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# CLOSED IDEALS WITH BOUNDED $\Delta$-WEAK APPROXIMATE IDENTITIES IN SOME CERTAIN BANACH ALGEBRAS

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*Abstract.* It is shown that a locally compact group $G$ is amenable if and only if some certain closed ideals of the Figà-Talamanca-Herz algebra $A_p(G)$ admit bounded $\Delta$-weak approximate identities. Also, similar results are obtained for the function algebras $\mathfrak{L}A(G)$ and $C_0^w(G)$.

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## 1. Introduction and preliminaries

Let $A$ be a Banach algebra, $\Delta(A)$ be the character space of $A$, that is, the space of all non-zero homomorphisms from $A$ into $\mathbb{C}$ and $A^*$ be the dual space of $A$ consisting of all bounded linear functionals on $A$. Throughout the paper, $A$ is a commutative and semi-simple Banach algebra, hence $\Delta(A)$ is non-empty.

Let $\{e_\alpha\}$ be a net in Banach algebra $A$. The net $\{e_\alpha\}$ is called,

(1) an *approximate identity* if for each $a \in A$, $\|ae_\alpha - a\| \to 0$,
(2) a *weak approximate identity* if for each $a \in A$, $|f(ae_\alpha) - f(a)| \to 0$ for all $f \in A^*$,
(3) a $\Delta$-*weak approximate identity* if for each $\phi \in \Delta(A)$, $|\phi(e_\alpha) - 1| \to 0$.

**Definition 1.** Let $A$ be a Banach algebra. A bounded $\Delta$-weak approximate identity for subspace $E \subseteq A$ is a bounded net $\{e_\alpha\}$ in $E$ such that for each $a \in E$,

$$\lim_\alpha |\phi(ae_\alpha) - \phi(a)| = 0 \qquad (\phi \in \Delta(A)).$$

For simplicity of notation, let b.$\Delta$-w.a.i stand for a bounded $\Delta$-weak approximate identity.

It was proved that every Banach algebra $A$ with a bounded $\Delta$-weak approximate identity has a bounded approximate identity (b.a.i) and conversely [4, Proposition 33.2].

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The notion of a $\Delta$-weak approximate identity introduced and studied in [15] where an example of a Banach algebra which has a $\Delta$-weak approximate identity but does not have an approximate identity, was given. Indeed, if $S = \mathbb{Q}^+$ is the semigroup of positive rationales under addition, it was shown that the semigroup algebra $l^1(S)$ has a b.$\Delta$-w.a.i, but it does not have any bounded or unbounded approximate identity.

As the second example to see the difference between bounded approximate and bounded $\Delta$-weak approximate identities, let $\mathbb{R}$ be the additive real line group and $1 < p \leq \infty$. Put $S_p(\mathbb{R}) = L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and define the following norm;

$$\|f\|_{S_p} = \max\{\|f\|_1, \|f\|_p\} \qquad (f \in S_p(\mathbb{R})).$$

Using [14, Theorem 2.1], we can see that $S_p(\mathbb{R})$ has a b.$\Delta$-w.a.i, but it has no b.a.i. Because we know that $S_p(\mathbb{R})$ is a Segal algebra and it is well-known that a Segal algebra $S$ in $L^1(\mathbb{R})$ has a b.a.i if and only if $S = L^1(\mathbb{R})$. But it is clear that $S_p(\mathbb{R}) \neq L^1(\mathbb{R})$.

These type of approximate identities have some interesting applications, for example; see [9, 16, 22]. In the past decades, B. E. Forrest studied the relations between the amenability of a group $G$ and closed ideals of $A(G)$ and $A_p(G)$ with a b.a.i; see [5–8], and the relations between some properties of $G$ and closed ideals of $A(G)$ with a b.$\Delta$-w.a.i; see [9].

In this paper, we try to improve some of the theorems in [5, 6, 8, 10, 16] with changing b.a.i by b.$\Delta$-w.a.i. As an application, we give the converse of [8, Corollary 4.2] due to B. Forrest, E. Kaniuth, A. T. Lau and N. Spronk.

## 2. Main results

Let $G$ be a locally compact group. For $1 < p < \infty$, let $A_p(G)$ denote the subspace of $C_0(G)$ consisting of all functions of the form $u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i}$ where $f_i \in L^p(G)$, $g_i \in L^q(G)$, $1/p + 1/q = 1$, $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$ and $\widetilde{g}(x) = \overline{g(x^{-1})}$ for all $x \in G$. With the pointwise operation and the following norm,

$$\|u\|_{A_p(G)} = \inf\{\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q : u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i}\},$$

$A_p(G)$ is a Banach algebra called the Figà-Talamanca-Herz algebra. It is clear that $\|u\| \leq \|u\|_{A_p(G)}$ where $\|u\|$ is the uniform norm of $u \in C_0(G)$. By [12, Theorem 3], we know that

$$\Delta(A_p(G)) = \{\phi_x : x \in G\} = G,$$

where $\phi_x$ is defined by $\phi_x(f) = f(x)$ for each $f \in A_p(G)$.

The dual of the Banach algebra $A_p(G)$ is the Banach space $PM_p(G)$ consisting of all limits of convolution operators associated to bounded measures. Indeed, $PM_p(G)$ is the $w^*$-closure of $\lambda_p(L^1(G))$ in $B(L^p(G))$ where $\lambda_p$ is the left regular representation of $G$ on $L^p(G)$; see [3] for more details.

The group $G$ is said to be *amenable* if, there exists an $m \in L^{\infty}(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^{\infty}(G)$ where $L_x f(y) = f(x^{-1}y)$.

**Theorem 1** (Leptin-Herz)**.** *Let $G$ be a locally compact group and $1 < p < \infty$. Then $A_p(G)$ has a b.a.i if and only if $G$ is amenable*

The proof of the above theorem in the case $p = 2$ is due to Leptin [19] and in general is due to Herz [12].

Forrest and Skantharajah in [9] showed that if $G$ is a discrete group, then $A_2(G) = A(G)$ has a b.$\Delta$-w.a.i if and only if $G$ is amenable. Kaniuth and Ülger in [18, Theorem 5.1], for the first time in our knowledge, announced that $A(G)$ has a b.$\Delta$-w.a.i if and only if $G$ is an amenable group, but the same result holds for the Figà-Talamanca-Herz algebras as follows. The proof is similar to the Fourier algebra case, but we give it for the convenience of reader.

**Theorem 2.** *Let $G$ be a locally compact group and $1 < p < \infty$. Then $A_p(G)$ has a b.$\Delta$-w.a.i if and only if $G$ is amenable.*

*Proof.* Let $\{e_\alpha\}$ be a b.$\Delta$-w.a.i for $A_p(G)$ and $e \in A_p(G)^{**}$ be a $w^*$-cluster point of $\{e_\alpha\}$. So, for each $\phi \in \Delta(A_p(G)) = G$, we have

$$e(\phi) = \lim_{\alpha} \phi(e_\alpha) = 1,$$

because $\{e_\alpha\}$ is a b.$\Delta$-w.a.i for $A_p(G)$. Therefore, by [23, Proposition 2.8] $G$ is weakly closed in $PM_p(G) = A_p(G)^*$. Now, by [2, Corollary 2.8] we conclude that $G$ is an amenable group. □

The following corollary immediately follows from the Leptin-Herz Theorem and Theorem 2.

**Corollary 1.** *Let $G$ be a locally compact group and $1 < p < \infty$. Then $A_p(G)$ has a b.$\Delta$-w.a.i. if and only if it has a b.a.i.*

The following theorem is a key tool in the sequel.

**Theorem 3.** *Let $A$ be a Banach algebra, $I$ be a closed two-sided ideal of $A$ which has a b.$\Delta$-w.a.i and the quotient Banach algebra $A/I$ has a b.l.a.i. Then $A$ has a b.$\Delta$-w.a.i.*

*Proof.* Let $\{e_\alpha\}$ be a b.$\Delta$-w.a.i for $I$ and $\{f_\delta + I\}$ be a b.l.a.i for $A/I$. We can assume that $\{f_\delta\}$ is bounded. Indeed, since $\{f_\delta + I\}$ is bounded, there exists a positive integer $K$ with $\|f_\delta + I\| < K$ for each $\delta$. So, there exists $y_\delta \in I$ such that $\|f_\delta + I\| < \|f_\delta + y_\delta\| < K$. Put $f_\delta^{'} = f_\delta + y_\delta$. Clearly, $\{f_\delta^{'} + I\}$ is a b.l.a.i for $A/I$ which $\{f_\delta^{'}\}$ is bounded.

Now, consider the bounded net $\{e_\alpha + f_\delta - e_\alpha f_\delta\}_{(\alpha,\delta)}$. For each $\phi \in \Delta(A)$ we have

$$\phi(e_\alpha + f_\delta - e_\alpha f_\delta) = \phi(e_\alpha) + \phi(f_\delta)(1 - \phi(e_\alpha)) \xrightarrow{(\alpha,\delta)} 1.$$

Therefore, $A$ has a b.$\Delta$-w.a.i. □

Let $G$ be a locally compact group, $E$ be a closed non-empty subset of $G$ and $1 < p < \infty$. Define

$$I_p(E) = \{u \in A_p(G) : u(x) = 0 \text{ for all } x \in E\}\cdot$$

The following result improves [5, Theorem 3.9].

**Theorem 4.** *Let $G$ be a locally compact group. Then the following assertions are equivalent.*

(1) *$G$ is an amenable group.*
(2) $\ker(\phi)$ *has a b.$\Delta$-w.a.i for each $\phi \in \Delta(A_p(G))$.*
(3) *$I_p(H)$ has a b.$\Delta$-w.a.i for some closed amenable subgroup $H$ of $G$.*

*Proof.* (1) $\Rightarrow$ (2): Let $G$ be an amenable group. Then $A_p(G)$ has a b.a.i by Leptin-Herz's Theorem. Now, the result follows from [17, Corollary 2.3].

(2) $\Rightarrow$ (3): Just take $H = \{e\}$, because we know that $I_p(\{e\}) = \ker(\phi_e)$.

(3) $\Rightarrow$ (1): Suppose that $I_p(H)$ for a closed amenable subgroup $H$ of $G$ has a b.$\Delta$-w.a.i. By [21, Lemma 3.19] we know that $A_p(H)$ is isometrically isomorphic to $A_p(G)/I_p(H)$. But $A_p(H)$ has a b.a.i, since $H$ is an amenable group. Therefore, $A_p(G)/I_p(H)$ also has a b.a.i. Now, the result follows from Theorems 3 and 2. □

Forrest in [6, Lemma 3.14] improved [5, Theorem 3.9]. Indeed he showed that $G$ is an amenable group if for some closed proper subgroup $H$ of $G$, $I_2(H)$ has a b.a.i. Also, using the operator space structure of $A(G)$, it was shown in [8, Theorem 1.5] that $G$ is an amenable group only if $I_2(H)$ for some closed subgroup $H$ of $G$ has a b.a.i.

Now, we give the following result which improves [8, Corollary 1.6] and Theorem 4. Our proof is a mimic of [6, Lemma 3.14].

**Theorem 5.** *Let $G$ be a locally compact group and $1 < p < \infty$. Then the following are equivalent.*

(1) *$G$ is an amenable group.*
(2) *$I_p(H)$ has a b.$\Delta$-w.a.i for some proper closed subgroup $H$ of $G$.*

*Proof.* In view of [8, Corollary 4.2], only (2) $\Rightarrow$ (1) needs proof.

Let $H$ be a proper closed subgroup of $G$ such that $I_p(H)$ has a b.$\Delta$-w.a.i. We will show that $H$ is an amenable group and this completes the proof by Theorem 4.

Since $H$ is a proper subgroup, there exists $x \in G \setminus H$. On the other hand, the mapping $I_p(H) \to I_p(xH)$ defined by $u \to L_x u$ is an isometric isomorphism, because for each $t \in G$ and $f \in A_p(G)$, we have,

$$L_t f \in A_p(G), \; \|L_t f\|_{A_p(G)} = \|f\|_{A_p(G)}.$$

Therefore, $I_p(xH)$ has a b.$\Delta$-w.a.i which we denote it by $(u_\alpha)$. For each $\alpha$, let $v_\alpha$ be the restriction of $u_\alpha$ to $H$. Using [12, Theorem 1a], we conclude that $(v_\alpha)$ is a bounded net in $A_p(H)$.

Let $\nu \in A_p(H) \cap C_c(H)$ and $K = \operatorname{supp} \nu \subseteq H$. Then there exists a neighborhood $V$ of $K$ in $G$ such that $V \cap xH = \varnothing$, because $K \cap xH = \varnothing$ (otherwise we conclude that $x$ is in $H$) and $G$ is completely regular by [13, Theorem 8.4] and hence it is a regular topological space. Indeed, for each $y \in xH$, let $V_y$ be a neighborhood of $K$ such that $y \notin V_y$. So, $V = \cap_{y \in xH} V_y$ satisfies $V \cap xH = \varnothing$.

By [3, Proposition 1, pp.34] there is $u \in A_p(G)$ such that $u(x) = 1$ for each $x \in K$ and $\operatorname{supp} u \subseteq V$, and by [12, Theorem 1b], there exists a $v \in A_p(G)$ such that $v_{|H} = \nu$. Now, put $w = vu$. Since $V \cap xH = \varnothing$ and $\operatorname{supp} u \subseteq V$, we have $w \in I_p(xH)$, and since $u(x) = 1$ for each $x \in K$ and $K = \operatorname{supp} \nu$, we have $w_{|H} = \nu$.

Now, for each $x \in H$, we have

$$\begin{aligned} \lim_\alpha |\phi_x(v_\alpha \nu) - \phi_x(\nu)| &= \lim_\alpha |v_\alpha(x)\nu(x) - \nu(x)| \\ &= \lim_\alpha |u_\alpha(x)w(x) - w(x)| = 0. \end{aligned}$$

Therefore, $(v_\alpha)$ is a b.$\Delta$-w.a.i for $A_p(H)$, since by [3, Corollary 7, pp. 38], $A_p(H) \cap C_c(H)$ is dense in $A_p(H)$. Hence, by Theorem 2, $H$ is amenable. □

As an application of the above theorem, we give the following corollary which is the converse of [8, Corollary 4.2].

**Corollary 2.** *Let $G$ be a locally compact group, $1 < p < \infty$ and $H$ be a proper closed subgroup of $G$. If $I_p(H)$ has a b.a.i, then $G$ is amenable.*

Ghahramani and Lau in [10] introduced and studied a new closed ideal of $A(G)$. Indeed, let $G$ be a locally compact group and put

$$\mathfrak{L}A(G) = L^1(G) \cap A(G)$$

with the norm

$$|||f||| = \|f\|_1 + \|f\|_{A(G)}.$$

Clearly $\mathfrak{L}A(G)$ with pointwise multiplication is a commutative Banach algebra with $\Delta(\mathfrak{L}A(G)) = G$ and it is called the *Lebesgue-Fourier algebra* of $G$.

It was shown that $\mathfrak{L}A(G)$ has a b.a.i if and only if $G$ is a compact group [10, Proposition 2.6]. Now, we give the following result concerning the b.$\Delta$-w.a. identities of this Banach algebra.

**Theorem 6.** *Let G be a locally compact group.*

(1) *for each $x \in G$, $\ker(\phi_x) \subseteq \mathfrak{L}A(G)$ has a b.$\Delta$-w.a.i.*
(2) *$\mathfrak{L}A(G)$ has a b.$\Delta$-w.a.i.*
(3) *$G$ is amenable.*
(4) *$G$ is compact.*

*Then* (1) ⇒ (2) ⇒ (3) *and* (4) ⇒ (1) *hold.*

*Proof.* (1) $\Rightarrow$ (2) : Follows from Theorem 3.

(2) $\Rightarrow$ (3) : Let $(u_\alpha)$ be a b.$\Delta$-w.a.i for $\mathfrak{L}A(G)$, Then $(u_\alpha)$ is a bounded net in $A(G)$. Now, the result follows from Theorem 2 and this fact that $\Delta(\mathfrak{L}A(G)) = G = \Delta(A(G))$.

(4) $\Rightarrow$ (1) : Let $G$ be a compact group. Then by [10, Proposition 2.6], we know that $\mathfrak{L}A(G) = A(G)$. On the other hand, by [17, Example 2.6] for each $x \in G$, $A(G)$ is $\phi_x$-amenable. Therefore, the result follows from [17, Proposition 2.2]. □

We do not know whether the implication (3) $\Rightarrow$ (2) in Theorem 6 remains true?

*Remark* 1. In [11], Granirer gave a $(p,r)$-version of the Lebesgue-Fourier algebra. Indeed, let $1 < p < \infty, 1 \leq r \leq \infty$ and put $A_p^r(G) = A_p(G) \cap L^r(G)$ with the norm,

$$\|u\|_{A_p^r(G)} = \|u\|_r + \|u\|_{A_p(G)}.$$

It was shown that $A_p^r(G)$ with pointwise multiplication is a commutative semi-simple Banach algebra such that $\Delta(A_p^r(G)) = G$ and for all $1 \leq r \leq \infty$, $A_p^r(G) = A_p(G)$ if $G$ is a compact group; see [11, Theorem 1, Theorem 2].

Therefore, in view of Theorems 2 and 3, Theorem 6 remains true if we replace $\mathfrak{L}A(G)$ with $A_p^r(G)$.

*Remark* 2. Runde in [20], by using the canonical operator space structure of $L^p(G)$, introduced and studied the algebra $\mathrm{O}A_p(G)$ for $1 < p < \infty$, the *operator Figà-Talamanca-Herz algebra*. It was shown that $A_p(G) \subseteq \mathrm{O}A_p(G)$ [20, Remark 4. pp. 159] and $\mathrm{O}A_p(G)$ has a b.a.i if and only if $G$ is an amenable group [20, Theorem 4.10]. That would be an interesting question: Are the preceding results remain true if we replace $A_p(G)$ by $\mathrm{O}A_p(G)$?

Now, let $G$ be a locally compact group and $w : G \to \mathbb{R}$ be an upper semicontinuous function such that $w(x) \geq 1$ for each $x \in G$. Put

$$C_0^w(G) = \{f \in C(G) : fw \in C_0(G)\}.$$

It is clear that $C_0^w(G)$ with pointwise operation and weighted supremum norm defined by

$$\|f\|_w = \sup_{x \in G} |f(x)|w(x) \quad (f \in C_0^w(G)),$$

is a commutative Banach algebra such that $\Delta(C_0^w(G)) = \Delta(C_0(G)) = G$; see [16, Section 4.3].

**Theorem 7.** *Let $G$ be a locally compact group and $t \in G$. Then the following are equivalent.*

(1) $\ker(\phi_t)$ *has a b.a.i.*
(2) $\ker(\phi_t)$ *has a b.$\Delta$-w.a.i.*
(3) $C_0^w(G)$ *has a b.$\Delta$-w.a.i.*
(4) *$w$ is bounded.*

*Proof.* (1) $\Rightarrow$ (2) : This part is clear. Applying Theorem 3, we conclude (2) $\Rightarrow$ (3). In view of [16, Corollary 4.7], we have (3) $\Rightarrow$ (4). Therefore, we only show (4) $\Rightarrow$ (1).

Suppose that $w$ is bounded by $M > 0$. Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Gamma}$ be a neighborhood base at $t$ directed by the reverse inclusion. For each $\alpha \in \Gamma$, by the Urysohn Lemma, there exists a continuous function $f_\alpha : X \to [0,1]$ such that $f_\alpha(t) = 1$ and $\operatorname{supp} f_\alpha \subseteq V_\alpha$.

Let $\epsilon > 0$ and $g \in C_0^w(G)$. For $\epsilon' = \epsilon/2M$, there exists a neighborhood $V$ of $t$ such that,

$$|g(y) - g(t)| < \epsilon' \quad (y \in V).$$

On the other hand, let $\alpha_0 \in \Gamma$ be such that $V_{\alpha_0} \subseteq V$. Therefore, there exists $x_0 \in G$ such that

$$\begin{aligned}\|gf_\alpha - \phi_t(g)f_\alpha\|_w &= \sup_{x\in G} |g(x)f_{\alpha_0}(x) - g(t)f_{\alpha_0}(x)|w(x)\\ &< |g(x_0)f_{\alpha_0}(x_0) - g(t)f_{\alpha_0}(x_0)|w(x_0) + \epsilon/2\\ &< M\epsilon' + \epsilon/2 = \epsilon.\end{aligned}$$

Therefore, for each $g \in C_0^w(G)$, $\|gf_\alpha - g(t)f_\alpha\|_w \longrightarrow 0$. Hence, by [17, Theorem 1.4, Proposition 2.2] and [1, Corollary 3.6] we conclude that $\ker(\phi_t)$ has a b.a.i. $\square$


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