



WEIGHTED BOUNDEDNESS OF MULTILINEAR OPERATOR ASSOCIATED TO MULTIPLIER OPERATOR FOR THE EXTREME CASES

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Abstract. We prove the weighted boundedness properties for the multilinear operator associated to the multiplier operator for the extreme cases.

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1. INTRODUCTION AND PRELIMINARIES

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1–4, 6, 13–15]). Let T be the Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$, a classical result of Coifman, Rochberg and Weiss (see [5]) stated that the commutator $[b, T](f) = T(bf) - bT(f)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [7], authors obtain the boundedness properties of the commutators for the extreme values of p are obtained (that is $p = 1$ and $p = \infty$). And note that $[b, T]$ is not bounded for the end point boundedness. The purpose of this paper is to introduce some multilinear operator associated to the multiplier operator and prove the weighted boundedness properties of the multilinear operator for the extreme cases.

First, let us introduce some preliminaries (see [6, 15]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable functions b and a weight function w (that is a non-negative locally integrable function), let $w(Q) = \int_Q w(x)dx$, $w_Q = |Q|^{-1} \int_Q w(x)dx$, the weighted sharp function of b is defined by

$$b^\#(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |b(y) - b_Q| w(y) dy.$$

We say that b belongs to $BMO(w)$ if $b^\#$ belongs to $L^\infty(w)$ and define $\|b\|_{BMO(w)} = \|b^\#\|_{L^\infty(w)}$. If $w = 1$, we denote $BMO(w) = BMO(R^n)$. It has

been known that (see [15])

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck\|b\|_{BMO}.$$

We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$\|f\|_{CMO} = \sup_{r>1} |Q(0, r)|^{-1} \int_Q |f(x) - f_Q| dx < \infty.$$

It is well-known that (see [15])

$$\|f\|_{CMO} \approx \sup_{r>1} \inf_{c \in \mathbb{C}} |Q(0, r)|^{-1} \int_Q |f(x) - c| dx.$$

Definition 1. Let $1 < p < \infty$ and w be a non-negative weight functions on R^n . We shall call $B_p(w)$ the space of those functions f on R^n such that

$$\|f\|_{B_p(w)} = \sup_{r>1} [w(Q(0, r))]^{-1/p} \|f \chi_{Q(0, r)}\|_{L^p(w)} < \infty.$$

The A_p weight is defined by (see [6]), for $1 < p < \infty$,

$$A_p = \left\{ 0 < w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \{0 < w \in L^1_{loc}(R^n) : \sup_{Q \ni x} \frac{1}{|Q|} \int_Q w(y) dy \leq Cw(x), a.e.\}.$$

2. THEOREMS

In this paper, we will study the multilinear operator as following (see [8]).

A bounded measurable function k defined on $R^n \setminus \{0\}$ is called a multiplier. The multiplier operator T associated with k is defined by

$$T(f)(x) = k(x)\hat{f}(x), \text{ for } f \in S(R^n),$$

where \hat{f} denotes the Fourier transform of f and $S(R^n)$ is the Schwartz test function class. Now, we recall the definition of the class $M(s, l)$. Denote by $|x| \sim t$ the fact that the value of x lies in the annulus $\{x \in R^n : at < |x| < bt\}$, where $0 < a \leq 1 < b < \infty$ are values specified in each instance.

Definition 2. Let $l \geq 0$ be a real number and $1 \leq s \leq 2$. we say that the multiplier k satisfies the condition $M(s, l)$, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi)|^s d\xi \right)^{\frac{1}{s}} < CR^{n/s - |\alpha|}$$

for all $R > 0$ and multi-indices α with $|\alpha| \leq l$, when l is a positive integer, and, in addition, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi) - D^\alpha k(\xi - z)|^s d\xi \right)^{\frac{1}{s}} \leq C \left(\frac{|z|}{R} \right)^\gamma R^{\frac{n}{s} - |\alpha|}$$

for all $|z| < R/2$ and all multi-indices α with $|\alpha| = [l]$, the integer part of l , i.e., $[l]$ is the greatest integer less than or equal to l , and $l = [l] + \gamma$ when l is not an integer.

Definition 3. For a real number $\tilde{l} \geq 0$ and $1 \leq \tilde{s} < \infty$, we say that K verifies the condition $\tilde{M}(\tilde{s}, \tilde{l})$, and write $K \in \tilde{M}(\tilde{s}, \tilde{l})$, if

$$\left(\int_{|x| \sim R} |D^{\tilde{\alpha}} K(x)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \leq C R^{n/\tilde{s} - n - |\tilde{\alpha}|}, \quad R > 0$$

for all multi-indices $|\tilde{\alpha}| \leq \tilde{l}$ and, in addition, if

$$\left(\int_{|x| \sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \leq C \left(\frac{|z|}{R} \right)^v R^{\frac{n}{\tilde{s}} - n - u}, \quad \text{if } 0 < v < 1,$$

$$\left(\int_{|x| \sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \leq C \left(\frac{|z|}{R} \right) (\log \frac{R}{|z|}) R^{\frac{n}{\tilde{s}} - n - u}, \quad \text{if } v = 1,$$

for all $|z| < \frac{R}{2}$, $R > 0$, and all multi-indices $\tilde{\alpha}$ with $|\tilde{\alpha}| = u$, where u denotes the largest integer strictly less than \tilde{l} with $\tilde{l} = u + v$.

Denote $D(R^n) = \{\phi \in S(R^n) : \text{supp}(\phi) \text{ is compact}\}$ and $\hat{D}_0(R^n) = \{\phi \in S(R^n) : \hat{\phi} \in D(R^n) \text{ and } \hat{\phi} \text{ vanishes in a neighbourhood of the origin}\}$. The following boundedness property of T on $L^p(R^n)$ is proved by Strömberg and Torkinsky (see [8, 12]).

Lemma 1 ([8]). *Let $k \in M(s, l)$, $1 \leq s \leq 2$, and $l > \frac{n}{s}$. Then the associated mapping T , defined a priori for $f \in \hat{D}_0(R^n)$, $T(f)(x) = (f * K)(x)$, extends to a bounded mapping from $L^p(R^n)$ into itself for $1 < p < \infty$ and $K(x) = \check{k}(x)$.*

Lemma 2 ([8]). *Suppose $k \in M(s, l)$, $1 \leq s \leq 2$. Given $1 \leq \tilde{s} < \infty$, let $r \geq 1$ be such that $\frac{1}{r} = \max\{\frac{1}{s}, 1 - \frac{1}{\tilde{s}}\}$. Then $K \in \tilde{M}(\tilde{s}, \tilde{l})$, where $\tilde{l} = l - \frac{n}{r}$.*

Lemma 3 ([8]). *Let $1 \leq s < \infty$, suppose that l is a positive real number with $l > n/r$, $1/r = \max\{1/s, 1 - 1/\tilde{s}\}$, and $k \in M(s, l)$. Then there is a positive constant a , such that*

$$\left(\int_{Q_k} |K(x - z) - K(x_Q - z)|^{\tilde{s}} dz \right)^{1/\tilde{s}} \leq C 2^{-ka} (2^k h)^{-n/\tilde{s}'}$$

Now we can define the multilinear operator associated to the multiplier operator as following.

Let m_j be the positive integers($j = 1, \dots, l$), $m_1 + \dots + m_l = m$ and b_j be the functions on $R^n(j = 1, \dots, l)$. Set, for $1 \leq j \leq l$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha.$$

Let T be the multiplier operator. By Lemma 1, $T(f)(x) = (K * f)(x)$ for $K(x) = \check{k}(x)$. The multilinear operator associated to T is defined by

$$T_b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x - y|^m} K(x - y) f(y) dy.$$

Note that when $m = 0$, T_b is just multilinear commutators of T and b (see [13,14]). It is well-known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [1–4]). In this paper, we will study the weighted boundedness properties of the multilinear operator T_b for the extreme cases(see [9–11, 16–19]).

We shall prove the following theorems in Section 3.

Theorem 1. *Let T be the multiplier operator, $w \in A_1$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Then T_b is bounded from $L^\infty(w)$ to $BMO(w)$.*

Theorem 2. *Let T be the multiplier operator, $1 < p < \infty$, $w \in A_1$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Then T_b is bounded from $B_p(w)$ to $CMO(w)$.*

3. PROOFS OF THEOREMS

We begin with some preliminaries lemmas.

Lemma 4 ([3]). *Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(b; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 5 ([8]). *Let T be the multiplier operator. Then T is bounded on $L^p(R^n, w)$ for $w \in A_p$ with $1 < p < \infty$.*

Proof of Theorem 1. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q |T_b(f)(x) - C_Q|w(x)dx \leq C \|f\|_{L^\infty(w)}$$

holds for any cube Q . Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha b_j) \tilde{Q} x^\alpha$,

then $R_m(b_j; x, y) = R_m(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
& T_b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} K(x-y) f(y) dy \\
= & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x-y) f_1(y) dy \\
- & \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1} D^{\alpha_1} \tilde{b}_1(y)}{|x-y|^m} K(x-y) f_1(y) dy \\
- & \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2} D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} K(x-y) f_1(y) dy \\
+ & \sum_{\substack{|\alpha_1|=m_1 \\ |\alpha_2|=m_2}} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} K(x-y) f_1(y) dy \\
+ & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} K(x-y) f_2(y) dy \\
= & T \left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_1 \right) \\
- & T \left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1}{|x-\cdot|^m} f_1 \right) \\
- & T \left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2} \tilde{b}_2}{|x-\cdot|^m} f_1 \right) \\
+ & T \left(\sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2}{|x-\cdot|^m} f_1 \right) \\
+ & T \left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_2 \right),
\end{aligned}$$

then

$$|T_b(f)(x) - T_{\tilde{b}}(f_2)(x_0)| \leq \left| T \left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_1 \right) \right|$$

$$\begin{aligned}
 &+ \left| T \left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1}{|x-\cdot|^m} f_1 \right) \right| \\
 &+ \left| T \left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2} \tilde{b}_2}{|x-\cdot|^m} f_1 \right) \right| \\
 &+ \left| T \left(\sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2}{|x-\cdot|^m} f_1 \right) \right| \\
 &+ |T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(x_0)| \\
 &= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x)
 \end{aligned}$$

and

$$\begin{aligned}
 &\frac{1}{w(Q)} \int_Q |T_b(f)(x) - T_{\tilde{b}}(f_2)(x_0)| w(x) dx \\
 \leq &\frac{1}{w(Q)} \int_Q I_1(x) w(x) dx + \frac{1}{w(Q)} \int_Q I_2(x) w(x) dx + \frac{1}{w(Q)} \int_Q I_3(x) w(x) dx \\
 &+ \frac{1}{w(Q)} \int_Q I_4(x) w(x) dx + \frac{1}{w(Q)} \int_Q I_5(x) w(x) dx \\
 = &I_1 + I_2 + I_3 + I_4 + I_5.
 \end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5 , respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 4, we get

$$R_m(\tilde{b}_j; x, y) \leq C|x-y|^m \sum_{|\alpha_j|=m} \|D^{\alpha_j} b_j\|_{BMO},$$

thus, by the the $L^p(w)$ -boundedness of T for $1 < p < \infty$ Lemma 5 and Hölder's inequality, we obtain

$$\begin{aligned}
 I_1 \leq &C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \frac{1}{w(Q)} \int_Q |T(f_1)(x)| w(x) dx \\
 \leq &C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{w(Q)} \int_{R^n} |T(f_1)(x)|^p w(x) dx \right)^{1/p} \\
 \leq &C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{w(Q)} \int_{R^n} |f_1(x)|^p w(x) dx \right)^{1/p}
 \end{aligned}$$

$$\begin{aligned} &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{w(\tilde{Q})}{w(Q)} \right)^{1/p} \|f\|_{L^\infty(w)} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}. \end{aligned}$$

For I_2 , since $w \in A_1$, w satisfies the reverse of Hölder' inequality:

$$\left(\frac{1}{|Q|} \int_Q w(x)^{p_0} dx \right)^{1/p_0} \leq \frac{C}{|Q|} \int_Q w(x) dx$$

for all cube Q and some $1 < p_0 < \infty$ (see [6]), thus, by the L^p -boundedness of T for $p > 1$, we get

$$\begin{aligned} I_2 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{w(Q)} \int_Q |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)| w(x) dx \\ &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{w(Q)} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)|^p w(x) dx \right)^{1/p} \\ &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{w(Q)} \int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) f_1(x)|^p w(x) dx \right)^{1/p} \\ &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} b_1(x) - (D^{\alpha_1} b_1)_{\tilde{Q}}|^{p p'_0} dx \right)^{1/p p'_0} \\ &\times w(Q)^{-1/p} |Q|^{1/p} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{p_0} dx \right)^{1/p p_0} \|f\|_{L^\infty(w)} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(Q)^{-1/p} |Q|^{1/p} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{1/p} \|f\|_{L^\infty(w)} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}. \end{aligned}$$

For I_3 , similar to the proof of I_2 , we get

$$I_3 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.$$

Similarly, for I_4 , choose $1 < r_1, r_2 < \infty$ such that $1/r_1 + 1/r_2 + 1/p_0 = 1$, we obtain, by Hölder' inequality and reverse of Hölder's inequality,

$$I_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{w(Q)} \int_Q |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)| w(x) dx$$

$$\begin{aligned}
 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{w(Q)} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)|^p w(x) dx \right)^{1/p} \\
 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} w(Q)^{-1/p} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x)|^p w(x) dx \right)^{1/p} \\
 &\leq C \sum_{\substack{|\alpha_1|=m_1 \\ |\alpha_2|=m_2}} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(x)|^{pr_1} dx \right)^{1/pr_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(x)|^{pr_2} dx \right)^{1/pr_2} \\
 &\times w(Q)^{-1/p} |Q|^{1/p} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{p_0} dx \right)^{1/pp_0} \|f\|_{L^\infty(w)} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.
 \end{aligned}$$

For I_5 , we write, for $x \in Q$,

$$\begin{aligned}
 &T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(x_0) \\
 = &\int_{R^n} (K(x-y) - K(x_0-y)) \frac{1}{|x-y|^m} \prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y) f_2(y) dy \\
 + &\int_{R^n} \left(\frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right) K(x_0-y) \prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y) f_2(y) dy \\
 + &\int_{R^n} (R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y)) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0-y) f_2(y) dy \\
 + &\int_{R^n} (R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y)) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|x_0-y|^m} K(x_0-y) f_2(y) dy \\
 - &\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x-y) \right. \\
 - &\left. \frac{R_{m_2}(\tilde{b}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0-y) \right] D^{\alpha_1} \tilde{b}_1(y) f_2(y) dy \\
 - &\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x-y) \right. \\
 - &\left. \frac{R_{m_1}(\tilde{b}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0-y) \right] D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x-y) \right. \\
& - \left. \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0-y) \right] D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \\
& = I_5^{(1)}(x) + I_5^{(2)}(x) + I_5^{(3)}(x) + I_5^{(4)}(x) + I_5^{(5)}(x) + I_5^{(6)}(x) + I_5^{(7)}(x).
\end{aligned}$$

By Lemma 4 and the following inequality (see [15])

$$|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}
|R_{m_j}(\tilde{b}_j; x, y)| & \leq C|x-y|^{m_j} \sum_{|\alpha|=m_j} (\|D^\alpha b_j\|_{BMO} + |(D^\alpha b_j)_{\tilde{Q}(x,y)} - (D^\alpha b_j)_{\tilde{Q}}|) \\
& \leq Ck|x-y|^{m_j} \sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}.
\end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on K and Lemma 3,

$$\begin{aligned}
& |I_5^{(1)}(x)| \\
& \leq \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)| |x_0-y|^{-m} \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)| |f(y)| dy \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 |K(x-y) - K(x_0-y)| |f(y)| dy \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^s dy \right)^{1/s'} \\
& \times \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)|^{s'} dy \right)^{1/s'} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 2^{-ka} \|f\|_{L^\infty(w)} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.
\end{aligned}$$

For $I_5^{(2)}(x)$, by Lemma 2, we obtain

$$|I_5^{(2)}(x)|$$

$$\begin{aligned}
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|}{|x_0-y|} |K(x_0-y)| |f(y)| dy \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^\infty k^2 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x_0-y)|^{s'} dy \right)^{1/s'} \\
 &\quad \times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^s dy \right)^{1/s} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^\infty k^2 2^{-k} \|f\|_{L^\infty(w)} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.
 \end{aligned}$$

For $I_5^{(3)}(x)$, by the formula (see [3]):

$$R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{b}_j; x, x_0)(x-y)^\beta$$

and Lemma 4, we have

$$\begin{aligned}
 &|R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y)| \\
 &\leq C \sum_{|\beta| < m_j} \sum_{|\alpha|=m_j} |x-x_0|^{m_j-|\beta|} |x-y|^{|\beta|} \|D^\alpha b_j\|_{BMO},
 \end{aligned}$$

thus, by Lemma 2, we obtain

$$\begin{aligned}
 &|I_5^{(3)}(x)| \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|}{|x_0-y|} |K(x_0-y)| |f(y)| dy \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^\infty k^2 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x_0-y)|^{s'} dy \right)^{1/s'} \\
 &\quad \times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^s dy \right)^{1/s} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^\infty k^2 2^{-k} (2^k d)^{-n/s} (2^k d)^{n/s} \|f\|_{L^\infty(w)} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.
 \end{aligned}$$

Similarly,

$$|I_5^{(4)}(x)| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.$$

For $I_5^{(5)}(x)$, similar to the proofs of $I_5^{(1)}(x)$, $I_5^{(2)}(x)$, $I_5^{(3)}(x)$ and I_4 , we get, for $1 < s_1, s_2 < \infty$ with $1/s_1 + 1/s_2 + 1/s = 1$,

$$\begin{aligned} |I_5^{(5)}(x)| &\leq C \sum_{|\alpha_1|=m_1} \int_{(\tilde{Q})^c} \frac{|(x-y)^{\alpha_1}|}{|x-y|^m} |R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y)| \\ &\quad \times |K(x-y) D^{\alpha_1} \tilde{b}_1(y)| |f(y)| dy \\ &\quad + C \sum_{|\alpha_1|=m_1} \int_{(\tilde{Q})^c} \left| \frac{(x-y)^{\alpha_1}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m} \right| \\ &\quad \times |R_{m_2}(\tilde{b}_2; x_0, y) K(x-y) D^{\alpha_1} \tilde{b}_1(y)| |f(y)| dy \\ &\quad + C \sum_{|\alpha_1|=m_1} \int_{(\tilde{Q})^c} \frac{|R_{m_2}(\tilde{b}_2; x_0, y)| |(x_0-y)^{\alpha_1}|}{|x_0-y|^m} \\ &\quad \times |K(x-y) - K(x_0-y)| |D^{\alpha_1} \tilde{b}_1(y)| |f(y)| dy \\ &\leq C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} k 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y)|^{s_1} dy \right)^{1/s_1} \\ &\quad \times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^s dy \right)^{1/s} \\ &\quad + C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \\ &\quad \times \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} k \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)|^{s_1} dy \right)^{1/s_1} \\ &\quad \times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^s dy \right)^{1/s} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-ka}) \|f\|_{L^\infty(w)} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}. \end{aligned}$$

Similarly,

$$|I_5^{(6)}(x)| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.$$

For $I_5^{(7)}(x)$, similar to the proofs of $I_5^{(5)}(x)$ and I_4 , we get, for $1 < s_1, s_2, s_3 < \infty$ with $1/s_1 + 1/s_2 + 1/s_3 + 1/s = 1$,

$$\begin{aligned} |I_5^{(7)}(x)| &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{(\tilde{Q})^c} \left| \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} \right| \\ &\times |K(x-y)D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy \\ &+ C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{(\tilde{Q})^c} \frac{|(x_0-y)^{\alpha_1+\alpha_2}|}{|x_0-y|^m} \\ &\times |K(x-y) - K(x_0-y)||D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=0}^{\infty} 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y)|^{s_1} dy \right)^{1/s_1} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^s dy \right)^{1/s} \\ &\times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_2}\tilde{b}_2(y)|^{s_3} dy \right)^{1/s_3} \\ &+ C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)|^{s_1} dy \right)^{1/s_1} \\ &\times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_2}\tilde{b}_2(y)|^{s_3} dy \right)^{1/s_3} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^s dy \right)^{1/s} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-ka}) \|f\|_{L^\infty(w)} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}. \end{aligned}$$

Thus

$$I_5 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q |T_b(f)(x) - C_Q|w(x)dx \leq C \|f\|_{B_p(w)}$$

holds for any cube $Q = Q(0, d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha b_j) \tilde{Q} x^\alpha$, then $R_{m_j}(b_j; x, y) = R_{m_j}(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j -$

$(D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. Similar to the proof of Theorem 1, we write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
& \frac{1}{w(Q)} \int_Q |T_b(f)(x) - T_{\tilde{b}}(f_2)(0)| w(x) dx \\
\leq & \frac{1}{w(Q)} \int_Q \left| T \left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x - \cdot|^m} f_1 \right) \right| w(x) dx \\
+ & \frac{1}{w(Q)} \int_Q \left| T \left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x - \cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1}{|x - \cdot|^m} f_1 \right) \right| w(x) dx \\
+ & \frac{1}{w(Q)} \int_Q \left| T \left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x - \cdot)^{\alpha_2} D^{\alpha_2} \tilde{b}_2}{|x - \cdot|^m} f_1 \right) \right| w(x) dx \\
+ & \frac{1}{w(Q)} \int_Q \left| T \left(\sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \frac{(x - \cdot)^{\alpha_1 + \alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2}{|x - \cdot|^m} f_1 \right) \right| w(x) dx \\
+ & \frac{1}{w(Q)} \int_Q |T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(0)| w(x) dx \\
= & L_1 + L_2 + L_3 + L_4 + L_5.
\end{aligned}$$

Similar to the proof of Theorem 1, we get

$$\begin{aligned}
L_1 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{w(Q)} \int_{R^n} |T(f_1)(x)|^p w(x) dx \right)^{1/p} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(\tilde{Q})^{-1/p} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.
\end{aligned}$$

For L_2 , taking $r, s, t > 1$ such that $r < p$, $t = pp_0/(p-r)$ and $1/s + 1/(p/r) + 1/t = 1$, then, by the reverse of Hölder's inequality,

$$\begin{aligned}
L_2 & \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{w(Q)} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)|^r w(x) dx \right)^{1/r} \\
& \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} w(Q)^{-1/r} \sum_{|\alpha_1|=m_1} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) f_1(x)|^r w(x) dx \right)^{1/r}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} w(Q)^{-1/r} \sum_{|\alpha_1|=m_1} \left(\int_{\tilde{Q}} |D^{\alpha} \tilde{b}_1(x)|^{rs} dx \right)^{1/rs} \\
&\times \left(\int_{\tilde{Q}} |f(x)|^p w(x) dx \right)^{1/p} \left(\int_{\tilde{Q}} w(x)^{(1-r/p)t} dx \right)^{1/rt} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(Q)^{-1/r} |\tilde{Q}|^{1/rs} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{p_0/rt} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(\tilde{Q})^{-1/p} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}, \\
L_3 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.
\end{aligned}$$

For L_4 , taking $r, s_1, s_2, t > 1$ such that $r < p$, $t = pp_0/(p-r)$ and $1/s_1 + 1/s_2 + 1/(p/r) + 1/t = 1$, then, by the reverse of Hölder's inequality,

$$\begin{aligned}
L_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{w(Q)} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)|^r w(x) dx \right)^{1/r} \\
&\leq C w(Q)^{-1/r} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x)|^r w(x) dx \right)^{1/r} \\
&\leq C w(Q)^{-1/r} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\int_{\tilde{Q}} |D^{\alpha} \tilde{b}_1(x)|^{rs_1} dx \right)^{1/rs_1} \left(\int_{\tilde{Q}} |D^{\alpha} \tilde{b}_2(x)|^{rs_2} dx \right)^{1/rs_2} \\
&\times \left(\int_{\tilde{Q}} |f(x)|^p w(x) dx \right)^{1/p} \left(\int_{\tilde{Q}} w(x)^{(1-r/p)t} dx \right)^{1/rt} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(Q)^{-1/r} |\tilde{Q}|^{1/rs_1+1/rs_2+1/rt} \\
&\times \|f\chi_{\tilde{Q}}\|_{L^p(w)} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{p_0/rt} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(\tilde{Q})^{-1/p} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.
\end{aligned}$$

For L_5 , similar to the proof of the proof of I_5 in Theorem 1, we have

$$T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(0) = \int_{R^n} \left(\frac{K(x-y)}{|x-y|^m} - \frac{K(-y)}{|y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y) f_2(y) dy$$

$$\begin{aligned}
& + \int_{\mathbb{R}^n} \left(R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; 0, y) \right) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|y|^m} K(-y) f_2(y) dy \\
& + \int_{\mathbb{R}^n} \left(R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; 0, y) \right) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|y|^m} K(-y) f_2(y) dy \\
& - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x-y) - \frac{R_{m_2}(\tilde{b}_2; 0, y)(-y)^{\alpha_1}}{|y|^m} K(-y) \right] \\
& \times D^{\alpha_1} \tilde{b}_1(y) f_2(y) dy \\
& - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x-y) - \frac{R_{m_1}(\tilde{b}_1; 0, y)(-y)^{\alpha_2}}{|y|^m} K(-y) \right] \\
& \times D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \\
& + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x-y) - \frac{(-y)^{\alpha_1+\alpha_2}}{|y|^m} K(-y) \right] \\
& \times D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \\
& = L_5^{(1)}(x) + L_5^{(2)}(x) + L_5^{(3)}(x) + L_5^{(4)}(x) + L_5^{(5)}(x) + L_5^{(6)}(x).
\end{aligned}$$

For $L_5^{(1)}(x)$, taking $1 < q, r < \infty$ such that $1/p + 1/q + 1/r = 1$, by $w \in A_1 \subset A_{p/r+1}$, we get

$$\begin{aligned}
|L_5^{(1)}(x)| & \leq C \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \left| \frac{1}{|x-y|^m} - \frac{1}{|y|^m} \right| \\
& \times |K(x-y)| \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)| |f(y)| w(y)^{1/p} w(y)^{-1/p} dy \\
& + \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(-y)| |y|^{-m} \\
& \times \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)| |f(y)| w(y)^{1/p} w(y)^{-1/p} dy \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathbf{BMO}} \right) \sum_{k=0}^{\infty} k^2 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y)|^q dy \right)^{1/q} \\
& \times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-r/p} dy \right)^{1/r} \\
& + C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathbf{BMO}} \right) \\
& \times \sum_{k=0}^{\infty} k^2 \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(-y)|^q dy \right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-r/p} dy \right)^{1/r} \\
 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \\
 & \times \sum_{k=1}^{\infty} k^2 2^{-k} w(2^k \tilde{Q})^{-1/p} \left(\int_{2^k \tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \\
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/p} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(p-1)} dy \right)^{(p-1)/p} \\
 & + C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \\
 & \times \sum_{k=1}^{\infty} k^2 2^{-ka} w(2^k \tilde{Q})^{-1/p} \left(\int_{2^k \tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \\
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/p} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-r/p} dy \right)^{1/r} \\
 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.
 \end{aligned}$$

Similarly, we get, for $1 < q, r_1, r_2, r_3, s < \infty$ with $1/p + 1/q + 1/r_1 + 1/s = 1$ and $1/p + 1/q + 1/r_2 + 1/r_3 + 1/s = 1$,

$$\begin{aligned}
 & |L_5^{(2)}(x) + L_5^{(3)}(x) + L_5^{(4)}(x) + L_5^{(5)}(x) + L_5^{(6)}(x)| \\
 & \leq C \left(\sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \right) \sum_{|\alpha|=m_1} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{k d}{2^k d} |K(-y)| \|D^{\alpha_1} \tilde{b}_1(y)\| |f(y)| dy \\
 & + C \left(\sum_{|\alpha|=m_1} \|D^\alpha b_1\|_{BMO} \right) \sum_{|\alpha|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{k d}{2^k d} |K(-y)| \|D^{\alpha_2} \tilde{b}_2(y)\| |f(y)| dy \\
 & + C \left(\sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \right) \sum_{|\alpha|=m_1} \sum_{k=0}^{\infty} k \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} |K(x-y) - K(-y)| \|D^{\alpha_1} \tilde{b}_1(y)\| |f(y)| dy \\
 & + C \left(\sum_{|\alpha|=m_1} \|D^\alpha b_1\|_{BMO} \right) \sum_{|\alpha|=m_2} \sum_{k=0}^{\infty} k \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} |K(x-y) - K(-y)| \|D^{\alpha_2} \tilde{b}_2(y)\| |f(y)| dy \\
 & + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} |K(x-y) - K(-y)| \|D^{\alpha_1} \tilde{b}_1(y)\| \|D^{\alpha_2} \tilde{b}_2(y)\| |f(y)| dy \\
 & \leq C \left(\sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \right) \sum_{|\alpha|=m_1} \sum_{k=0}^{\infty} k 2^{-k} \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{r_1} dy \right)^{1/r_1} \\
 & \times \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} |K(-y)|^q dy \right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p} dy \right)^{1/s}
 \end{aligned}$$

$$\begin{aligned}
 &+ C \left(\sum_{|\alpha|=m_1} \|D^\alpha b_1\|_{BMO} \right) \sum_{|\alpha|=m_2} \sum_{k=0}^{\infty} k 2^{-k} \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(y)|^{r_1} dy \right)^{1/r_1} \\
 &\times \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(-y)|^q dy \right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p} dy \right)^{1/s} \\
 &+ C \left(\sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \right) \sum_{|\alpha|=m_1} \sum_{k=0}^{\infty} k \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(0, y)|^q dy \right)^{1/q} \\
 &\times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{r_1} dy \right)^{1/r_1} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p} dy \right)^{1/s} \\
 &+ C \left(\sum_{|\alpha|=m_1} \|D^\alpha b_1\|_{BMO} \right) \sum_{|\alpha|=m_2} \sum_{k=0}^{\infty} k \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(0, y)|^q dy \right)^{1/q} \\
 &\times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(y)|^{r_1} dy \right)^{1/r_1} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p} dy \right)^{1/s} \\
 &+ C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(0, y)|^q dy \right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p} dy \right)^{1/s} \\
 &\times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{r_2} dy \right)^{1/r_2} \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(y)|^{r_3} dy \right)^{1/r_3} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-ka}) w(2^k \tilde{Q})^{-1/p} \left(\int_{2^k \tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.
 \end{aligned}$$

Thus

$$L_5 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.$$

This finishes the proof of Theorem 2. □

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