Miskolc Mathematical Notes
Vol. 18 (2017), No. 1, pp. 265–276
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1528

MULTIPLICATIVE GENERALIZED DERIVATIONS ON LIE IDEALS IN SEMIPRIME RINGS II

EMINE KOÇ AND ÖZNUR GÖLBAŞI

Received 02 February, 2015

Abstract. Let R be a semiprime ring and L is a Lie ideal of R such that $L \nsubseteq Z(R)$. A map $F: R \to R$ is called a multiplicative generalized derivation if there exists a map $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. In the present paper, we shall prove that d is a commuting map on L if any one of the following holds: i) $F(uv) = \pm uv$, ii) $F(uv) = \pm vu$, iii) $F(u)F(v) = \pm uv$, iv) $F(u)F(v) = \pm vu$, v) $F(u)F(v) \pm uv \in Z$, vi) $F(u)F(v) \pm vu \in Z$, vii) $[F(u), v] \pm [u, G(v)] = 0$, for all $u, v \in L$.

2010 *Mathematics Subject Classification:* 16N60; 16W25

Keywords: semiprime ring, Lie ideal, generalized derivation, multiplicative generalized derivation

1. Introduction

Let R will be an associative ring with center Z. For any $x, y \in R$, as usual $[x, y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan products respectively. Recall that a ring R is prime if for $x, y \in R$, $xRy = 0$ implies either $x = 0$ or $y = 0$ and R is semiprime if for $x \in R$, $xRx = 0$ implies $x = 0$. An additive subgroup L of R is said to be a Lie ideal of R if $[u, r] \in L$, for all $u \in L, r \in R$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized derivation if there exists a derivation $d: R \to R$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

This definition was given by M. Bresar in [4].

Following [5], a multiplicative derivation is a map $d: R \to R$ which satisfies $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. The concept of multiplicative derivation was introduced by Daif. Of course these maps are not additive. Motivated by this work, the definition of a multiplicative generalized derivation was extended by Daif and Tamman El-Sayiad in [6] as follows:

$F: R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Dhara and

Ali gave a slight generalization of this definition taking d is any map (not necessarily an additive map or a derivation) in [7]. Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, multiplicative derivation and the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations. Every generalized derivation is a multiplicative generalized derivation. But the converse is not ture in general.

During the past few years several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations on appropriate subsets of R. In [1], Ashraf and Rehman showed that R is prime ring with a nonzero ideal L of R and d is a derivation of R such that $d(xy) \pm xy \in Z$, for all $x, y \in L$, then R is commutative. This theorem considered for generalized derivations in [10]. Being inspired by these results, recently Dhara and Ali discuss the commutativity theorems for prime rings or semiprime rings involving multiplicative generalized derivations in [7].

In the present paper, we shall extend the above result for Lie ideals of semiprime rings with multiplicative generalized derivation of R. Our obtained results are more usefull then [7]. Throught this paper, R will be a $2-$torsion free semiprime ring admitting two multiplicative generalized derivations F and G, L a square-closed Lie ideal of R such that $F(2x) = 2F(x)$, $G(2x) = 2G(x)$ for all $x \in R$.

2. RESULTS

We will make some extensive use of the basic commutator identities:
$[x, yz] = y[x,z] + [x, y]z$
$[xy, z] = [x,z]y + x[y,z]$
$xo(yz) = (xoy)z - y[x,z] = y(xoz) + [x, y]z$
$(xy)oz = x(yoz) - [x,z]y = (xoz)y + x[y,z]$.
Moreover, we shall require the following lemmas.

Lemma 1 ([3, Lemma 4]). *If $L \nsubseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aLb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2 ([3, Lemma 2]). *Let R be a prime ring with characteristic not two. If L a noncentral Lie ideal of R, then $C_R(L) = Z$.*

Lemma 3 ([3, Lemma 5]). *Let R be a prime ring with characteristic not two and L a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $d(L) = 0$, then $L \subseteq Z$.*

Lemma 4 ([2, Theorem 7]). *Let R be a prime ring with characteristic not two and L a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $[d(u), u] \in Z$ for all $u \in L$, then $L \subseteq Z$.*

Lemma 5 ([9, Lemma 2]). *Let R be a 2-torsion free semiprime ring, L is a Lie ideal of R such that $L \nsubseteq Z(R)$ and $a \in L$. If $aLa = 0$, then $a^2 = 0$ and there exists a nonzero ideal $K = R[L,L]R$ of R generated by $[L,L]$ such that $[K,R] \subseteq L$ and $Ka = aK = 0$.*

Corollary 1 ([8, Corollary 2.1]). *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \nsubseteq Z(R)$ and $a,b \in L$.*
(i) If $aLa = 0$, then $a = 0$.
(ii) If $aL = 0$(or $La = 0$), then $a = 0$
(ii) If L is square-closed and $aLb = 0$, then $ab = 0$ and $ba = 0$.

Theorem 1. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(uv) = \pm uv$, for all $u,v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u,v \in L$.*

Proof. By the hypothesis, we have

$$F(uv) = \pm uv \text{ for all } u,v \in L. \tag{2.1}$$

Replacing v by vw, $w \in L$ in (2.1), we get

$$F(uvw) = \pm uvw \text{ for all } u,v,w \in L.$$

This implies that

$$F(uv)w + uvd(w) = \pm uvw.$$

Using equation (2.1), we obtain

$$uvd(w) = 0, \text{ for all } u,v,w \in L.$$

By Corollary 1, we have

$$ud(w) = 0, \text{ for all } u,w \in L.$$

Again using Corollary 1, the last expression forces that $d(L) = (0)$. Hence we obtain that $F(uv) = F(u)v + ud(v) = F(u)v$, for all $u,v \in L$. □

Corollary 2. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d. If $F(uv) = \pm uv$, for all $u,v \in L$, then $L \subseteq Z$.*

Proof. By the same techniques in the proof of Theorem 1, we obtain that

$$uvd(w) = 0, \text{ for all } u,v,w \in L.$$

By Lemma 1, we have $u = 0$ or $d(w) = 0$ for all $u,w \in L$. In the first case, we have $u = 0$. Hence, $d(u) = 0$. As a result of both two situation, we get $d(u) = 0$, for all $u \in L$. By Lemma 3, we conclude that $L \subseteq Z$. □

Theorem 2. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(uv) = \pm vu$, for all $u, v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u, v \in L$.*

Proof. Suppose that

$$F(uv) - vu = 0, \text{ for all } u, v \in L. \tag{2.2}$$

Replacing v by vw, $w \in L$ in the above equation, we get

$$0 = F(uvw) - vwu = F(uv)w + uvd(w) - vwu = (F(uv) - vu)w + v[u,w] + uvd(w).$$

Using (2.2), we have

$$v[u,w] + uvd(w) = 0 \tag{2.3}$$

Writing w by u in the last equation, we get

$$uvd(u) = 0, \text{ for all } u, v \in L. \tag{2.4}$$

By Corollary 1, we get

$$ud(u) = 0, \text{ for all } u \in L.$$

Replacing w by v in (2.3) and using the last equation, we obtain

$$v[u,v] = 0 \text{ for all } u, v \in L. \tag{2.5}$$

Writing u by wu, we have

$$0 = v[wu, v] = vw[u,v] + v[w,v]u = vw[u,v], \text{ for all } u, v, w \in L.$$

This implies that

$$vw[u,v] = 0 \text{ for all } u, v, w \in L. \tag{2.6}$$

Replacing w by uw in the above equation, we get

$$vuw[u,v] = 0 \text{ for all } u, v, w \in L.$$

Left multiplying by u this (2.6), we obtain that

$$uvw[u,v] = 0 \text{ for all } u, v, w \in L.$$

Commuting the two last equation, we have

$$[u,v]w[u,v] = 0 \text{ for all } u, v, w \in L.$$

By Corollary 1, we have

$$[u,v] = 0, \text{ for all } u, v \in L.$$

Returning the equation (2.3) and using the last equation, we obtain

$$uvd(w) = 0, \text{ for all } u, v, w \in L.$$

Again using Corollary 1, we conclude that $d(L) = (0)$, and so $F(uv) = F(u)v$.

In a similar manner, we can prove that the same conclusion holds for $F(uv) + vu = 0$ for all $u, v \in L$. ☐

Corollary 3. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d. If $F(uv) = \pm vu$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. Appliying the same techniques in the proof of Theorem 2, we get

$$uvd(u) = 0, \text{ for all } u, v \in L.$$

By Lemma 1, we have $u = 0$ or $d(u) = 0$, and so, $d(u) = 0$, for all $u \in L$. By Lemma 3, we conclude that $L \subseteq Z$. ☐

Theorem 3. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) = \pm uv$, for all $u, v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u, v \in L$.*

Proof. By the hypothesis

$$F(u)F(v) = \pm\, uv \text{ for all } u, v \in L. \tag{2.7}$$

Replacing v by vw in (2.7) and using the hypothesis, we find that

$$0 = F(u)F(vw) \mp uvw = F(u)(F(v)w + vd(w)) \mp uvw = (F(u)F(v) \mp uv)w \\ + F(u)vd(w).$$

Using (2.7), we have

$$F(u)vd(w) = 0, \text{ for all } u, v, w \in L. \tag{2.8}$$

Replacing u by tu, $t \in L$, we have

$$0 = F(tu)vd(w) = (F(t)u + td(u))vd(w), \text{ for all } u, v, w, t \in L.$$

Application (2.8), we have

$$td(u)vd(w) = 0, \text{ for all } u, v, w, t \in L.$$

In particular, we get $vd(u)wvd(u) = 0$, for all $u, v \in L$. This implies that $vd(u) = 0$, for all $u, v \in L$ by Corollary 1, and so $d(u) = 0$, for all $u \in L$. Hence we obtain $F(uv) = F(u)v$. ☐

Corollary 4. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d. If $F(u)F(v) = \pm uv$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. Using the same methods in the proof of Theorem 3, we have

$$vd(u) = 0, \text{ for all } u, v \in L.$$

By Lemma 1, we have $d(u) = 0$, for all $u \in L$, and so, $L \subseteq Z$, by Lemma 3. ☐

Theorem 4. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) = \pm vu$, for all $u,v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u,v \in L$.*

Proof. Assume that

$$F(u)F(v) - vu = 0 \text{ for all } u,v \in L. \tag{2.9}$$

Replacing v with vu in (2.9), we obtain

$$0 = F(u)F(vu) - vu^2 = F(u)(F(v)u + vd(u)) - vu^2 = (F(u)F(v) - vu)u + F(u)vd(u).$$

Using (2.9), we get

$$F(u)vd(u) = 0, \text{ for all } u,v \in L.$$

Left multiplying by $F(w)$ for $w \in L$ this equation and using this, we obtain

$$F(w)F(u)vd(u) = 0, \text{ for all } u,v,w \in L.$$

By the hypothesis, we can write the last equation

$$uwvd(u) = 0, \text{ for all } u,v,w \in L.$$

Replacing w by $d(u)$, we have

$$ud(u)vd(u) = 0, \text{ for all } u,v,w \in L.$$

Taking v by vu, we get

$$ud(u)vud(u) = 0, \text{ for all } u,v \in L.$$

Hence we arrive at $ud(u) = 0$, for all $u \in L$, by Corollary 1.

Now, by the hypothesis, we get

$$\begin{aligned} 0 &= F(u)F(vw) - (vw)u = F(u)(F(v)w + vd(w)) - vwu \\ &= (F(u)F(v) - vu)w + v[u,w] + F(u)vd(w), \end{aligned}$$

and so

$$v[u,w] + F(u)vd(w) = 0, \text{ for all } u,v,w \in L. \tag{2.10}$$

Replacing w by v in this equation and using $vd(v) = 0$ for all $v \in L$, we get

$$v[u,v] = 0, \text{ for all } u,v \in L.$$

This equation is same as equation (2.5) in the proof of Theorem 2. Hence, using the same arguments in there, we get

$$[u,v] = 0, \text{ for all } u,v \in L.$$

Using the last equation in (2.10), we have

$$F(u)vd(w) = 0, \text{ for all } u,v,w \in L.$$

Putting u by uw, we get

$$0 = F(uw)vd(w) = (F(u)w + ud(w))vd(w) = ud(w)vd(w)$$

and so

$$ud(w)vd(w) = 0, \text{ for all } u,v,w \in L.$$

Writing vu instead of v, we get

$$ud(w)vud(w) = 0, \text{ for all } u,v,w \in L.$$

By Corollary 1, we have $ud(w) = 0$, for all $u,w \in L$, and so $d(L) = 0$. Hence, we conclude that $F(uv) = F(u)v$, for all $u,v \in L$.

In a similar manner, we can prove that the same conclusion holds for $F(u)F(v) + vu = 0$ for all $u,v \in L$. □

Corollary 5. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R. Supposed that R admits a multiplicative generalized derivation F associated with the derivation d. If $F(u)F(v) = \pm vu$, for all $u,v \in L$, then $L \subseteq Z$.*

Proof. By the same techniques in the proof of Theorem 4, we obtain that

$$uwvd(u) = 0, \text{ for all } u,v,w \in L.$$

By Lemma 1, we have $u = 0$ or $d(u) = 0$ for each $u \in L$, and so $d(u) = 0$, for all $u \in L$. By Lemma 3, we conclude that $L \subseteq Z$. □

Theorem 5. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) \pm uv \in Z$, for all $u,v \in L$, then d is a commuting map on L.*

Proof. We have

$$F(u)F(v) \pm uv \in Z, \text{ for all } u,v \in L. \tag{2.11}$$

Replacing v with vw, $w \in L$, we have

$$F(u)F(vw) \pm u(vw) \in Z, \text{ for all } u,v,w \in L. \tag{2.12}$$

That is

$$F(u)(F(v)w + vd(w)) \pm uvw \in Z, \text{ for all } u,v,w \in L.$$

Commuting the last equation with w and using (2.12), we get

$$[F(u)vd(w), w] = 0, \text{ for all } u,v,w \in L. \tag{2.13}$$

Putting v by wv in the above relation, we obtain

$$[F(u)wvd(w), w] = 0, \text{ for all } u,v,w \in L. \tag{2.14}$$

Replacing u by uw in (2.13), we get

$$[(F(u)w + ud(w))vd(w), w] = 0.$$

Using (2.14), we have

$$[ud(w)vd(w),w] = 0, \text{ for all } u,v,w \in L. \tag{2.15}$$

That is

$$ud(w)v[d(w),w] + [ud(w)v,w]d(w) = 0, \text{ for all } u,v,w \in L.$$

Replacing u with ut in the last equation and using the last equation, we have

$$[u,w]td(w)vd(w) = 0, \text{ for all } u,v,w,t \in L.$$

Taking v by $v[u,w]t$ in this equation, we obtain

$$[u,w]td(w)v[u,w]td(w) = 0, \text{ for all } u,v,w,t \in L.$$

By Corollary 1, we have

$$[u,w]td(w) = 0, \text{ for all } u,w,t \in L.$$

Writing u bu $d(w)$ in the above equation, we see that

$$[d(w),w]td(w) = 0, \text{ for all } w,t \in L. \tag{2.16}$$

Multiplying (2.16) on the right by w, we get

$$[d(w),w]td(w)w = 0, \text{ for all } w,t \in L.$$

Replacing t by tw in equation (2.16), we obtain that

$$[d(w),w]twd(w) = 0, \text{ for all } w,t \in L.$$

Commuting two last equation, we have $[d(w),w]L[d(w),w] = 0$, for all $w \in L$. By Corollary 1, we see that $[d(w),w] = 0$, for all $w \in L$. Hence, d is a commuting map on L. □

Corollary 6. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R. Supposed that R admits a multiplicative generalized derivation F associated with the derivation d. If $F(u)F(v) \pm uv \in Z$, for all $u,v \in L$, then $L \subseteq Z$.*

Proof. Using the same methods in the proof of Theorem 5, we have

$$[u,w]td(w) = 0, \text{ for all } u,w,t \in L.$$

By Lemma 1, we get either $[u,w] = 0$ or $d(w) = 0$, for each $w \in L$. We set $K = \{w \in L \mid [u,w] = 0, \text{ for all } u \in L\}$ and $T = \{w \in L \mid d(w) = 0\}$. Clearly each of K and T is additive subgroup of L. Morever, L is the set-theoretic union of K and T. But a group can not be the set-theoretic union of two proper subgroups, hence $K = L$ or $T = L$. In the first case, we have $L \subseteq Z$ by Lemma 2. In the latter case, we have $L \subseteq Z$ by Lemma 3. This completes the proof. □

Theorem 6. *Let R be a semiprime ring with characteristic not two and L a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) \pm vu \in Z$, for all $u,v \in L$, then d is commuting map on L.*

Proof. By the hypothesis, we have

$$F(u)F(v) - vu \in Z, \text{ for all } u, v \in L.$$

Replacing v with vw, we get

$$F(u)(F(v)w + vd(w)) - vwu \in Z$$

That is

$$(F(u)F(v) - vu)w + v[u,w] + F(u)vd(w) \in Z$$

Commuting this equation with w and using the hypothesis, we obtain

$$[v[u,w],w] + [F(u)vd(w),w] = 0 \tag{2.17}$$

Replacing u by uw in the above relation, we get

$$[v[u,w],w]w + [(F(u)w + ud(w))vd(w),w] = 0 \tag{2.18}$$

Taking v by wv in equation (2.17), we get

$$w[v[u,w],w] + [F(u)wvd(w),w] = 0 \tag{2.19}$$

Subtracting (2.18) from (2.19), we have

$$[[v[u,w],w],w] + [ud(w)vd(w),w] = 0 \tag{2.20}$$

Writing u by uw in the above relation, we have

$$[[v[u,w],w],w]w + [uwd(w)vd(w),w] = 0 \tag{2.21}$$

Right multiplying (2.20) by w and subtracting it from (2.21), we get

$$[u[d(w)vd(w),w],w] = 0, \text{ for all } u,v,w \in L. \tag{2.22}$$

Replacing u by $d(w)vd(w)u$ in the last equation, we have

$$\begin{aligned} 0 &= [d(w)vd(w)u[d(w)vd(w),w],w] \\ &= d(w)vd(w)[u[d(w)vd(w),w],w] + [d(w)vd(w),w]u[d(w)vd(w),w]. \end{aligned}$$

Using (2.22), it reduces to

$$[d(w)vd(w),w]u[d(w)vd(w),w] = 0, \text{ for all } u,v,w \in L.$$

By Corollary 1, we have

$$[d(w)vd(w),w] = 0, \text{ for all } v,w \in L.$$

That is,

$$d(w)vd(w)w - wd(w)vd(w) = 0, \text{ for all } v,w \in L. \tag{2.23}$$

Taking v by $vd(w)u, u \in L$ in this equation, we have

$$d(w)vd(w)ud(w)w - wd(w)vd(w)ud(w) = 0.$$

Using (2.23), we obtain

$$d(w)vwd(w)ud(w) - d(w)vd(w)wud(w) = 0.$$

That is,

$$d(w)v[d(w),w]ud(w) = 0, \text{ for all } u,v,w \in L.$$

This implies that

$$[d(w),w]v[d(w),w]u[d(w),w] = 0, \text{ for all } u,v,w \in L.$$

Right multiplying this equation by $v[d(w),w]$, we get

$$[d(w),w]v[d(w),w]u[d(w)w]v[d(w),w] = 0, \text{ for all } u,v,w \in L.$$

By Corollary 1, we obtain

$$[d(w),w]v[d(w),w] = 0, \text{ for all } v,w \in L.$$

That is, $[d(w),w] = 0$, for all $w \in L$ by Corollary 1. Hence, d is commuting on L. In a similar manner, we can prove that the same conclusion holds for $F(u)F(v)+vu \in Z$ for all $u,v \in L$. □

Corollary 7. *Let R be a prime ring with characteristic not two and L a square-closed Lie ideal of R. Supposed that R admits a multiplicative generalized derivation F associated with the derivation d. If $F(u)F(v) \pm vu \in Z$, for all $u,v \in L$, then $L \subseteq Z$.*

Proof. By the same methods in the proof of Theorem 6, we have

$$[u[d(w)vd(w),w],w] = 0, \text{ for all } u,v,w \in L. \tag{2.24}$$

Replacing u by $d(w)vd(w)u$, $w \in [L,L]$ in the last equation, we have

$$[d(w)vd(w),w] = 0, \text{ for all } v \in L, \ w \in [L,L].$$

Again using the same techniques after the equation (2.23) in the proof of Theorem 6, we have $[d(w),w] = 0$, for all $w \in [L,L]$. By Lemma 5 and Lemma 2, we conclude that $L \subseteq Z$. □

Theorem 7. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$. Suppose that R admit two multiplicative generalized derivations F,G associated with nonzero maps d and g such that $g(x) \in L$, for all $x \in L$. If $[F(u),v] \pm [u,G(v)] = 0$ for all $u,v \in L$, then g is commuting map on L.*

Proof. By the hypothesis, we have

$$[F(u),v] \pm [u,G(v)] = 0, \text{ for all } u,v \in L.$$

Replacing v by vu in the hypothesis, we have

$$v[F(u),u] + [F(u),v]u \pm [u,G(v)]u \pm [u,v]g(v) \pm v[u,g(u)] = 0, \text{ for all } u,v \in L.$$

Application the hypothesis, we get

$$v[F(u),u] \pm [u,v]g(u) \pm v[u,g(u)] = 0, \text{ for all } u,v \in L. \tag{2.25}$$

Writing v by wv in (2.25) and using this equation, we get

$$[u,w]vg(u) = 0, \text{ for all } u,v \in L.$$

Taking w by $g(u)$ in the last equation, we obtain

$$[u,g(u)]vg(u) = 0, \text{ for all } u,v \in L. \tag{2.26}$$

Replacing v by vu in (2.26), we find that

$$[u,g(u)]vug(u) = 0, \text{ for all } u,v \in L. \tag{2.27}$$

Multipliying (2.26) on the right by u, we get

$$[u,g(u)]vg(u)u = 0, \text{ for all } u,v \in L. \tag{2.28}$$

Subtracting (2.27) and (2.28), we have

$$[u,g(u)]v[u,g(u)] = 0, \text{ for all } u,v \in L.$$

By Corollary 1, we obtain that $[u,g(u)] = 0$, for all $u,v \in L$. □

Corollary 8. *Let R be a 2-torsion free prime ring, L be a square-closed Lie ideal of R and $F,G : R \to R$ two generalized derivations associated with derivations d and g. If $[F(u),v] \pm [u,G(v)] = 0$ for all $u,v \in L$, then $L \subseteq Z$.*

Proof. Using the same methods in the proof of Theorem 7, we obtain that

$$[u,w]vg(u) = 0, \text{ for all } u,v \in L.$$

By Lemma 1, we get $[u,w] = 0$ or $g(u) = 0$, for all $u,v,w \in L$. We set $K = \{u \in L \mid [u,w] = 0, \text{ for all } w \in L\}$ and $T = \{u \in L \mid g(u) = 0\}$. We had done Corollary 6. Hence we obtain that $L \subseteq Z$. □

REFERENCES

[1] M. Ashraf and N. Rehman, "On derivations and commutativity in prime rings." *East-West J. Math.*, vol. 3, no. 1, pp. 87–91, 2001.

[2] R. Awtar, "Lie structure in prime rings with derivations." *Publ. Math.*, vol. 31, pp. 209–215, 1984.

[3] J. Bergen, I. Herstein, and W. Kerr, "Lie ideals and derivation of prime rings." *J. Algebra*, vol. 71, pp. 259–267, 1981, doi: 10.1016/0021-8693(81)90120-4.

[4] M. Bresar, "On the distance of the compositions of two derivations to the generalized derivations." *Glasgow Mathematical Journal*, vol. 33, no. 1, pp. 89–93, 1991, doi: 10.1017/S0017089500008077.

[5] M. N. Daif, "When is a multiplicative derivation additive?" *Int. J. Math. Math. Sci.*, vol. 14, no. 3, pp. 615–618, 1991, doi: 10.1155/S0161171291000844.

[6] M. N. Daif and M. S. Tammam El-Sayiad, "Multiplicative generalized derivation which are additive." *East-West J. Math.*, vol. 9, no. 1, pp. 31–37, 1997.

[7] B. Dhara and S. Ali, "On multiplicative (generalized)-derivations in prime and semiprime rings." *Aequationes Math.*, vol. 86, no. 1-2, pp. 65–79, 2013, doi: 10.1007/s00010-013-0205-y.

[8] M. Hongan, Nadeem ur Rehman, and R. M. AL-Omary, “Lie ideals and Jordan triple derivations in rings.” *Rend. Semin. Mat. Univ. Padova*, vol. 125, pp. 147–156, 2011, doi: 10.4171/RSMUP/125-9.

[9] Nadeem ur Rehman and M. Hongan, “Generalized Jordan derivations on Lie ideals associate with Hochschild 2-cocycles of rings.” *Rend. Circ. Mat. Palermo (2)*, vol. 60, no. 3, pp. 437–444, 2011, doi: 10.1007/s12215-011-0069-8.

[10] M. Quadri, M. Shadab Khan, and N. Rehman, “Generalized derivations and commutativity of prime rings.” *Indian J. Pure Appl. Math.*, vol. 34, no. 9, pp. 1393–1396, 2003.

Authors' addresses

Emine Koç
Cumhuriyet University, Faculty of Science, Department of Mathematics, Sivas, TURKEY
E-mail address: eminekoc@cumhuriyet.edu.tr

Öznur Gölbaşı
Cumhuriyet University, Faculty of Science, Department of Mathematics, Sivas, TURKEY
Current address: Cumhuriyet University, Faculty of Science, Department of Mathematics, Sivas, TURKEY
E-mail address: ogolbasi@cumhuriyet.edu.tr