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PERIODIC SOLUTIONS FOR A SYSTEM OF TOTALLY NONLINEAR DYNAMIC EQUATIONS ON TIME SCALE

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Abstract. Let $\mathbb{T}$ be a periodic time scale. We use a reformulated version of Krasnoselskii's fixed point theorem to show that the system of nonlinear neutral dynamic equation with delay

$$x^{\Delta}(t) = -A(t)H(x^{\sigma}(t)) + (Q(t,x(t-r(t))))^{\Delta} + G\big(t,x(t),x(t-r(t))\big), t \in \mathbb{T},$$

has periodic solutions on the time scale $\mathbb{T}$.

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1. Introduction

Motivated by the papers [1–6, 10–12] and the references therein, we consider the system of dynamic equation

$$x^{\Delta}(t) = -A(t)H(x^{\sigma}(t)) + (Q(t,x(t-r(t))))^{\Delta} + G\big(t,x(t),x(t-r(t))\big),\ t \in \mathbb{T}, \tag{1.1}$$

where $x^{\Delta}(t)$ is $n \times 1$ column vector determined by Δ-derivative components of $x(t)$, $A(t) = \text{diag}\ [a_1(t), a_2(t), ..., a_n(t)]$, $H : \mathbb{R}^n \to \mathbb{R}^n$, $Q : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $G : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

If $n = 1$ and $(Q(t,x(t-r(t))))^{\Delta} = c(t)x^{\Delta}(t-r(t))$ then equation (1.1) reduces to the equation considered in [4]. On the other hand, if $n = 1$ and $h(x^{\sigma}(t)) = x^{\sigma}(t)$, then equation (1.1) reduces to the equation considered in [11]. Thus, in this paper we not only generalize the results obtained in [4] and [11] to systems of equations, but even for $n = 1$ our results also extends the work of Ardjouni and Djoudi [4] and Kaufmann and Raffoul [11].

We assume in this work that $r : \mathbb{T} \to \mathbb{R}$ and that $id - r : \mathbb{T} \to \mathbb{T}$ is strictly increasing so that the function $x(t - r(t))$ is well defined over $\mathbb{T}$.

Some preliminary material is presented in the next section. In particular, we will provide some facts about the exponential function on time scale and also state a reformulated version of Krasnoselskii's fixed point theorem. Our main results on the existence of periodic solutions for equation (1.1) is presented in Section 3.

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2. PRELIMINARIES

We begin this section by giving some definitions introduced by Actici et al. in [6] and Kaufman and Raffoul in [10].

Definition 1. We say that a time scale $\mathbb{T}$ is *periodic* if there exist a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the *period* of the time scale.

For example, the following time scales taken from [10] are periodic.

(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty}[2(i-1)h, 2ih]$, $h > 0$ has period $p = 2h$.
(2) $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.
(3) $\mathbb{T} = \mathbb{R}$.
(4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $p = 1$.

As pointed out in [10], all periodic time scales are unbounded above and below.

Definition 2. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

As established in [10], if $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

Most of the following definitions, lemmas and theorems can be found in [7, 8]. Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales [7, Theorem 1.93].

Theorem 1 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta} = (w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our second theorem is the substitution rule [7, Theorem 1.98].

Theorem 2 (Substitution). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\,\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\,\tilde{\Delta}s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\Big(\int_s^t \frac{1}{\mu(z)}\text{Log}(1 + \mu(z)p(z))\,\Delta z\Big),$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma, [7, Theorem 2.36].

Lemma 1. *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*
(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$*;*
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$*;*
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*
(vi) $\big(\frac{1}{e_p(\cdot,s)}\big)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

Lemma 2 ([6])**.** *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\Big(\int_s^t p(u)\Delta u\Big),\ \forall\, t \in \mathbb{T}.$$

Corollary 1 ([6])**.** *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\Big(\int_s^t p(u)\Delta u\Big) < 1,\ \forall\, t \in \mathbb{T}.$$

Lastly in this section, we state Krasnoselskii-Burton's fixed point theorem (see [9]) which is employed in establishing our results.

Theorem 3 (Krasnoselskii-Burton)**.** *Let $\mathbb{M}$ be a bounded convex non-empty subset of a Banach space $(S, ||.||)$. Suppose that A, B map $\mathbb{M}$ into $\mathbb{M}$ and that*

(i) *for all* $x, y \in \mathbb{M} \Rightarrow Ax + By \in \mathbb{M}$,
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is a $z \in \mathbb{M}$ with $z = Az + Bz$.

3. Existence of periodic solutions

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$, and (a,b) are defined similarly. Define $P_T = \{\varphi \in C(\mathbb{T}, R^n) : \varphi(t+T) = \varphi(t)\}$. Then P_T is a Banach space when it is endowed with the usual linear structure as well as the norm

$$\|x\| = \sum_{j=1}^{n} |x_j|_0, \text{ for } x = (x_1, x_2, ..., x_n) \in P_T,$$

where

$$|x_j|_0 = \sup_{t\in[0,T]} |x(t)|, \ j = 1, ..., n.$$

Also, define the set

$$\mathbb{M} = \{\phi \in \mathbb{P}_{\mathbb{T}} : ||\phi|| \le L \text{ with } |\phi_j|_0 \le \frac{L}{n}, \ j = 1, 2, ..., n.\},$$

where L is a positive constant.

We next state the following lemma which will be used in subsequent sections.

Lemma 3 ([10]). *Let $x \in P_T$. Then $|x_j^\sigma|_0$ exists and $|x_j^\sigma|_0 = |x_j|_0$.*

In this paper we assume that h_j, is continuous, $a_j \in \mathcal{R}^+$ is continuous, $a_j(t) > 0$ for all $t \in \mathbb{T}$ and

$$a_j(t+T) = a_j(t), \quad \big(id - r\big)(t+T) = \big(id - r\big)(t), \tag{3.1}$$

where, *id* is the identity function on $\mathbb{T}$. We also require that $q_j(t,x)$ and $g_j(t,x,y)$ are continuous and periodic in t and Lipschitz continuous in x and y. That is,

$$q_j(t+T,x) = q_j(t,x), \ g_j(t+T,x,y) = g_j(t,x,y), \tag{3.2}$$

and there are positive constants E_1, E_2, E_3 such that

$$|q_j(t,x) - q_j(t,y)| \le E_1|x-y|_0, \text{ for } x, y \in \mathbb{R}, \tag{3.3}$$

and

$$|g_j(t,x,y) - g_j(t,z,w)| \le E_2|x-z|_0 + E_3|y-w|_0, \text{ for } x,y,z,w \in \mathbb{R}. \tag{3.4}$$

For our next lemma we consider the neutral dynamic equation

$$\begin{aligned} x^\Delta(t) = &-a_j(t)h_j(x(\sigma(t)) + \big(q_j(t, x(t-r(t)))\big)\big)^\Delta \\ &+ g_j\big(t, x(t), x(t-r(t))\big), t \in \mathbb{T}, \ j = 1, 2, ..., n. \end{aligned} \tag{3.5}$$

Lemma 4. *Suppose (3.1), (3.2) hold. If $x \in P_T$, then x is a solution of equation (3.5) if and only if,*

$$\begin{aligned} x(t) = q_j\big(t, x(t-r(t))\big) &+ \big(1 - e_{\ominus a_j}(t, t-T)\big)^{-1} \\ \times \int_{t-T}^{t} &\Big[a_j(s)[x^\sigma(s) - h_j(x(\sigma(s)))] - a_j(s)q_j^\sigma\big(s, x(s-r(s))\big) \\ &+ g_j\big(s, x(s), x(s-r(s))\big)\Big] e_{\ominus a_j}(t,s)\,\Delta s. \end{aligned} \tag{3.6}$$

Proof. Let $x \in P_T$ be a solution of (3.5). First we write (3.5) as

$$\begin{aligned}\{x(t)-q_j(t,x(t-g(t)))\}^{\Delta} &= -a_j(t)\{x^{\sigma}(t)-q_j^{\sigma}(t,x(t-r(t)))\}\\ &\quad + a_j(t)[x^{\sigma}(t)-h_j(x(\sigma(t)))]\\ &\quad - a_j(t)q_j^{\sigma}(t,x(t-r(t))) + g_j(t,x(t),x(t-r(t))).\end{aligned}$$

Multiply both sides by $e_{a_j}(t,0)$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned}&\int_{t-T}^{t}[e_{a_j}(s,0)\{x(s)-q_j(s,x(s-r(s)))\}]^{\Delta}\,\Delta s\\ &= \int_{t-T}^{t}\Big[a_j(s)[x^{\sigma}(s)-h_j(x(\sigma(s)))]-a_j(s)q_j^{\sigma}(s,x(s-r(s)))\\ &\quad + g_j(s,x(s),x(s-r(s)))\Big]e_{a_j}(s,0)\,\Delta s.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&e_{a_j}(t,0)\Big(x(t)-q_j(t,x(t-r(t)))\Big)\\ &\quad - e_{a_j}(t-T,0)\Big(x(t-T)-q_j(t-T,x(t-T-r(t-T)))\Big)\\ &= \int_{t-T}^{t}\big[a_j(s)[x^{\sigma}(s)-h_j(x(\sigma(s)))]-a_j(s)q_j^{\sigma}(s,x(s-r(s)))\\ &\quad + g_j(s,x(s),x(s-r(s)))\big]e_{a_j}(s,0)\,\Delta s.\end{aligned}$$

After making use of (3.1), (3.2) and $x \in P_T$, we divide both sides of the above equation by $e_{a_j}(t,0)$ to obtain

$$\begin{aligned}x(t) &= q_j(t,x(t-r(t))) + \big(1-e_{\ominus a_j}(t,t-T)\big)^{-1}\\ &\quad \times \int_{t-T}^{t}\Big[a_j(s)[x^{\sigma}(s)-h_j(x(\sigma(s)))]-a_j(s)q_j^{\sigma}(s,x(s-r(s)))\\ &\quad + g_j(s,x(s),x(s-r(s)))\Big]e_{\ominus a_j}(t,s)\,\Delta s.\end{aligned}$$

Since each step is reversible, the converse follows. This completes the proof. □

Let $\rho(t,t-T) = \operatorname{diag}[\rho_1,\rho_2,\dots,\rho_n]$ where $\rho_j = \big(1-e_{\ominus a_j}(t,t-T)\big)^{-1}$ for $j = 1,2,\dots,n$. Also, we let $\mu(t,s) = \operatorname{diag}[e_{\ominus a_1}(t,s),\dots,e_{\ominus a_n}(t,s)]$.

Define the mapping $F : P_T \to P_T$ by

$$\begin{aligned}(F\varphi)(t) &= Q(t,\varphi(t-g(t))) + \rho(t,t-T)\int_{t-T}^{t}\mu(t,s)\\ &\Big[A(s)[\varphi^{\sigma}(s)-H(\varphi(\sigma(s)))]-A(s)Q^{\sigma}(s,\varphi(s-g(s)))+G(s,\varphi(s),\varphi(s-g(s)))\Big]\,\Delta s.\end{aligned} \tag{3.7}$$

We express equation (3.7) as

$$(F\varphi)(t) = (B\varphi)(t) + (A\varphi)(t)$$

where, A, B are given by

$$(B\varphi)(t) = \rho(t,t-T)\int_{t-T}^{t} \mu(t,s)A(s)[\varphi^{\sigma}(s) - H(\varphi(\sigma(s)))]\,\Delta s. \tag{3.8}$$

and

$$(A\varphi)(t) = Q\big(t,\varphi(t-g(t))\big)$$
$$+\rho(t,t-T)\int_{t-T}^{t}\mu(t,s)\Big[-A(s)Q^{\sigma}\big(s,\varphi(s-g(s))\big) + G\big(s,\varphi(s),\varphi(s-g(s))\big)\Big]\,\Delta s. \tag{3.9}$$

In the rest of the section we require the following conditions.

$$E_1\frac{L}{n} + |q_j(t,0)|_0 \le \alpha\frac{L}{n} \tag{3.10}$$

$$E_2\frac{L}{n} + E_2\frac{L}{n} + |g_j(t,0,0)|_0 \le \frac{L}{n}\gamma a_j(t), \tag{3.11}$$

and

$$J(2\alpha+\gamma) \le 1, \tag{3.12}$$

where α, γ, L and J are constants with $J \ge 3$.

Lemma 5. *Suppose (3.1)–(3.4) and (3.10)–(3.12) hold. Then $A : \mathbb{M} \to \mathbb{M}$, as defined by (3.9), is continuous in the supremum norm and maps $\mathbb{M}$ into a compact subset of $\mathbb{M}$.*

Proof. We first show that $A : \mathbb{M} \to \mathbb{M}$. Evaluate (3.9) at $t+T$.

$$\begin{aligned}(A\varphi)(t+T) = {} & Q\big(t+T,\varphi(t+T-g(t+T))\big)\\ & +\rho(t+T,t)\int_{t}^{t+T}\mu(t+T,s)\Big[-A(s)Q^{\sigma}\big(s,\varphi(s-r(s))\big)\\ & +G\big(s,\varphi(s),\varphi(s-r(s))\big)\Big]\,\Delta s.\end{aligned} \tag{3.13}$$

With $u = s-T$ and using conditions (3.1) – (3.2) we obtain

$$\begin{aligned}(A\varphi)(t+T) = {} & Q\big(t,\varphi(t-r(t))\big) + \rho(t+T,t)\\ & \times\int_{t-T}^{t}\mu(t+T,u+T)\Big[-A(u+T)Q^{\sigma}\big(u-T,\varphi(u-T-r(u-T))\big)\\ & +G\big(s,\varphi(u-T),\varphi(u-T-r(u-T))\big)\Big]\,\Delta u.\end{aligned}$$

But we have that $e_{\ominus a_j}(t+T,u+T) = e_{\ominus a_j}(t,u)$ thus, $\mu(t+T,u+T) = \mu(t,u)$. Moreover, $e_{\ominus a_j}(t+T,t) = e_{\ominus a_j}(t,t-T)$ and so $\rho(t+T,t) = \rho(t,t-T)$. Thus (3.13) becomes

$$\begin{aligned}(A\varphi)(t+T) &= Q\big(t,\varphi(t-r(t))\big) + \rho(t,t-T)\\ &\quad \times \int_{t-T}^{t} \mu(t,u)\Big[-A(u)Q^{\sigma}\big(u,\varphi(u-r(u))\big)\\ &\quad + G\big(u,\varphi(u),\varphi(u-r(u))\big)\Big]\Delta u\\ &= (A\varphi)(t).\end{aligned}$$

Note that in view of (3.3) and (3.4) we have that

$$\begin{aligned}|q_j(t,x)| &= |q_j(t,x) - q_j(t,0) + q_j(t,0)|\\ &\le |q_j(t,x) - q_j(t,0)| + |q_j(t,0)|\\ &\le E_1|x|_0 + |q_j(t,0)|_0.\end{aligned}$$

Similarly,

$$\begin{aligned}|g_j(t,x,y)| &= |g_j(t,x,y) - g_j(t,0,0) + g_j(t,0,0)|\\ &\le |g_j(t,x,y) - g_j(t,0,0)| + |g_j(t,0,0)|\\ &\le E_2|x|_0 + E_3|y|_0 + |g_j(t,0,0)|_0.\end{aligned}$$

Thus, for any $\varphi \in \mathbb{M}$ we have

$$||(A\varphi)|| = \sum_{j=1}^{n} \sup_{t\in[0,T]} |(A_j\varphi)(t)|$$

But

$$\begin{aligned}&|(A_j\varphi)(t)| = \Big|q_j\big(t,\varphi(t-g(t))\big) + \big(1-e_{\ominus a_j}(t,t-T)\big)^{-1}\\ &\quad \times \int_{t-T}^{t}\Big[-a_j(s)q_j^{\sigma}\big(s,\varphi(s-r(s))\big) + g_j\big(s,\varphi(s),\varphi(s-r(s))\big)\Big]e_{\ominus a_j}(t,s)\,\Delta s\Big|\\ &\le |q_j\big(t,\varphi(t-r(t))\big)| + \big(1-e_{\ominus a_j}(t,t-T)\big)^{-1}\int_{t-T}^{t}\big|-a_j(s)\big|\,\big|q_j^{\sigma}\big(s,\varphi(s-r(s))\big)\big|\\ &\quad + \big|g_j\big(s,\varphi(s),\varphi(s-r(s))\big)\big|e_{\ominus a_j}(t,s)\,\Delta s\\ &\le E_1\frac{L}{n} + |q_j(t,0)|_0 + \big(1-e_{\ominus a_j}(t,t-T)\big)^{-1}\\ &\quad \times \int_{t-T}^{t}\Big[a_j(s)(E_1\frac{L}{n} + |q_j(s,0)|_0) + (E_2+E_3)\frac{L}{n} + |g_j(s,0,0)|_0\Big]e_{\ominus a_j}(t,s)\,\Delta s\\ &\le \alpha\frac{L}{n} + \big(1-e_{\ominus a}(t,t-T)\big)^{-1}\end{aligned}$$

$$\times \int_{t-T}^{t} \Big[\alpha \frac{L}{n} + \gamma \frac{L}{n}\Big] a(s) e_{\ominus a}(t,s)\,\Delta s$$
$$\leq (2\alpha+\gamma)\frac{L}{n} \leq \frac{L}{nJ}.$$

Thus,

$$||(A\varphi)|| \leq \sum_{j=1}^{n} \frac{L}{nJ} \leq \frac{L}{J} < L,$$

showing that A maps $\mathbb{M}$ into itself. To see that A is continuous, let $\varphi, \psi \in \mathbb{M}$ and define

$$\eta := \sup_{t\in[0,T]} \left|\big(1-e_{\ominus a_j}(t,t-T)\big)^{-1}\right|, \;\; \sigma := \sup_{t\in[0,T]} |a_j(t)|,$$
$$\gamma := \sup_{u\in[t-T,t]} e_{\ominus a_j}(t,u), \;\; \lambda := \sup_{t\in[0,T]} |\big(q_j(t,x(t),x(t-r(t)))\big)^{\Delta}|,$$
$$\alpha := \sup_{t\in[0,T]} |q_j(t,0)|, \;\; \beta := \sup_{t\in[0,T]} |g_j(t,0,0)|. \tag{3.14}$$

Given $\varepsilon > 0$, take $\delta = \varepsilon/nM$ with $M = E_1 + \eta\,\gamma\,T(\sigma\,E_1 + E_2 + E_3)$ where, E_1, E_2 and E_3 are given in (3.3) and (3.4) such that $\|\varphi-\psi\| < \delta$. Using (3.9) we get

$$\left\|A\varphi - A\psi\right\| = \sum_{j=1}^{n} \sup_{t\in[0,T]} \left|(A_j\varphi)(t) - (A_j\psi)(t)\right|.$$

But,

$$\left|A_j\varphi - A_j\psi\right|_0 \leq E_1|\varphi-\psi|_0 + \eta\gamma \int_0^T \Big[\sigma\,E_1|\varphi-\psi|_0 + (E_2+E_3)|\varphi-\psi|_0\Big]\Delta u$$
$$\leq M|\varphi-\psi|_0.$$

Thus,

$$\left\|A\varphi - A\psi\right\| \leq nM\|\varphi-\psi\| < \varepsilon.$$

This proves that A is continuous.

We next show that A is compact. Consider the sequence of periodic functions $\{\varphi_n\} \subset \mathbb{M}$. Thus as before we have that

$$\|A(\varphi_n)\| \leq L,$$

showing that the sequence $\{A\varphi_n\}$ is uniformly bounded. Now, it can be easily checked that

$$(A_j\varphi_n)^{\Delta}(t) = \big(q_j(t,\varphi_n(t),\varphi_n(t-r(t)))\big)^{\Delta} - a_j(t)q_j^{\sigma}\big(t,\varphi_n(t-r(t))\big)$$
$$+ g_j\big(t,\varphi_n(t),\varphi_n(t-r(t))\big) - a_j(t)\Big\{\big(1-e_{\ominus a}(t,t-T)\big)^{-1}$$

$$\times\int_{t-T}^{t}\Big[-a_j(s)q_j^\sigma\big(s,\varphi_n(s-r(s))\big)+g_j\big(s,\varphi_n(s),\varphi_n(s-r(s))\big)\Big]e_{\ominus a}(t,s)\,\Delta s\Big\}$$

$$\begin{aligned}&=\big(q_j(t,\varphi_n(t),\varphi_n(t-r(t))))\big)^\Delta-a_j(t)q_j^\sigma\big(t,\varphi_n(t-r(t))\big)\\&\quad+g_j\big(t,\varphi_n(t),\varphi_n(t-r(t))\big)-a_j(t)\Big\{\big(1-e_{\ominus a}(t,t-T)\big)^{-1}\\&\quad\times\int_{t-T}^{t}\Big[-a_j(s)q_j^\sigma\big(s,\varphi_n(s-r(s))\big)+g_j\big(s,\varphi_n(s),\varphi_n(s-r(s))\big)\Big]e_{\ominus a}(t,s)\,\Delta s\\&\quad+q_j(t,\varphi_n(t-r(t)))\Big\}+a_j(t)q_j(t,\varphi(t-r(t))).\end{aligned}$$

$$\begin{aligned}(A_j\varphi_n)^\Delta(t)=&\;\big(q_j(t,\varphi_n(t),\varphi_n(t-r(t))))\big)^\Delta\\&-a_j(t)(A_j\varphi_n)^\sigma(t)-a_j(t)q_j^\sigma\big(t,\varphi_n(t-r(t))\big)\\&+g_j\big(t,\varphi_n(t),\varphi_n(t-r(t))\big)+a_j(t)q_j(t,\varphi_n(t-r(t))).\end{aligned}$$

Consequently,

$$|(A_j\varphi_n)^\Delta(t)|\le\lambda+\sigma L+2\sigma(E_1\frac{L}{n}+\alpha)+E_2\frac{L}{n}+E_3\frac{L}{n}+\beta$$

for all n.

Thus,

$$||(A\varphi_n)^\Delta||\le\sum_{j=1}^{n}\Big(\lambda+\sigma L+2\sigma(E_1\frac{L}{n}+\alpha)+E_2\frac{L}{n}+E_3\frac{L}{n}+\beta\Big)=F.$$

That is $\|(A\varphi_n)^\Delta\|\le F$, for some positive constant F. Thus the sequence $\{A\varphi_n\}$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that there is a subsequence $\{A\varphi_{n_k}\}$ which converges uniformly to a continuous T-periodic function φ^*. Thus A is compact. □

We next state the following proposition (see [1]), in which the following assumptions are made on the function $h:\mathbb{R}\to\mathbb{R}$.

(H1) h is continuous on $U_l=[-l,l]$ and differentiable on U_l.
(H2) h is strictly increasing on U_l.
(H3) $\sup_{s\in U_l}h^\Delta(s)\le 1$.

Proposition 1 ([1]). *Let h be a function satisfying $(H1)-(H3)$. Then the mapping $\mathfrak{h}(\varphi)(t)=\varphi(t)-h(\varphi(t))$ is a large contraction on the set $\mathbb{M}_l$.*

The next result gives a relationship between the mappings $\mathfrak{h}_j$ and B in the sense of large contraction.

Lemma 6. *If $\mathfrak{h}_j$ is a large contraction on $\mathbb{M}$, then so is the mapping B.*

Proof. If $\mathfrak{h}_j$ is a large contraction on $\mathbb{M}$, then for $x, y \in \mathbb{M}$, with $x \neq y$, we have $\|\mathfrak{h}_j x - \mathfrak{h}_j y\| \leq |x-y|_0$. Then it follows from the equality

$$a_j(u)e_{\ominus a_j}(t+T,\sigma(u)) = [e_{\ominus a_j}(t+T,u)]^{\Delta_s},$$

where Δ_s indicates the delta derivative with respect to s that

$$\begin{aligned}|B_j x(t) - B_j y(t)| &\leq \int_t^{t+T} \frac{e_{\ominus a_j}(t+T,\sigma(u))}{1-e_{\ominus a_j}(t,t+T)} a_j(u)\big|\mathfrak{h}_j(x(u)) - \mathfrak{h}_j(y(u))\big|\Delta u \\ &\leq \frac{|x-y|_0}{1-e_{\ominus a_j}(t,t+T)} \int_t^{t+T} a_j(u)e_{\ominus a_j}(t+T,\sigma(u))\Delta u \\ &= |x-y|_0.\end{aligned}$$

Thus,

$$\begin{aligned}\|Bx - By\| &= \sum_{j=1}^n \sup_{t\in[0,T]} |B_j x(t) - B_j y(t)| \\ &\leq \sum_{j=1}^n |x-y|_0 = \|x-y\|\end{aligned}$$

One may also show in a similar way that

$$\|Bx - By\| \leq \delta\|x-y\|$$

holds if we know the existence of a $0 < \delta < 1$, such that for all $\epsilon > 0$

$$\big[x, y \in \mathbb{M},\ \|x-y\| \geq \epsilon\big] \Rightarrow \|Bx - By\| \leq \delta\|x-y\|.$$

The proof is complete. □

Lemma 7. *Suppose* (3.1)-(3.4)*, and* (3.10)-(3.12) *hold. Suppose also that*

$$\max\Big(|\mathfrak{h}_j(-L)|, |\mathfrak{h}_j(L)|\Big) \leq \frac{(J-1)L}{Jn}.$$

For B, A defined by (3.8) *and* (3.9)*, if $\varphi, \psi \in \mathbb{M}$ are arbitrary, then*

$$A\varphi + B\psi : \mathbb{M} \to \mathbb{M}.$$

Proof. Let $\varphi, \psi \in \mathbb{M}$ be arbitrary. Using the definition of B and the result of Lemma 5 we obtain

$$\begin{aligned}\|A_j(\varphi) + B_j(\psi)\| \leq{}& |q_j\big(t,\varphi(t-r(t))\big)| \\ &+ \big(1-e_{\ominus a_j}(t,t-T)\big)^{-1}\int_{t-T}^t \big|-a_j(s)\big|\big|q_j^\sigma\big(s,\varphi(s-r(s))\big)\big| \\ &+ \big|g_j\big(s,\varphi(s),\varphi(s-r(s))\big)\big|e_{\ominus a_j}(t,s)\,\Delta s \\ &+ \max\Big(|\mathfrak{h}_j(-L)|, |\mathfrak{h}_j(L)|\Big)\int_t^{t+T} \frac{e_{\ominus a_j}(t+T,\sigma(u))}{1-e_{\ominus a_j}(t,t+T)} a_j(u)\Delta u\end{aligned}$$

$$\leq \frac{L}{Jn} + \frac{(J-1)L}{Jn} = \frac{L}{n}.$$

Thus,

$$\|A(\varphi) + B(\psi)\| \leq \sum_{j=1}^{n} \frac{L}{n} = L.$$

This completes the proof. □

Theorem 4. *Suppose* (3.1)-(3.4) *and* (3.10)-(3.12) *hold. Suppose further that the hypotheses of Lemma 5, Lemma 6 and Lemma 7 hold. Then equation* (1.1) *has a periodic solution in the subset* $\mathbb{M}$.

Proof. By Lemma 5, $A : \mathbb{M} \to \mathbb{M}$ is completely continuous. By Lemma 7, $A\varphi + B\psi \in \mathbb{M}$ whenever $\varphi, \psi \in \mathbb{M}$. Moreover, $B : \mathbb{M} \to \mathbb{M}$ is a large contraction by Lemma 6. Thus all the hypotheses of Theorem 3 are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}$ such that $\varphi = A\varphi + B\varphi$. Hence (1.1) has a $T-$ periodic solution. □

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