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A GENERALIZATION OF J-QUASIPOLAR RINGS

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Abstract. In this paper we introduce a class of quasipolar rings which is a generalization of J-quasipolar rings. Let R be a ring with identity. An element $a \in R$ is called δ-*quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p$ is contained in $\delta(R)$, and the ring R is called δ-*quasipolar* if every element of R is δ-quasipolar. We use δ-quasipolar rings to extend some results of J-quasipolar rings. Then some of the main results of J-quasipolar rings are special cases of our results for this general setting. We give many characterizations and investigate general properties of δ-quasipolar rings.

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1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. According to Koliha and Patricio [10], the *commutant* and *double commutant* of an element $a \in R$ are defined by $comm(a) = \{x \in R \mid xa = ax\}$, $comm^2(a) = \{x \in R \mid xy = yx$ for all $y \in comm(a)\}$, respectively. If $R^{qnil} = \{a \in R \mid 1 + ax \in U(R)$ for every $x \in comm(a)\}$ and $a \in R^{qnil}$, then a is said to be *quasinilpotent* (see [9]). The element a is called *quasipolar* if there exists $p^2 = p \in R$ such that $p \in comm^2(a)$, $a + p$ is invertible in R and $ap \in R^{qnil}$. Any idempotent p satisfying the above conditions is called a *spectral idempotent* of a, and this term is borrowed from spectral theory in Banach algebra and it is unique for a.

Quasipolar rings have been studied by many ring theorists (see [1, 2, 5–7, 9, 10] and [15]). In [7], the element $a \in R$ is called *nil-quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p$ is nilpotent, the idempotent p is called a *nil-spectral idempotent* of a. The ring R is said to be *nil-quasipolar* if every element of R is nil-quasipolar. Recently, J-quasipolar rings are studied in [4]. The element a is called J-*quasipolar* if there exists $p^2 = p \in R$ such that $p \in comm^2(a)$ and $a + p \in J(R)$, p is called a J-*spectral idempotent* of a. The ring R is said to be J-*quasipolar* if

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every element of R is J-quasipolar. Motivated by these, we introduce a new class of quasipolar rings which is a generalization of J-quasipolar rings. By using δ-quasipolar rings, we extend some results of J-quasipolar rings.

An outline of the paper is as follows: Section 2 deals with δ-quasipolar rings. We prove various basic characterizations and properties of δ-quasipolar rings. It is proven that every J-quasipolar ring is δ-quasipolar. We supply an example to show that all δ-quasipolar rings need not be J-quasipolar. Among others the δ-quasipolarity of Dorroh extensions and some classes of matrix rings are investigated. In Section 3, we introduce an upper class of δ-quasipolar rings, namely, weakly δ-quasipolar rings. We show that every direct summand of a weakly δ-quasipolar ring is weakly δ-quasipolar and every direct product of weakly δ-quasipolar rings is weakly δ-quasipolar, and we give some properties of such rings.

In what follows, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. For a positive integer n, let $Mat_n(R)$ denote the ring of all $n \times n$ matrices and $T_n(R)$ the ring of all $n \times n$ upper triangular matrices with entries in R. We write $J(R)$ and $nil(R)$ for the Jacobson radical of R and the set of nilpotent elements of R, respectively.

2. δ-QUASIPOLAR RINGS

In this section we introduce the concept of δ-quasipolar rings and investigate some properties of such rings. We show that every quasipolar ring need not be δ-quasipolar (Example 2). It is proven that every J-quasipolar ring is δ-quasipolar and the converse does not hold in general (see Example 3). Among others we extend some results of J-quasipolar rings for this general setting.

A right ideal I of the ring R is said to be δ-*small* in R if whenever $R = I + K$ with R/K singular right R-module for any right ideal K then $R = K$. In [16], the ideal $\delta(R)$ is introduced as a sum of δ-small right ideals of R. We begin with the equivalent conditions for $\delta(R)$ which is proved in [16, Theorem 1.6] for an easy reference for the reader.

Lemma 1. *Given a ring R, each of the following sets is equal to $\delta(R)$.*

(1) R_1 = *the intersection of all essential maximal right ideals of R.*
(2) R_2 = *the unique largest δ-small right ideal of R.*
(3) $R_3 = \{x \in R \mid xR + K_R = R$ *implies K_R is a direct summand of R_R*$\}$.
(4) $R_4 = \bigcap\{$*ideals P of $R \mid R/P$ has a faithful singular simple module*$\}$.
(5) $R_5 = \{x \in R \mid$ *for all $y \in R$ there exists a semisimple right ideal Y of R such that* $(1 + xy)R \oplus Y = R_R\}$.

Now we give our main definition.

Definition 1. Let R be a ring. An element $a \in R$ is called *δ-quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in \delta(R)$ and p is called a *δ-spectral idempotent*. The ring R is called *δ-quasipolar* if every element of R is δ-quasipolar.

The following are examples for δ-quasipolar rings.

Example 1. (1) Every semisimple ring and every Boolean ring is δ-quasipolar.
(2) Since $\delta(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}$ is δ-quasipolar. On the other hand, $\mathbb{Z}$ is not δ-quasipolar since $\delta(\mathbb{Z}) = 0$.

One may suspects that every quasipolar ring is δ-quasipolar. But the following example erases the possibility.

Example 2. Let p be a prime integer with $p \geq 3$ and $R = \mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at the ideal (p). By [4, Example 2.8], R is a quasipolar ring. Since $J(R) = \delta(R)$, it is not δ-quasipolar.

Let S_r denote the right socle of the ring R, that is, S_r is the sum of minimal right ideals of R. We now prove that the class of J-quasipolar rings is a subclass of δ-quasipolar rings.

Lemma 2. *If R is a J-quasipolar ring, then R is δ-quasipolar. The converse holds if $S_r \subseteq J(R)$.*

Proof. The first assertion is clear since $J(R) \subseteq \delta(R)$. Assume that R is δ-quasipolar. If $S_r \subseteq J(R)$, then $J(R)/S_r = J(R/S_r) = \delta(R)/S_r$ by [16, Corollary 1.7] and we have $J(R) = \delta(R)$. Hence, R is J-quasipolar. □

The converse of Lemma 2 is not true in general as the following example shows.

Example 3. Let F be a field and consider the ring $R = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$. Then R is a semisimple ring and $R = \delta(R)$ and $J(R) = 0$. Hence R is δ-quasipolar and it is not J-quasipolar.

Lemma 3. *Let R be a ring. Then we have the following.*

(1) *If $a, u \in R$ and u is invertible, then a is δ-quasipolar if and only if $u^{-1}au$ is δ-quasipolar.*
(2) *The element $a \in R$ is δ-quasipolar if and only $-1-a$ is δ-quasipolar.*
(3) *If R is a δ-quasipolar ring with $\delta(R) = J(R)$, then the spectral idempotent for any invertible element in R is the identity of R.*

Proof. (1) Assume that a is δ-quasipolar. Let $p^2 = p \in comm^2(a)$ such that $a + p \in \delta(R)$. Let $x \in comm(u^{-1}au)$. Then $(uxu^{-1})a = a(uxu^{-1})$. Since $p \in comm^2(a)$, $(uxu^{-1})p = p(uxu^{-1})$. Hence $(u^{-1}pu)^2 = u^{-1}pu \in comm^2(u^{-1}au)$. Since $\delta(R)$ is an ideal of R, $u^{-1}(a + p)u = u^{-1}au + u^{-1}pu \in \delta(R)$. Thus $u^{-1}au$ is δ-quasipolar. Conversely, if $u^{-1}au$ is δ-quasipolar, then by the preceding proof

$u(u^{-1}au)u^{-1} = a$ is δ-quasipolar.
(2) Assume that a is δ-quasipolar. Let $p^2 = p \in comm^2(a)$ such that $a + p = r \in \delta(R)$. Then $-1 - a + (1 - p) = -r \in \delta(R)$. Then $1 - p \in comm^2(-1 - a)$ and $1 - p$ is the spectral idempotent of $-1 - a$. Conversely, if $-1 - a$ is δ-quasipolar, then from what we have proved that $-1 - (-1 - a) = a$ is quasipolar.
(3) Assume that $\delta(R) = J(R)$. Then δ-quasipolarity of R implies J-quasipolarity of R. So its proof can be directly obtained from [4, Example 2.2]. □

In [4, Corollary 2.3], it is proved that if R is a J-quasipolar ring, then $2 \in J(R)$. In this direction we prove the following.

Lemma 4. *If R is a δ-quasipolar ring, then $2 \in \delta(R)$.*

Proof. For the identity 1, there exists $p^2 = p \in R$ such that $1 + p \in \delta(R)$. Multiplying the latter by p, we have $2p \in \delta(R)$. So $2 = 2(1 + p) - 2p \in \delta(R)$. □

Lemma 4 can be used to determine whether given rings are δ-quasipolar.

Example 4. (1) The ring $\mathbb{Z}_3$ is a semisimple ring and δ-quasipolar but the ring $R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ is not δ-quasipolar since $\delta(R) = \begin{bmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ and 2 does not contained in $\delta(R)$.
(2) Let $R = \{(a_{ij}) \in \mathrm{T}_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \cdots = a_{nn}\}$. $\mathbb{Z}_3$ is δ-quasipolar but R is not since $\delta(R) = \{(a_{ij}) \in \mathrm{T}_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \cdots = a_{nn} = 0\}$ and 2 does not contained in $\delta(R)$.

Recall that a ring R is called *local* if it has only one maximal left ideal, equivalently, maximal right ideal.

Proposition 1. *Let R be a local ring. If $R/J(R) \cong \mathbb{Z}_2$, then R is δ-quasipolar.*

Proof. Let $a \in R$. If $a \in J(R)$, it is clear. Assume that $a \notin J(R)$. Since R is local, a is invertible. Hence $a + 1 \in \delta(R)$ by $\delta(R) = J(R)$. □

A ring R is said to be *clean* [12] if for each $a \in R$ there exists $e^2 = e \in R$ such that $a - e$ is invertible, and R is called *strongly clean* [13] provided that every element of R can be written as the sum of an idempotent and an invertible element that commute.

Example 5. Let $R = \{(q_1, q_2, q_3, \ldots, q_n, a, a, a, \ldots) \mid n \geq 1; q_i \in \mathbb{Q};\ a \in \mathbb{Z}_{(2)}\}$. Then R is strongly clean but not quasipolar (see [15, Example 3.4(3)]). Therefore R is not J-quasipolar since every J-quasipolar ring is quasipolar. On the other hand, since $S_r = 0$ and $\delta(R)/S_r = J(R)/S_r$, $\delta(R) = J(R)$. Thus R is not δ-quasipolar.

In [4, Theorem 2.9], it is shown that if the ring R is J-quasipolar, then $R/J(R)$ is Boolean and idempotents in $R/J(R)$ lift R. We have the following result for δ-quasipolar rings.

Theorem 1. *If R is a δ-quasipolar ring, then $R/\delta(R)$ is a Boolean ring and idempotents in $R/\delta(R)$ lift R.*

Proof. Let $\overline{a} \in R/\delta(R)$. There exists $p^2 = p \in comm^2(-1+a)$ such that $-1+a+p \in \delta(R)$. Hence $\overline{a} = \overline{1-p}$ is an idempotent in $R/\delta(R)$ and $R/\delta(R)$ is a Boolean ring. Let $\overline{a}^2 = \overline{a} \in R/\delta(R)$. Then there exists $p^2 = p \in comm^2(-a)$ such that $-a + p \in \delta(R)$. This yields $\overline{a} = \overline{p}$, as asserted. □

The concept of δ_r-clean rings are defined in [8]. A ring R is called δ_r*-clean* if for every element $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in \delta(R)$. A ring is *abelian* if all idempotents are central.

Lemma 5. *If R is a δ-quasipolar ring, then it is δ_r-clean. The converse holds if R is abelian.*

Proof. Let R be a δ-quasipolar ring and $a \in R$. There exists $p^2 = p \in comm^2$ $(-1+a)$ such that $-1+a+p \in \delta(R)$. Then $a-(1-p) \in \delta(R)$. For the converse, assume that R is abelian. Let $a \in R$. There exists an idempotent e such that $1+a-e \in \delta(R)$. By assumption, $1-e$ is a central idempotent and so $1-e \in comm^2(a)$. □

Recall that a ring R is *exchange* if for every $a \in R$, there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$. Namely, von Neumann regular rings and clean rings are exchange.

Corollary 1. *Let R be a δ-quasipolar ring. Then*

(1) *R is an exchange ring.*
(2) *$R/\delta(R)$ is a clean ring.*

Proof. (1) Let R be a δ-quasipolar ring. By Lemma 5, R is a δ_r-clean ring. By [8, Theorem 2.2(2)], every δ_r-clean ring is an exchange ring.
(2) By Theorem 1, $R/\delta(R)$ is Boolean, therefore, it is clean. □

Corollary 2. *Consider following conditions for a ring R.*

(1) *R is δ-quasipolar and $\delta(R) = 0$.*
(2) *R is Boolean.*
(3) *R is von Neumann regular and δ-quasipolar.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Assume that R is δ-quasipolar and $\delta(R) = 0$. By Theorem 1, R is Boolean.
(2) $\Rightarrow$ (3) Assume that R is Boolean. Then it is commutative with characteristic 2 and $a^2 + a = 0 \in \delta(R)$ and $a^2 = a = a^3$ for all $a \in R$. Hence R is von Neumann regular and δ-quasipolar. □

Strongly J-clean rings were introduced by Chen in [3]. For a ring R the element $a \in R$ is called J*-clean* if a is the sum of an idempotent and a radical element in its Jacobson radical. The ring R is called *J-clean* if every element is a sum of an idempotent and a radical element.

Theorem 2. *If R is an abelian J-clean ring, then it is δ-quasipolar.*

Proof. Let $a \in R$. Then we have $-a \in R$. Since R is J-clean, there exist $e^2 = e \in R$ and $j \in J(R)$ such that $-a = e + j$. Hence $a + e \in J(R)$. Since R is abelian, $e^2 = e \in comm^2(a)$ and $J(R) \subseteq \delta(R)$, R is δ-quasipolar as asserted. □

All δ-quasipolar rings need not be Boolean and the converse statement of Theorem 2 is not true in general.

Example 6. The ring $\mathbb{Z}_3$ is semisimple and so $\mathbb{Z}_3 = \delta(\mathbb{Z}_3)$. Therefore $\mathbb{Z}_3$ is δ-quasipolar, but it is neither Boolean nor J-clean.

In [4, Proposition 2.11], it is shown that a ring R is local and J-quasipolar if and only if R is J-quasipolar with only trivial idempotents if and only if $R/J(R) \cong \mathbb{Z}_2$. We have the following for δ-quasipolar rings.

Proposition 2. *Let R be a ring with only trivial idempotents. Then R is δ-quasipolar if and only if $R/\delta(R) \cong \mathbb{Z}_2$.*

Proof. Assume that R is δ-quasipolar. Let $a \in R$. There exists an idempotent $p \in comm^2(a)$ such that $-a + p \in \delta(R)$. By hypothesis $p = 1$ or $p = 0$. If $\delta(R) = 0$, then $R/\delta(R) \cong \mathbb{Z}_2$. Suppose that $\delta(R) \neq 0$. For any $a \in R \setminus \delta(R)$, $\bar{a} = \bar{1} \in R/\delta(R)$. Hence $R/\delta(R) \cong \mathbb{Z}_2$. Conversely, suppose that $R/\delta(R)$ is isomorphic to $\mathbb{Z}_2$ by isomorphism f. Let $a \in R \setminus \delta(R)$. Then $f(-\bar{a}) = \bar{1} \in \mathbb{Z}_2$. Then $f(-\bar{a}) = f(\bar{1})$ implies $-\bar{a} - \bar{1} \in \operatorname{Ker} f = 0$. Hence $-\bar{a} = \bar{1}$. That is, $a + 1 \in \delta(R)$. Thus R is δ-quasipolar. □

Recall that a ring R is called *strongly π-regular* if for every element a of R there exist a positive integer n (depending on a) and an element x of R such that $a^n = a^{n+1}x$, equivalently, an element y of R such that $a^n = ya^{n+1}$. In spite of the fact that $J(R)$ is contained in both $\delta(R)$ and R^{qnil}, no comparings between $\delta(R)$ and R^{qnil} exist. Strongly π-regular rings play crucial role in this direction.

Proposition 3. *Let R be a δ-quasipolar ring and $\delta(R) = J(R)$. Then R is strongly π-regular if and only if $J(R) = R^{qnil} = nil(R) = \delta(R)$.*

Proof. Necessity. Let $a \in R^{qnil}$. Then for any $x \in comm(a)$, $1 - ax$ is invertible. By hypothesis, there exist a positive integer m and $b \in R$ such that $a^m = a^{m+1}b$. Since $b \in comm(a)$ by [11, Page 347, Exercise 23.6(1)], $a^m = 0$. Hence $a \in nil(R)$ and so $R^{qnil} \subseteq nil(R)$. To prove $nil(R) \subseteq \delta(R)$, let $a \in nil(R)$. By hypothesis there exists $p^2 = p \in comm^2(1 - a)$ such that $1 - a + p \in \delta(R)$. Since $1 - a$ is invertible, $p = 1$ by Lemma 3 (3). Hence $2 - a \in \delta(R)$. Also $2 \in \delta(R)$ by Lemma 4, we then have $a \in \delta(R)$.

Sufficiency. Let $a \in R$. There exists $p^2 = p \in comm^2(-1+a)$ such that $-1+a+p \in \delta(R)$. Set $u = -1+a+p \in nil(R)$. Then $a+p$ is invertible and $ap = up$ is nilpotent so that $a^n p = 0$ for some positive integer n. So $a^n = a^n(1-p) = (u+(1-p))^n(1-p) = (u+1)^n(1-p) = (a+p)^n(1-p) = (1-p)(a+p)^n$. By [13, Proposition 1], a is strongly π-regular. This completes the proof. □

Let R and V be rings and V be an (R,R)-bimodule that is also a ring with $(vw)r = v(wr)$, $(vr)w = v(rw)$, and $(rv)w = r(vw)$ for all v, $w \in V$ and $r \in R$. The *Dorroh extension* $D(R,V)$ of R by V defined as the ring consisting of the additive abelian group $R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw+vs+vw)$ where r, $s \in R$ and v, $w \in V$.

Uniquely clean rings were introduced by Nicholson and Zhou in [14]. A ring R is *uniquely clean* in case for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a-e \in R$ is invertible. In [8], among others, uniquely δ_r-clean rings are studied. A ring R is called *uniquely δ_r-clean* if for every element $a \in R$ there exists a unique idempotent $e \in R$ such that $a-e \in \delta(R)$. Uniquely clean Dorroh extensions in [14, Proposition 7] and uniquely δ_r-clean Dorroh extensions in [8, Proposition 3.11] are considered. Now we consider δ-quasipolar Dorroh extensions.

Proposition 4. *Let R be a ring. Then we have the following.*

(1) *If $D(R,V)$ is δ-quasipolar, then R is δ-quasipolar.*
(2) *If the following conditions are satisfied, then $D(R,V)$ is δ-quasipolar.*
  (i) *R is δ-quasipolar;*
  (ii) *$e^2 = e \in R$, then $ev = ve$ for all $v \in V$;*
  (iii) *$V = \delta(V)$.*

Proof. (1) Let $r \in R$. There exists $e^2 = e \in D(R,V)$ such that $e \in comm^2(r,0)$ and $(r,0)+e \in \delta(D(R,V))$. Since $e \in D(R,V)$, e has the form such that $(p,v)^2 = (p,v)$ and $p^2 = p$. Then $e = (p,v) \in comm^2(r,0)$ implies that $p \in comm^2(r)$ and $r+p \in \delta(R)$ since $(r+p,v) \in \delta(D(R,V))$ and by [8, Proposition 3.11]. Hence R is δ-quasipolar.

(2) Assume that (i), (ii) and (iii) hold. Let $(r,v) \in D(R,V)$. There exists $p^2 = p \in comm^2(r)$ such that $r+p \in \delta(R)$. By (iii), $(0,V) \subseteq \delta(D(R,V))$. Then $(r,v) + (p,0) = (r+p,v) \in \delta(D(R,V))$. To see that $(p,0) \in comm^2((r,v))$, let $(a,b) \in D(R,V)$ and $(a,b)(r,v) = (r,v)(a,b)$. Then $ar = ra$ and so $ap = pa$ since $p \in comm^2(r)$. Also $pb = bp$ by (ii). Therefore we have $(p,0)(a,b) = (a,b)(p,0)$ that is $(p,0) \in comm^2((r,v))$. □

As an application of Dorroh extensions we consider the following example. This example also shows that in Proposition 4 (2), the conditions (i), (ii) and (iii) are not superfluous.

Example 7. Consider the ring $D(\mathbb{Z},\mathbb{Q})$. Then $D(\mathbb{Z},\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Q}$. Then $\delta(\mathbb{Z} \times \mathbb{Q}) = (0) \times \mathbb{Q}$. Since $\mathbb{Z}$ is not δ-quasipolar, $D(\mathbb{Z},\mathbb{Q})$ is not δ-quasipolar.

Let R and S be any ring and M an (R,S)-bimodule. Consider the ring of the formal upper triangular matrix ring $T=\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. It is well known that $\delta(T)\subseteq \begin{bmatrix} \delta(R) & M \\ 0 & \delta(S) \end{bmatrix}$. However, if $M=R=S=F$ is a field, then $\delta(T)=\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$.

The following example illustrates the δ-quasipolarity of full matrix rings and upper triangular matrix rings depend on the coefficient ring.

Example 8. (1) Consider the ring $R=\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then $J(R)=\begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $\delta(R)=\begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. R is δ-quasipolar.

(2) As noted in Example 4, the ring $\mathbb{Z}_3$ is semisimple and therefore δ-quasipolar. However, the ring $\begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ is not δ-quasipolar.

(3) Let $A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\in \mathrm{Mat}_2(\mathbb{Z})$. For any $P^2=P\in comm^2(A)$, the matrix P has the form $P=\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ with $x^2=x$ and $2xy=y$ where $x,y\in\mathbb{Z}$. This would imply that P is the zero matrix or the identity matrix. Since $\delta(\mathbb{Z})=0$, $\delta(\mathrm{Mat}_2(\mathbb{Z}))=0$. In consequence, $A+P$ can not be in $\delta(\mathrm{Mat}_2(\mathbb{Z}))$. Therefore $\mathrm{Mat}_2(\mathbb{Z})$ is not δ-quasipolar.

(4) Let $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\in T_2(\mathbb{Z})$. The idempotents of $T_2(\mathbb{Z})$ are zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$ where y is an arbitrary integer. Since A commutes with only zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, among these idempotents there is no idempotent P such that $A+P\in\delta(T_2(\mathbb{Z}))$ since $\delta(T_2(\mathbb{Z}))=\begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Hence $T_2(\mathbb{Z})$ is not δ-quasipolar.

3. WEAKLY δ-QUASIPOLAR RINGS

In this section, we introduce an upper class of δ-quasipolar rings, namely, weakly δ-quasipolar rings, and we give some properties of such rings.

Definition 2. Let R be a ring and $a\in R$. The element a is called *weakly δ-quasipolar* if there exists $p^2=p\in comm(a)$ such that $a+p\in\delta(R)$, and p is called a *weakly δ-spectral idempotent*. A ring R is called *weakly δ-quasipolar* if every element of R is weakly δ-quasipolar.

An element of a ring is called *strongly J-clean* [3] provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is *strongly J-clean* in case each of its elements is strongly J-clean.

Example 9. (1) Every semisimple ring and every Boolean ring is weakly δ-quasipolar, since δ-quasipolar rings are weakly δ-quasipolar.
(2) Every strongly J-clean ring is weakly δ-quasipolar.

Proposition 5. *Let $f : R \to S$ be a surjective ring homomorphism. If R is weakly δ-quasipolar, then S is weakly δ-quasipolar.*

Proof. Let $s \in S$ with $s = f(r)$ where $r \in R$. There exists an idempotent $p \in comm(r)$ such that $r + p \in \delta(R)$. Let $q = f(p)$. Then $q^2 = q \in comm(f(r)) = comm(s)$. By [16], $f(\delta(R)) \subseteq \delta(S)$. Then $s + q = f(r) + f(p) = f(r + p) \in f(\delta(R)) \subseteq \delta(S)$. Hence S is weakly δ-quasipolar. □

Corollary 3. *Every direct summand of a weakly δ-quasipolar ring is weakly δ-quasipolar.*

Proposition 6. *Let $R = \prod_{i=1}^{n} R_i$ be a finite direct product of rings. R is weakly δ-quasipolar if and only if each R_i is weakly δ-quasipolar for $(i = 1, 2, \ldots, n)$.*

Proof. One way is clear from Corollary 3. We may assume that $n = 2$ and R_1 and R_2 are weakly δ-quasipolar. Let $a = (x_1, x_2) \in R$. There exist idempotents $p_i \in comm(x_i)$ such that $x_i + p_i \in \delta(R_i)$ for $(i = 1, 2)$. Then $p = (p_1, p_2)$ is an idempotent in R and $p \in comm(a)$ and $a + p \in \delta(R)$. Hence R is weakly δ-quasipolar. □

In [8], Gurgun and Ozcan introduce and investigate properties of δ_r-clean rings. Motivated by this work strongly δ_r-clean rings can be defined as follows.

Definition 3. An element $x \in R$ is called *strongly δ_r-clean* provided that there exist an idempotent $e \in R$ and an element $w \in \delta_r$ such that $x = e + w$ and $ew = we$. A ring R is called *strongly δ_r-clean* in case every element in R is strongly δ_r-clean.

Any strongly J-clean ring is strongly δ_r-clean. But the converse need not be true, for example any commutative semisimple ring which is not a Boolean ring is such a ring.

Note that in the following theorem it is proved that the notions of strongly δ_r-clean rings and weakly δ-quasipolar rings coincide.

Theorem 3. *Let R be a ring. Then R is a weakly δ-quasipolar ring if and only if it is strongly δ_r-clean.*

Proof. Let R be a weakly δ-quasipolar ring and $a \in R$. There exits $p^2 = p \in comm(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1 - p) \in \delta(R)$ and $a(1 - p) = (1 - p)a$. Hence R is a strongly δ_r-clean ring. Conversely, assume that R is a strongly

δ_r-clean ring. Let $a \in R$. Since $-a \in R$, by assumption there exists an idempotent $p \in R$ such that $-a - p \in \delta(R)$ and $(-a)p = p(-a)$. So R is a weakly δ-quasipolar ring. □

Theorem 3 states that the weakly δ-quasipolarity of a ring is equivalent to the strongly δ_r-cleanness of this ring. The following example reveals that a weakly δ-quasipolar element is different from a strongly δ_r-clean element.

Example 10. Let $R = \mathbb{Z}$ and $a = 1 \in R$. There exists no idempotent p such that $a + p \in \delta(R)$. Then a is not weakly δ-quasipolar. Let $p = 1 \in R$. Since $a - p \in \delta(R)$, a is strongly δ_r-clean. On the other hand, if $a = -1 \in R$, then there exists no idempotent p such that $a - p \in \delta(R)$. Then a is not strongly δ_r-clean. Let $p = 1 \in R$. Since $a + p \in \delta(R)$, a is weakly δ-quasipolar.

Theorem 4. *Let R be a local ring with non-zero maximal ideal. Then the following are equivalent.*

(1) *R is weakly δ-quasipolar;*
(2) *R is strongly J-clean;*
(3) *R is uniquely clean;*
(4) *$R/J(R) \cong \mathbb{Z}_2$;*
(5) *$R/\delta(R) \cong \mathbb{Z}_2$.*

Proof. Let R be a local ring with non-zero maximal ideal.
(1) ⇔ (2) Assume that R is weakly δ-quasipolar. Let $a \in R$. There exists $p^2 = p \in comm(-1+a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1-p) \in \delta(R)$. Since $p \in comm(-1+a)$, $pa = ap$. Hence R is strongly J-clean by $J(R) = \delta(R)$. Similarly, the rest is clear.
(2) ⇔ (3) follows from [3, Lemma 4.2].
(3) ⇔ (4) follows from [14, Theorem 15].
(1) ⇒ (5) Let R be weakly δ-quasipolar and $\overline{0} \neq \overline{a} = a + \delta(R) \in R/\delta(R)$, we show that $\overline{a} = \overline{1}$. Then there exists an idempotent $p \in R$ such that $-a + p \in \delta(R)$ and $p^2 = p \in comm(-a)$. Since R is a local, $p = 0$ or $p = 1$. If $p = 0$, this contradicts $\overline{0} \neq \overline{a}$. Therefore $p = 1$. It follows that $\overline{a} = \overline{1}$.
(5) ⇒ (1) It follows from Proposition 2. □

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