GENERALIZED DERIVATIONS ACTING AS HOMOMORPHISM OR ANTI-HOMOMORPHISM WITH CENTRAL VALUES IN SEMIPRIME RINGS

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Abstract. Let $R$ be a semiprime ring with center $Z(R)$. A mapping $F : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper, our main object is to study the situations: (1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$, (3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) - F(y)F(x) \in Z(R)$; for all $x, y$ in some suitable subset of $R$.

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1. INTRODUCTION

Let $R$ be an associative ring with center $Z(R)$. For $x, y \in R$, $[x, y]$ denotes the commutator element $xy - yx$. We use the notation to define the Engel type polynomials $[x, y]_{n+1} = [[x, y]_n, y]$ instead of $[x, y, y, \ldots, y]$ for $n \geq 1$ and $[x, y]_1 = [x, y]$. Recall that a ring $R$ is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is called semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $F : R \rightarrow R$ is called a generalized derivation of $R$ if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for any $x, y \in R$. If $d = 0$, then $F$ is said to be a left centralizer map of $R$. For any subset $S$ of $R$, $r_R(S)$ denotes the right annihilator of $S$ in $R$, that is, $r_R(S) = \{x \in R | Rx = 0\}$ and $l_R(S)$ denotes the left annihilator of $S$ in $R$ that is, $l_R(S) = \{x \in R | xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of $R$ and is written as $ann_R(S)$.

Let $S$ be a nonempty subset of a ring $R$. The mapping $F : R \rightarrow R$ is said to be a homomorphism (anti-homomorphism) acting on $S$ if $F(xy) = F(x)F(y)$ holds for all $x, y \in S$ (respectively $F(xy) = F(y)F(x)$ holds for all $x, y \in S$).
A series of papers in literature studied the homomorphism or anti-homomorphism of some specific type of additive mappings in prime and semiprime rings under certain conditions (see [1–4, 7, 10–12, 14, 15]).

In [4], Bell and Kappe showed that if a derivation \( d \) of a prime ring \( R \) can act as homomorphism or anti-homomorphism on a nonzero right ideal of \( R \), then \( d = 0 \) on \( R \). Then Ali, Rehman and Ali in [2] proved a similar result to Lie ideal case. They proved that if \( R \) is a 2-torsion free prime ring, \( L \) a nonzero Lie ideal of \( R \) such that \( u^2 \in L \) for all \( u \in L \) and \( d \) acts as a homomorphism or anti-homomorphism on \( L \), then either \( d = 0 \) or \( L \subseteq Z(R) \).

On the other hand, the authors developed above results, replacing the derivation \( d \) with a generalized derivation \( F \) of \( R \). In this view, Rehman [14] proved the following:

Let \( R \) be a 2-torsion free prime ring and \( I \) be a nonzero ideal of \( R \). Suppose \( F : R \to R \) is a nonzero generalized derivation with \( d \).

(i) If \( F \) acts as a homomorphism on \( I \) and if \( d \neq 0 \), then \( R \) is commutative.

(ii) If \( F \) acts as an anti-homomorphism on \( I \) and if \( d \neq 0 \), then \( R \) is commutative.

Recently, in [3] Ali and Huang studied the case when a generalized Jordan \((\alpha, \beta)\)-derivation \( F \) acts as homomorphism or anti-homomorphism on a square closed Lie ideal \( U \) in prime ring \( R \).

It is natural to investigate the above situations in semiprime rings. Recently, in [7] the first author of this article has studied the situations, when a generalized derivation \( F \) of a semiprime ring \( R \) acts as homomorphism or anti-homomorphism in a nonzero left ideal of \( R \).

From above results, it is natural to consider the situations, when the generalized derivations \( F \) satisfies the identities: (1) \( F(xy) - F(x)F(y) \in Z(R) \), (2) \( F(xy) + F(x)F(y) \in Z(R) \), (3) \( F(xy) - F(y)F(x) \in Z(R) \), (4) \( F(xy) + F(y)F(x) \in Z(R) \); for all \( x, y \) in some suitable subset of \( R \).

Recently, Albas [1] studied the above mentioned identities in prime rings. Albas proved the following theorems:

**Theorem A.** Let \( R \) be a prime ring with center \( Z(R) \) and \( I \) be a nonzero ideal of \( R \). If \( R \) admits a nonzero generalized derivation \( F \) of \( R \), with associated derivation \( d \) such that \( F(xy) - F(x)F(y) \in Z(R) \) or \( F(xy) + F(x)F(y) \in Z(R) \) for all \( x, y \in I \), then either \( R \) is commutative or \( F = I_{id} \) or \( F = -I_{id} \), where \( I_{id} \) denotes the identity map of the ring \( R \).

**Theorem B.** Let \( R \) be a prime ring with center \( Z(R) \) and \( I \) be a nonzero ideal of \( R \). If \( R \) admits a nonzero generalized derivation \( F \) of \( R \), with associated derivation \( d \) such that \( F(xy) - F(y)F(x) \in Z(R) \) or \( F(xy) + F(y)F(x) \in Z(R) \) for all \( x, y \in I \), then \( R \) is commutative.
In the present paper our main object is to investigate the situations in semiprime rings.

2. PRELIMINARIES

We shall use following basic identities which will be used frequently: for \( x, y, z \in R \),
\[
[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z.
\]
We need the following facts which will be used to prove our theorems:

Fact-1. [5, Theorem 3] Let \( R \) be a semiprime ring and \( U \) a nonzero left ideal of \( R \). If \( R \) admits a derivation \( d \) which is nonzero on \( U \) and \( [d(x), x] \in Z(R) \) for all \( x \in U \), then \( R \) contains a nonzero central ideal.

Fact-2. [8, Fact-4] Let \( R \) be a semiprime ring, \( d \) a nonzero derivation of \( R \) such that \( x[\{d(x), x\}, x] = 0 \) for all \( x \in R \). Then \( d \) maps \( R \) into its center.

Fact-3. [13, Corollary 2] If \( R \) is a semiprime ring and \( I \) is an ideal of \( R \), then \( I \cap \text{ann}_R(I) = 0 \).

Fact-4. [6, Lemma 2] (a) If \( R \) is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of \( R \); in particular, any commutative one-sided ideal is contained in the center of \( R \).

(b) If \( R \) is a prime ring with a nonzero central ideal, then \( R \) must be commutative.

Fact-5. [9, Corollary 2.6] Let \( R \) be a prime ring, \( I \) a nonzero ideal of \( R \) and \( F : R \to R \) a nonzero left centralizer map. (1) If \( F(x)F(y) = F(xy) \in Z(R) \) for all \( x, y \in I \), then either \( R \) is commutative or \( F(r) = r \) for all \( r \in R \). (2) If \( F(x)F(y) + F(xy) \in Z(R) \) for all \( x, y \in I \), then either \( R \) is commutative or \( F(r) = -r \) for all \( r \in R \).

3. MAIN RESULTS

**Theorem 1.** Let \( R \) be a semiprime ring with center \( Z(R) \) and \( I \) a nonzero ideal of \( R \). Let \( F : R \to R \) be a generalized derivation associated with the derivation \( d : R \to R \). If \( F(xy) - F(x)F(y) \in Z(R) \) for all \( x, y \in I \), then one of the following holds:

1. \( R \) contains a nonzero central ideal;
2. \( d(I) = (0) \) and \( F \) is a left centralizer map on \( I \) such that \( [F(x), x] = 0 \) for all \( x \in I \).

**Proof.** By our assumption, we have
\[
F(xy) - F(x)F(y) \in Z(R) \quad (3.1)
\]
for all \( x, y \in I \). Replacing \( y \) with \( yz \), where \( z \in I \), we get
\[
F(xy)z - F(x)F(yz) \in Z(R)
\] (3.2)
which implies
\[
F(xy)z + yzd(z) - F(x)\{F(y)z + yd(z)\} \in Z(R)
\] (3.3)
that is
\[
(F(xy) - F(x)F(y))z + (x - F(x))yd(z) \in Z(R).
\] (3.4)
Commuting both sides with \( z \), we get
\[
[(F(xy) - F(x)F(y))z + (x - F(x))yd(z), z] = 0
\] (3.5)
for all \( x, y, z \in I \). By using (3.1), above relation yields
\[
[(x - F(x))yd(z), z] = 0
\] (3.6)
for all \( x, y, z \in I \). Now we put \( x = xz \), and then obtain that
\[
[(xz - F(x)z - xd(z))yd(z), z] = 0
\] (3.7)
which is
\[
[(x - F(x))z yd(z), z] - [xd(z)yd(z), z] = 0
\] (3.8)
for all \( x, y, z \in I \). In (3.6), replacing \( y \) with \( yz \), we get
\[
[(x - F(x))zyd(z), z] = 0
\] (3.9)
for all \( x, y, z \in I \). Using (3.9), (3.8) implies
\[
[xd(z)yd(z), z] = 0
\] (3.10)
for all \( x, y, z \in I \). Now we put \( x = d(z)x \) in (3.10), and then we see that
\[
0 = [d(z)xd(z)yd(z), z] = d(z)[zd(z)yd(z), z] + [d(z), z]xd(z)yd(z)
\] (3.11)
for all \( x, y, z \in I \). As an application of (3.10), (3.11) reduces to
\[
[d(z), z]xd(z)yd(z) = 0
\] (3.12)
for all \( x, y, z \in I \). Replacing \( x \) with \( xz \) and \( y \) with \( yz \) respectively in (3.12), we get
\[
[d(z), z]xzbd(z)yd(z) = 0
\] (3.13)
and
\[
[d(z), z]xd(z)zd(yd(z)) = 0
\] (3.14)
for all \( x, y, z \in I \). Subtracting one from another yields
\[
[d(z), z]xd(z)yd(z) = 0
\] (3.15)
for all \( x, y, z \in I \). Replacing \( y \) with \( yz \) in (3.15) and right multiplying (3.15) by \( z \) respectively and then subtracting one from another yields
\[
[d(z), z]zd(z)y[d(z), z] = 0
\] (3.16)
for all \(x, y, z \in I\), which implies \((I[d(z), z])^3 = (0)\) for all \(z \in I\). Since \(R\) is semiprime, it contains no nonzero nilpotent left ideal, implying \(I[d(z), z] = (0)\) for all \(z \in I\). Thus, \([d(z), z] \in Ann_R(I)\) for all \(z \in I\). Since \(I\) is an ideal, we conclude that \([d(z), z] \in I\) for all \(z \in I\). This implies that \([d(z), z] \in I \cap Ann_R(I)\) for all \(z \in I\). In view of Fact-3, \([d(z), z] = 0\) for all \(z \in I\). Further, if \(d\) is derivation such that \(d(I) \neq (0)\), then by Fact-1, \(R\) contains a nonzero central ideal.

Let \(d(I) = (0)\). Then \(F(xy) = F(x)y + xd(y) = F(x)y\) for all \(x, y \in I\), i.e., \(F\) is a left centralizer map on \(I\). Then by our hypothesis, we have

\[
F(x)(y - F(y)) \in Z(R) \tag{3.17}
\]

for all \(x, y \in I\). Replacing \(y\) with \(yu\), where \(u \in I\), we get

\[
F(x)(y - F(y))u \in Z(R) \tag{3.18}
\]

for all \(x, y, u \in I\). Commuting both sides with \(v\), where \(v \in I\), we get

\[
F(x)(y - F(y))uv - vF(x)(y - F(y))u = 0. \tag{3.19}
\]

By using (3.18), it reduces to

\[
vF(x)(y - F(y))u \in Z(R) \tag{3.20}
\]

for all \(u, v, x, y \in I\). We choose \(x, y \in I\) such that \(a = F(x)(y - F(y)) \neq 0\). Then from above, we have \(I \cap I \subseteq Z(R)\), that is, \(R\) contains a central ideal. If this ideal is zero ideal, then

\[
I(F(x)(y - F(y))) = (0)
\]

for all \(x, y \in I\). Replacing \(x\) with \(xz, z \in I\), this gives

\[
I(F(x)z(y - F(y))) = (0)
\]

for all \(x, y, z \in I\). Thus \(F(x)z(y - F(y)) \in I \cap Ann_R(I) = (0)\) for all \(x, y, z \in I\). This gives

\[
[F(x), x]z(y - F(y)) = 0
\]

for all \(x, y, z \in I\). Putting \(y = y^2\) and \(z = zy\) respectively and then subtracting one from another, we get

\[
[F(x), x]z[F(y), y] = 0
\]

for all \(x, y, z \in I\). Since \(I\) is an ideal of \(R\), it follows that \((F(x), x)^2 = (0)\) for all \(x \in I\). Since semiprime ring contains no nonzero nilpotent ideal, we have \([F(x), x]I = (0)\) for all \(x \in I\). Thus by Fact-3, \([F(x), x] \in I \cap Ann_R(I) = (0)\) for all \(x \in I\). Thereby, the proof is completed. \(\square\)

**Theorem 2.** Let \(R\) be a semiprime ring with center \(Z(R)\) and \(I\) a nonzero ideal of \(R\). Let \(F : R \to R\) be a generalized derivation associated with the derivation \(d : R \to R\). If \(F(xy) + F(x)F(y) \in Z(R)\) for all \(x, y \in I\), then one of the following holds:

1. \(R\) contains a nonzero central ideal;
(2) \( d(I) = (0) \) and \( F \) is a left centralizer map on \( I \) such that \([F(x), x] = 0\) for all \( x \in I \).

Proof. If we replace \( F \) with \(-F\) and \( d \) with \(-d\) in Theorem 1, we conclude that 
\[ (-F)(xy) - (-F)(x)(-F)(y) \in Z(R) \]
for all \( x, y \in I \), implies \([(-d)(x), x] = 0\) for all \( x \in I \), that is, \( F(xy) + F(x)F(y) \in Z(R) \) for all \( x, y \in I \), implies \([d(x), x] = 0\) for all \( x \in I \). Hence conclusion follows by Theorem 1. \(\square\)

The following corollary is immediate consequences of the Theorem 1 and Theorem 2 by using Fact-4 and Fact-5.

Corollary 1. Let \( R \) be a prime ring with center \( Z(R) \) and \( F : R \to R \) be a generalized derivation associated with the derivation \( d : R \to R \). 
(1) If \( R \) satisfies \( F(xy) + F(x)F(y) \in Z(R) \), then either \( R \) is commutative or \( F(x) = x \) for all \( x \in R \). 
(2) If \( R \) satisfies \( F(xy) - F(x)F(y) \in Z(R) \), then either \( R \) is commutative or \( F(x) = x \) for all \( x \in R \).

Theorem 3. Let \( R \) be a semiprime ring with center \( Z(R) \). Let \( F : R \to R \) be a generalized derivation associated with the derivation \( d : R \to R \). If \( F(xy) - F(y)F(x) \in Z(R) \) for all \( x, y \in R \), then one of the following holds: (1) \( R \) contains a nonzero central ideal; (2) \( F \) is a left centralizer map of \( R \) such that \( F : R \to Z(R) \).

Proof. By hypothesis, we have
\[ F(xy) - F(y)F(x) \in Z(R) \quad (3.21) \]
for all \( x, y \in R \). Putting \( x = xz \), we have
\[ F(xz) - F(x)(x)(z) + xd(z) \in Z(R) \quad (3.22) \]
which gives
\[ F(x)y + xd(z)y - F(y)F(x)z - F(y)xd(z) \in Z(R). \quad (3.23) \]
Commuting both sides with \( z \), we have
\[ [F(x)y - F(y)F(x)z - F(y)xd(z) + xd(z)y, z] = 0 \quad (3.24) \]
that is
\[ [F(x)y, z] - [F(y)F(x), z]z - [F(y)xd(z) - xd(z)y], z = 0 \quad (3.25) \]
for all \( x, y, z \in R \). From (3.21), we can write that \([F(xy) - F(y)F(x), z] = 0\) for all \( x, y, z \in R \), that is, \([F(xy), z] = [F(y)F(x), z] \) for all \( x, y, z \in R \). Thus (3.25) reduces to
\[ [F(xy), z] - [F(y)F(x), z]z - [F(y)xd(z) - xd(z)y], z = 0 \quad (3.26) \]
for all \( x, y, z \in R \). Putting \( y = z^2 \) in (3.26), we have
\[ [F(x)z^3, z] - [F(x)z^2 + xd(z^2), z]z - [(F(z)c)zd(z)]zd(z) - xd(z^3), z] = 0 \quad (3.27) \]
that is,
\[
[F(z)zd(z) + zd(z)zd(z) - zd(z^3) + zd(z^2)z, z] = 0
\] (3.28)
for all \( x, z \in R \). Putting \( x = xz \) in (3.26), we have
\[
[(F(z)x + zd(x))z, z] - [F(z)xy + zd(xy), z] - [F(y)zd(z) - zd(z), z] = 0
\] (3.29)
that is,
\[
[F(z)xyz, z] - [F(z)xy, z] - [F(y)zd(z) - zd(z), z] = 0
\] (3.30)
for all \( x, y, z \in R \). Assuming \( y = z \), we have
\[
[F(z)zd(z) - zd(x)z^2 - zd(xz)z, z] = 0
\] (3.31)
for all \( x, z \in R \). Subtracting (3.31) from (3.28), we get
\[
[zd(z)zd(z) - xd(z^3) + zd(z^2)z + zd(z^2) - zd(xz)z, z] = 0
\] (3.32)
for all \( x, z \in R \). This reduces to
\[
[zd(z)zd(z), z] + [-xd(z^3) + zd(z^2) + zd(z^2) - zd(xz)z, z] = 0
\] (3.33)
for all \( x, z \in R \). Left multiplying (3.33) by \( z \) and then subtracting from (3.34), we get
\[
[z[d(z), z]zd(z), z] = 0
\] (3.35)
for all \( x, z \in R \). Again putting \( x = xz \) in (3.33), we get
\[
[z[d(z), z]xz[d(z), z] = 0
\] (3.36)
for all \( x, z \in R \). Now right multiplying (3.35) by \( z \) and then subtracting from (3.36), we obtain
\[
[z[d(z), z]x[d(z), z], z] = 0
\] (3.37)
and hence
\[
[z[d(z), z]xz[d(z), z] = 0
\] (3.38)
for all \( x, z \in R \). This implies
\[
z[d(z), z]xz[d(z), z] - z^2[d(z), z]xz[d(z), z] = 0
\] (3.39)
for all \( x, z \in R \). In (3.39), replacing \( x \) with \( xz[d(z), z], z \) and \( u \), we obtain
\[
z[d(z), z]xz[d(z), z] - z^2[d(z), z]xz[d(z), z] = 0
\] (3.40)
for all \( x, u, z \in R \). Using (3.39), (3.40) gives
\[
z[d(z), z]xz^2[d(z), z]uz[d(z), z] - z[d(z), z]xz[d(z), z]uz[d(z), z] = 0
\] (3.41)
that is
\[ z[d(z), z]x[z[d(z), z], z]uz[d(z), z] = 0 \]  
for all \( x, u, z \in R \). This implies \([z[d(z), z], z]x[z[d(z), z], z]u[z[d(z), z], z] = 0\) for all \( x, u, z \in R \), which is \((R[z[d(z), z], z])^3 = 0\) for all \( z \in R \). Since \( R \) is semiprime, we conclude that \( R[z[d(z), z], z] = 0 \) for all \( z \in R \). Hence, \( z[(d(z), z), z] = 0 \) for all \( z \in R \). Then by Fact-2, either \( d(R) = (0) \) or \( d(R) \subseteq Z(R) \). If \( d(R) \neq (0) \), then the second case implies \([d(x), x] = 0\) for all \( x \in R \). Hence in view of Fact-1, \( R \) contains a nonzero central ideal.

Let \( d(R) = (0) \). Then \( F(xy) = F(x)y + xd(y) = F(x)y \) for all \( x, y \in R \), i.e., \( F \) is a left centralizer map of \( R \). Then by our hypothesis, we have
\[ F(xy) - F(y)F(x) \in Z(R) \]
for all \( x, y \in R \). Replacing \( y \) with \( uu \), where \( u \in R \), we get
\[ F(xy)u - F(y)uF(x) \in Z(R) \]
that is
\[ (F(xy) - F(y)F(x))u + F(y)[F(x), u] \in Z(R) \]
for all \( x, y, u \in R \). Commuting both sides with \( u \), we get
\[ [(F(xy) - F(y)F(x))u, u] + [F(y), F(x), u] = 0 \]
for all \( x, y, u \in R \). Since \( F(xy) - F(y)F(x) \in Z(R) \) for all \( x, y \in R \), we have from (3.46) that
\[ [F(y), F(x), u] = 0 \]
for all \( x, y, u \in R \). We put \( y = yr \) in above and get
\[ [F(y)r, F(x), u] = 0 \]
for all \( x, y, u, r \in R \). Now putting \( y = yu \) in above, we have
\[ [F(y)ur, F(x), u] = 0 \]
for all \( x, y, u, r \in R \). Left multiplying (3.48) by \( u \), we get
\[ u[F(y)r, F(x), u] = 0 \]
for all \( x, y, u, r \in R \). Subtracting (3.50) from (3.49), we obtain that
\[ [[F(y), u]r, F(x), u] = 0 \]
for all \( x, y, u, r \in R \). In particular, above relation yields
\[ [[F(x), u]r, F(x), u] = 0 \]
that is
\[ [F(x), u]r[F(x), u] - u[F(x), u]r[F(x), u] = 0 \]
for all \( x, u, r \in R \). In (3.53), replacing \( r \) with \( r[F(x), u]v \) we get
\[ [F(x), u]r[F(x), u]v[F(x), u] - u[F(x), u]r[F(x), u]v[F(x), u] = 0. \]
By using (3.53), (3.54) becomes
\[ [F(x), u]u[F(x), u]v[F(x), u] - [F(x), u]r[F(x), u]uv[F(x), u] = 0, \] (3.55)
which is
\[ [F(x), u]r[F(x), u]uv[F(x), u] = 0 \] (3.56)
for all \( x, r, u, v \in R \). Replacing \( v \) with \( vu \) in (3.56) and right multiplying (3.56) by \( u \) respectively and then subtracting one from another, we have
\[ [F(x), u]r[F(x), u]uv[F(x), u] = 0. \] (3.57)
Similarly, just from above relation, we can write
\[ [F(x), u]r[F(x), u]uv[F(x), u] = 0. \] (3.58)
Thus \( [F(x), u]u[F(x), u]u[F(x), u] = 0 \) for all \( x, u \in R \). Since \( R \) is semiprime, \( R \) contains no nonzero nilpotent ideals. Hence \( [F(x), u]u[F(x), u]u[F(x), u] = 0 \) for all \( x, u \in R \). Then by Fact-1, we conclude that either \( [F(x), u] = 0 \) for all \( x, u \in R \), then \( F \) maps \( R \) into its center. Thus we obtain our all conclusions. □

**Theorem 4.** Let \( R \) be a semiprime ring with center \( Z(R) \). Let \( F : R \to R \) be a generalized derivation associated with the derivation \( d : R \to R \). If \( F(xy) + F(y)F(x) \in Z(R) \) for all \( x, y \in R \), then one of the following holds: (1) \( R \) contains a nonzero central ideal; (2) \( F \) is a left centralizer map of \( R \) such that \( F : R \to Z(R) \).

**Proof.** If we replace \( F \) with \( -F \) and \( d \) with \( -d \) in Theorem 3, we conclude that \( (F(xy) - F(y)(-F)(x)) \in Z(R) \) for all \( x, y \in R \) implies \( x[(d)(x), x]_2 = 0 \) for all \( x \in I \), that is, \( F(xy) + F(y)F(x) \in Z(R) \) for all \( x, y \in R \), implies \( x[d(x), x]_2 = 0 \) for all \( x \in R \). Hence the conclusion follows by Theorem 3. □

We conclude our paper with the following example which shows that the above theorems do not hold for arbitrary rings.

**Example:** Consider the ring \( R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{Z} \right\} \). Obviously, \( R \) is not semiprime, because \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0) \).

We define maps \( F, d : R \to R \) by \( F \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) and \( d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} \). Then \( F \) is a generalized derivation of \( R \) associated with the derivation \( d \) of \( R \). For \( I = R \), we have that \( F(xy) - F(x)F(y) \in Z(R) \) for all \( x, y \in I \) and \( F(xy) - F(y)F(x) \in Z(R) \) for all \( x, y \in I \). Since \( d(R) \neq (0) \) and \( R \) contains no nonzero central ideal for \( Z(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \), the semiprimeness hypothesis in Theorem 1 and Theorem 3 is not superfluous.
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