\mathbb{Z}_{(p)}$ of the ring $\mathbb{Z}$ of integers at a prime ideal $p\mathbb{Z} \neq 0$. Then, the completion of $\mathbb{Z}_{(p)}$, the ring $J_{(p)}$ of $p$-adic integers, is a maximal valuation domain which is not field. Hence, $J_{(p)}$ has the property $(WEE)$ but not perfect.

### 3. Modules with the properties $(WE^*)$ and $(WEE^*)$

In this section, we study on modules with the property $(WE^*)$ (resp. $(WEE^*)$), which have a weak supplement (resp. ample weak supplements) in every coatomic extension, as a generalization of modules with the property $(WE)$ (resp. $(WEE)$). We prove that over a left $V$-ring $R$, every left $R$-module with the property $(WE^*)$ is injective.

**Proposition 8.** Let $M$ be a module. If $M$ has the property $(WE^*)$, then every direct summand of $M$ has the property $(WE^*)$. 
Proof. Let $M_1$ be a direct summand of $M$ and $N$ be a coatomic extension of $M_1$. Then there exists a submodule $M_2$ of $M$ such that $M = M_1 \oplus M_2$. Let $N'$ be the external direct sum $N \oplus M_2$ and $\varphi : M \to N'$ be the canonical embedding. Then $M \cong \varphi(M)$ has the property $\langle WE^* \rangle$. Note that $\frac{N}{M_1} \cong \frac{N \oplus M_2}{\varphi(M)} = \frac{N'}{\varphi(M)}$ is coatomic. Since $\varphi(M)$ has the property $\langle WE^* \rangle$, there exists a submodule $V$ of $N'$ such that $N' = \varphi(M) + V$ and $\varphi(M) \cap V \ll N$. For the projection $\phi : N' \to N$, we have that $M_1 + \phi(V) = N$. Also since $\text{Ker} \phi \subseteq \varphi(M)$, $\phi(\varphi(M) \cap V) \subseteq \phi(\varphi(M)) \cap \phi(V) = M_1 \cap \phi(V) \ll \phi(N') = N$. Hence $\phi(V)$ is a weak supplement of $M_1$ in $N$. □

**Proposition 9.** A module $M$ has the property $\langle WE^* \rangle$ if and only if every submodule of $M$ has the property $\langle WE^* \rangle$.

Proof. Assume that every submodule of $M$ has the property $\langle WE^* \rangle$. For a coatomic extension $N$ of $M$, let $N = M + V$ for some submodule $V$ of $N$. Then $\frac{N}{M} \cong \frac{V}{M \cap V}$ is coatomic and so $V$ is a coatomic extension of $M \cap V$. Since $M \cap V$ has the property $\langle WE^* \rangle$, there exists a submodule $K$ of $V$ such that $V = M \cap V + K$ and $M \cap K \ll V$. Note that $N = M + V = M + (M \cap V + K) = M + K$. It follows that $K$ is a weak supplement of $M$ in $N$.

Conversely, let $M$ be a module with the property $\langle WE^* \rangle$ and let $M_1$ be any submodule of $M$. For a coatomic extension $N$ of $M_1$, let $S = \frac{M \oplus N}{L}$, where the submodule $L$ is the set of all elements $(m', m)$ of $M \oplus N$ with $m' \in M_1$ and let $f : M \to S$ via $f(m) = (m, 0) + L$, $g : N \to S$ via $g(n) = (0, n) + L$ for all $m \in M, n \in N$. For the inclusion homomorphisms $\tau_1 : M_1 \to N$ and $\tau_2 : M_1 \to M$, we can draw the following pushout:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\tau_1} & N \\
\downarrow{\tau_2} & & \downarrow{g} \\
M & \xrightarrow{f} & S
\end{array}
$$

It is clear that $S = \text{Im}(f) + \text{Im}(g)$. Now we define $\theta : S \to \frac{N}{M}$ by $\theta((m, n) + L) = n + M_1$ for all $(m, n) + L \in S$. Note that $\theta$ is an epimorphism and $\text{Ker} \theta = \text{Im}(f)$. It follows that $\frac{N}{M} \cong \frac{S}{\text{Im}(f)}$ is coatomic. Since $f$ is monomorphism, by assumption, $\text{Im}(f)$ has the property $\langle WE^* \rangle$. Then it follows immediately that $\text{Im}(f)$ has a weak supplement $V$ in $S$ such that $V \subseteq \text{Im}(g)$, i.e. $S = \text{Im}(f) + V$ and $\text{Im}(f) \cap V \ll S$. Then we obtain that $N = g^{-1}(\text{Im}(f)) + g^{-1}(V) = M_1 + g^{-1}(V)$ and $M_1 \cap g^{-1}(V) \ll N$. Hence $g^{-1}(V)$ is a weak supplement of $M_1$ in $N$. □

Recall from [2] a module $M$ is called cofinitely weak supplemented if every cofinite submodule of $M$ has a weak supplement in $M$. It is clear from Proposition 9 that if a module $M$ has the property $\langle WE^* \rangle$, then every maximal submodule of $M$ has a weak supplement in $M$, equivalently $M$ is cofinitely weak supplemented by [2, Theorem 2.16].
In [13], a module $M$ is called weakly radical supplemented (namely wrs-module) if every submodule $U$ of $M$ with $\text{Rad}(M) \subseteq U$ has a weak supplement in $M$. A module $M$ is called semilocal if $\frac{M}{\text{Rad}(M)}$ is semisimple. A ring $R$ is semilocal if the left $R$-module $R$ is semilocal.

**Corollary 3.** Let $R$ be a semilocal ring and $M$ be an $R$-module. If $M$ has the property $(WE^*)$, then $M$ is wrs-module.

**Proof.** Let $U$ be a submodule of $M$ with $\text{Rad}(M) \subseteq U$. Since $R$ is semilocal ring, it follows from [9, Theorem 3.5] that $\frac{M}{U}$ is semisimple as a factor module of the semisimple module $\frac{M}{\text{Rad}(M)}$. Hence $\frac{M}{U}$ is coatomic. By assumption and Proposition 9, $U$ has a weak supplement in $M$. Hence $M$ is a wrs-module. □

**Proposition 10.** Over a left $V$-ring $R$, every left $R$-module with $(WE^*)$ is injective.

**Proof.** Let $M$ be an $R$-module with $(WE^*)$. Let $N$ be any extension of $M$. Suppose that $\text{Rad}(N) = K$ for a submodule $K$ of $N$. Since $R$ is a left $V$-ring, $\text{Rad}(N) = 0$. Then it immediately follows that $N = K$. Hence $N$ is coatomic. Then, by assumption, $M$ has a weak supplement $V$ in $N$, i.e. $N = M + V$ and $M \cap V \ll N$. Since $R$ is a left $V$-ring, we obtain that $M \cap V \subseteq \text{Rad}(N) = 0$. This completes the proof. □

The next result can be directly obtained from Proposition 10 and Lemma 2.

**Corollary 4.** Let $R$ be a left $V$-ring and $M$ be an $R$-module. The following statements are equivalent:

1. $M$ has the property $(WE)$.
2. $M$ has the property $(WE^*)$.
3. $M$ is injective.

Now we shall give a characterization for semilocal rings via the modules that have a weak supplement in every extension with small radical.

**Theorem 2.** For any ring $R$ the following statements are equivalent:

1. $R$ is semilocal.
2. Every left $R$-module with small radical is weakly supplemented.
3. Every left $R$-module has a weak supplement in every extension with small radical.

**Proof.** (1) $\Rightarrow$ (2) follows from [9, Theorem 3.5].

(2) $\Rightarrow$ (3) Let $M$ be a left $R$-module and $N$ be an extension of $M$ with small radical. By hypothesis, $M$ has a weak supplement in $N$. Conversely, let $M$ be an $R$-module with small radical and $U$ be a submodule of $M$. By assumption, $U$ has a weak supplement in $M$. □
Finally, we give an example of modules that have a weak supplement in every extension with small radical but not have the property $(CEE)$. 

Example 3. (see [14, Section 42.13, Exercise 4]). Let $R$ be the following subring of the rational numbers:

$$R = \{ \frac{m}{n} | m, n \in \mathbb{Z}, (m, n) = 1, 2 \text{ and } 3 \text{ are not divisors of } n \}$$

Since $\frac{R}{\text{Rad}(R)}$ is semisimple, the left $R$-module $RR$ is a module which has a weak supplement in every extension with small radical by Theorem 2. Whereas, since $R$ is not semiperfect, $RR$ does not have the property $(CEE)$ by [5, Theorem 2.12].

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