



ESTIMATE FOR INITIAL MACLAURIN COEFFICIENTS OF CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, estimates for second and third MacLaurin coefficients of certain subclasses of bi-univalent functions in the open unit disk defined by convolution are determined, and certain special cases are also indicated. The main result extends and improve a recent one obtained by Srivastava et al.

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1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} be the class of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Koebe one-quarter theorem [3] ensures that the image of \mathbb{D} under every univalent function $f \in \mathcal{A}$ contains the disk with the center in the origin and the radius $1/4$. Thus, every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\mathbb{D}) \rightarrow \mathbb{D}$, satisfying $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$, and

$$f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq \frac{1}{4}.$$

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots, w \in f(\mathbb{D}). \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent*, if both f and f^{-1} are univalent in \mathbb{D} , in the sense that f^{-1} has a univalent analytic continuation to \mathbb{D} , and we denote by σ this class of bi-univalent functions.

In [8] the authors defined the classes of functions $\mathcal{P}_m(\beta)$ as follows: let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions P , normalized with $P(0) = 1$, and satisfying

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} P(z) - \beta}{1 - \beta} \right| d\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{D}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class \mathcal{P}_m represents the class of functions p analytic in \mathbb{D} , normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} d\mu(t), \quad (1.2)$$

where μ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m, \quad m \geq 2. \quad (1.3)$$

Clearly, $\mathcal{P} := \mathcal{P}_2$ is the well-known class of *Carathéodory functions*, i.e. the normalized functions with positive real part in the open unit disk \mathbb{D} .

Lewin [6] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the concept of bi-starlike functions and the bi-convex functions, and obtained estimates for the initial coefficients. Recently, Ali et al. [1], Srivastava et al. [9], Frasin and Aouf [4], Goyal and Goswami [5] and many others have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Motivated by work of Srivastava et al. [9], we introduce a new subclass of bi-univalent functions, as follows.

For the functions $f, h \in \mathcal{A}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad h(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

we recall the *Hadamard (or convolution) product* of f and h , defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Definition 1. For a given function $k \in \sigma$, a function $f \in \sigma$ is said to be in the class $\mathcal{BR}^k(m; \beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied

$$\frac{(f * k)(z)}{z} \in \mathcal{P}_m(\beta),$$

$$\frac{(g * k)(w)}{w} \in \mathcal{P}_m(\beta),$$

where $g = f^{-1}$ and $z, w \in \mathbb{D}$.

Remark 1. Taking $k(z) = z/(1-z)^2$ and $m = 2$ in the Definition 1 we obtain the class $\mathcal{B}(\beta) := \mathcal{BR}^{z/(1-z)^2}(2; \beta)$ studied by Srivastava et al. [9, Definition 2].

Definition 2. For a given function $k \in \sigma$ and a number $\alpha \in \mathbb{C}$, a function $f \in \sigma$ is said to be in the class $\mathcal{BV}^k(m; \alpha, \beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied

$$(1-\alpha) \frac{z(f * k)'(z)}{(f * k)(z)} + \alpha \left(1 + \frac{z(f * k)''(z)}{(f * k)'(z)} \right) \in \mathcal{P}_m(\beta),$$

$$(1-\alpha) \frac{w(g * k)'(w)}{(g * k)(w)} + \alpha \left(1 + \frac{w(g * k)''(w)}{(g * k)'(w)} \right) \in \mathcal{P}_m(\beta),$$

where $g = f^{-1}$.

Remark 2. (i) Taking $\alpha = 0$ and $\alpha = 1$ in the above class $\mathcal{BV}^k(m; \alpha, \beta)$ we obtain the classes $\mathcal{S}_m^k(\beta) := \mathcal{BV}^k(m; 0, \beta)$ and $\mathcal{C}_m^k(\beta) := \mathcal{BV}^k(m; 1, \beta)$, respectively.

(ii) Moreover, if we take $k(z) = z/(1-z)$ and $m = 2$, the classes $\mathcal{S}_m^k(\beta)$ and $\mathcal{C}_m^k(\beta)$ reduces to the well-known classes of *bi-starlike* and *bi-convex functions*, respectively (see also [2]).

The object of the paper is to find estimates for the coefficients a_2 and a_3 for functions in the subclass $\mathcal{BR}^k(m; \beta)$ and $\mathcal{BV}^k(m; \alpha, \beta)$, and these bounds are obtained by employing the techniques used earlier by Srivastava et al. [9].

2. MAIN RESULTS

In order to prove our main result for the functions $f \in \mathcal{BR}^k(m; \beta)$, first we will prove the following lemma:

Lemma 1. Let the function $\Phi(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$, $z \in \mathbb{D}$, such that $\Phi \in \mathcal{P}_m(\beta)$.

Then,

$$|h_n| \leq m(1-\beta), \quad n \geq 1.$$

Proof. From (1.2) and (1.3), like in [8] and [7], we can see that if $p \in \mathcal{P}_m$, then

$$p(z) = \left(\frac{m}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2} \right) p_2(z), \quad (2.1)$$

where $p_1, p_2 \in \mathcal{P}$.

Further, if $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in \mathbb{D}$, where $p_1(z) = 1 + \sum_{n=1}^{\infty} p_n^{(1)} z^n$ and $p_2(z) = 1 + \sum_{n=1}^{\infty} p_n^{(2)} z^n$ for all $z \in \mathbb{D}$, comparing the coefficients of both sides of (2.1) we get

$$p_n = \left(\frac{m}{4} + \frac{1}{2}\right) p_n^{(1)} - \left(\frac{m}{4} - \frac{1}{2}\right) p_n^{(2)}, \quad n \geq 1.$$

Since $p_1, p_2 \in \mathcal{P}$, where \mathcal{P} is the class of Carathéodory functions, it is well-known that $|p_n^{(1)}| \leq 2$ and $|p_n^{(2)}| \leq 2$ for all $n \geq 1$, and thus

$$\begin{aligned} |p_n| &\leq \left(\frac{m}{4} + \frac{1}{2}\right) |p_n^{(1)}| + \left(\frac{m}{4} - \frac{1}{2}\right) |p_n^{(2)}| \leq \\ &2 \left(\frac{m}{4} + \frac{1}{2}\right) + 2 \left(\frac{m}{4} - \frac{1}{2}\right) = m, \quad n \geq 1. \end{aligned} \quad (2.2)$$

Now, the proof of this lemma is straight forward, if we write

$$\Phi(z) = (1-\beta)p(z) + \beta, \quad \text{where } p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}_m.$$

Then,

$$\Phi(z) = 1 + (1-\beta) \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D},$$

which gives

$$h_n = (1-\beta)p_n, \quad n \geq 1,$$

and using the inequality (2.2) we obtain the desired result. \square

Theorem 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{BR}^k(m; \beta)$, where $k \in \sigma$ has the form $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. If $k_2, k_3 \neq 0$, then

$$|a_2| \leq \min \left\{ \sqrt{\frac{m(1-\beta)}{|k_3|}}; \frac{m(1-\beta)}{|k_2|} \right\}, \quad |a_3| \leq \frac{m(1-\beta)}{|k_3|},$$

and

$$|2a_2^2 - a_3| \leq \frac{m(1-\beta)}{|k_3|}.$$

Proof. Since $f \in \mathcal{BR}^k(m; \beta)$, from the Definition 1 we have

$$\frac{(f * k)(z)}{z} = p(z) \quad (2.3)$$

and

$$\frac{(g * k)(w)}{w} = q(w), \quad (2.4)$$

where $p, q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Using the fact that the functions p and q have the following Taylor expansions

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots, \quad z \in \mathbb{D}, \quad (2.5)$$

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \dots, \quad w \in \mathbb{D}, \quad (2.6)$$

and equating the coefficients in (2.3) and (2.4), from (1.1) we get

$$k_2a_2 = p_1, \quad (2.7)$$

$$k_3a_3 = p_2, \quad (2.8)$$

and

$$\begin{aligned} -k_2a_2 &= q_1, \\ k_3(2a_2^2 - a_3) &= q_2. \end{aligned} \quad (2.9)$$

Since $p, q \in \mathcal{P}_m(\beta)$, according to Lemma 1, the next inequalities hold:

$$|p_k| \leq m(1 - \beta), \quad k \geq 1, \quad (2.10)$$

$$|q_k| \leq m(1 - \beta), \quad k \geq 1, \quad (2.11)$$

and thus, from (2.8) and (2.9), by using the inequalities (2.10) and (2.11), we obtain

$$|a_2|^2 \leq \frac{|q_2| + |p_2|}{2|k_3|} \leq \frac{m(1 - \beta)}{|k_3|},$$

which gives

$$|a_2| \leq \sqrt{\frac{m(1 - \beta)}{|k_3|}}. \quad (2.12)$$

From (2.7), by using (2.10) we obtain immediately that

$$|a_2| = \left| \frac{p_1}{k_2} \right| \leq \frac{m(1 - \beta)}{|k_2|},$$

and combining this with the inequality (2.12), the first inequality of the conclusion is proved.

According to (2.8), from (2.10) we easily obtain

$$|a_3| = \left| \frac{p_2}{k_3} \right| \leq \frac{m(1 - \beta)}{|k_3|},$$

and from (2.9), by using (2.10) and (2.11) we finally deduce

$$|2a_2^2 - a_3| = \left| \frac{q_2}{k_3} \right| \leq \frac{m(1 - \beta)}{|k_3|},$$

which completes our proof. \square

Setting $\beta = 0$ in Theorem 1 we get the following special case:

Corollary 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{BR}^k(m; 0)$, where $k \in \sigma$ has the form $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. If $k_2, k_3 \neq 0$, then

$$|a_2| \leq \min \left\{ \sqrt{\frac{m}{|k_3|}}, \frac{m}{|k_2|} \right\}, \quad |a_3| \leq \frac{m}{|k_3|}, \quad \text{and} \quad |2a_2^2 - a_3| \leq \frac{m}{|k_3|}.$$

For $k(z) = z/(1-z)^2$ the above corollary reduces to the next result:

Example 1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{BR}^{z/(1-z)^2}(m; 0)$, then

$$|a_2| \leq \sqrt{\frac{m}{3}}, \quad |a_3| \leq \frac{m}{3}, \quad \text{and} \quad |2a_2^2 - a_3| \leq \frac{m}{3}.$$

Taking $k(z) = z/(1-z)$ in Corollary 1, we get:

Example 2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{BR}^{z/(1-z)}(m; 0)$, then

$$|a_2| \leq \sqrt{m}, \quad |a_3| \leq m, \quad \text{and} \quad |2a_2^2 - a_3| \leq m.$$

If we put $k(z) = z/(1-z)^2$ in Theorem 1, we deduce the next corollary:

Corollary 2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{B}(\beta)$, then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & \text{if } 0 \leq \beta \leq \frac{1}{3}, \\ 1-\beta, & \text{if } \frac{1}{3} < \beta < 1, \end{cases} \quad |a_3| \leq \frac{2(1-\beta)}{3},$$

and

$$|2a_2^2 - a_3| \leq \frac{2(1-\beta)}{3}.$$

Remark 3. For the special case $\frac{1}{3} < \beta < 1$, the above first inequality, and the second one for all $0 \leq \beta < 1$, improve the estimates given by Srivastava et al. in [9, Theorem 2].

Theorem 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{BV}^k(m; \alpha, \beta)$, with $\alpha \in \mathbb{C} \setminus \{-1\}$, where $k \in \sigma$ has the form $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. If $k_2, k_3 \neq 0$ and

$$2(1+2\alpha)k_3 - (1+3\alpha)k_2^2 \neq 0,$$

then

$$|a_2| \leq \min \left\{ \sqrt{\frac{m(1-\beta)}{|2(1+2\alpha)k_3 - (1+3\alpha)k_2^2|}}; \frac{m(1-\beta)}{|1+\alpha||k_2|} \right\},$$

and

$$|a_3| \leq \min \left\{ \frac{m(1-\beta)}{|2(1+2\alpha)k_3 - (1+3\alpha)k_2^2|} + \frac{m(1-\beta)}{2|1+2\alpha||k_3|}; \right. \\ \left. \frac{m(1-\beta)}{2|1+2\alpha||k_3|} \left(1 + \frac{m(1-\beta)|1+3\alpha|}{|1+\alpha|^2} \right); \right. \\ \left. \frac{m(1-\beta)}{2|1+2\alpha||k_3|} \left(1 + \frac{m(1-\beta)|4(1+2\alpha)k_3 - (1+3\alpha)k_2^2|}{|k_2|^2|1+\alpha|^2} \right) \right\},$$

whenever $\alpha \in \mathbb{C} \setminus \left\{ -\frac{1}{2} \right\}$.

Proof. If $f \in \mathcal{BV}^k(m; \alpha, \beta)$, according to the Definition 2 we have

$$(1-\alpha) \frac{z(f * k)'(z)}{(f * k)(z)} + \alpha \left(1 + \frac{z(f * k)''(z)}{(f * k)'(z)} \right) = p(z)$$

and

$$(1-\alpha) \frac{w(g * k)'(w)}{(g * k)(w)} + \alpha \left(1 + \frac{w(g * k)''(w)}{(g * k)'(w)} \right) = q(w),$$

where $p, q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Since

$$(1-\alpha) \frac{z(f * k)'(z)}{(f * k)(z)} + \alpha \left(1 + \frac{z(f * k)''(z)}{(f * k)'(z)} \right) = \\ 1 + (1+\alpha)a_2k_2z + [2(1+2\alpha)a_3k_3 - (1+3\alpha)a_2^2k_2^2]z^2 + \dots \quad z \in \mathbb{D},$$

and according to (1.1)

$$(1-\alpha) \frac{z(g * k)'(w)}{(g * k)(w)} + \alpha \left(1 + \frac{z(g * k)''(w)}{(g * k)'(w)} \right) = 1 - (1+\alpha)a_2k_2w + \\ \left\{ [4(1+2\alpha)k_3 - (1+3\alpha)k_2^2]a_2^2 - 2(1+2\alpha)a_3k_3 \right\} w^2 + \dots, \quad w \in \mathbb{D},$$

from (2.5) and (2.6) combined with the above two expansion formulas, it follows that

$$(1+\alpha)a_2k_2 = p_1, \quad (2.13)$$

$$2(1+2\alpha)a_3k_3 - (1+3\alpha)a_2^2k_2^2 = p_2, \quad (2.14)$$

and

$$-(1+\alpha)a_2k_2 = q_1, \\ [4(1+2\alpha)k_3 - (1+3\alpha)k_2^2]a_2^2 - 2(1+2\alpha)a_3k_3 = q_2. \quad (2.15)$$

Now, from (2.14) and (2.15) we deduce that

$$a_2^2 = \frac{p_2 + q_2}{4(1 + 2\alpha)k_3 - 2(1 + 3\alpha)k_2^2}, \quad (2.16)$$

whenever $2(1 + 2\alpha)k_3 - (1 + 3\alpha)k_2^2 \neq 0$, and

$$4(1 + 2\alpha)k_3(a_3 - a_2^2) = p_2 - q_2.$$

Using (2.16) in the above relation, we obtain

$$a_3 = \frac{p_2 + q_2}{4(1 + 2\alpha)k_3 - 2(1 + 3\alpha)k_2^2} + \frac{p_2 - q_2}{4(1 + 2\alpha)k_3}, \quad (2.17)$$

$$\text{whenever } 2(1 + 2\alpha)k_3 - (1 + 3\alpha)k_2^2 \neq 0, \alpha \in \mathbb{C} \setminus \left\{ -\frac{1}{2} \right\}.$$

From (2.13) and (2.14) we get

$$a_3 = \frac{1}{2(1 + 2\alpha)k_3} \left[p_2 + \frac{1 + 3\alpha}{(1 + \alpha)^2} p_1^2 \right], \quad \text{for } \alpha \in \mathbb{C} \setminus \left\{ -1; -\frac{1}{2} \right\}, \quad (2.18)$$

while from (2.13) and (2.15) we deduce that

$$a_3 = \frac{1}{2(1 + 2\alpha)k_3} \left[-q_2 + \frac{4(1 + 2\alpha)k_3 - (1 + 3\alpha)k_2^2}{k_2^2(1 + \alpha)^2} p_1^2 \right], \quad (2.19)$$

$$\text{for } \alpha \in \mathbb{C} \setminus \left\{ -1; -\frac{1}{2} \right\}.$$

Combining (2.13) and (2.16) for the computation of the upper-bound of $|a_2|$, and (2.17), (2.18) and (2.19) for the computation of $|a_3|$, by using Lemma 1 we easily find the estimates of our theorem. \square

Taking $\alpha = 0$ and $\alpha = 1$ in Theorem 2 we obtain the following two special cases, respectively:

Corollary 3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_m^k(\beta)$, where $k \in \sigma$ has the form $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. If $k_2, k_3 \neq 0$ and

$$2k_3 - k_2^2 \neq 0,$$

then

$$|a_2| \leq \min \left\{ \sqrt{\frac{m(1-\beta)}{|2k_3 - k_2^2|}}, \frac{m(1-\beta)}{|k_2|} \right\},$$

and

$$|a_3| \leq \min \left\{ \frac{m(1-\beta)}{|2k_3 - k_2^2|} + \frac{m(1-\beta)}{2|k_3|}, \frac{m(1-\beta)(1 + m(1-\beta))}{2|k_3|} \right\};$$

$$\frac{m(1-\beta)}{2|k_3|} \left(1 + \frac{m(1-\beta)|4k_3 - k_2^2|}{|k_2|^2} \right) \Bigg\}.$$

Corollary 4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{E}_m^k(\beta)$, where $k \in \sigma$ has the form $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. If $k_2, k_3 \neq 0$ and

$$3k_3 - 2k_2^2 \neq 0,$$

then

$$|a_2| \leq \min \left\{ \sqrt{\frac{m(1-\beta)}{|6k_3 - 4k_2^2|}}; \frac{m(1-\beta)}{2|k_2|} \right\},$$

and

$$|a_3| \leq \min \left\{ \frac{m(1-\beta)}{|6k_3 - 4k_2^2|} + \frac{m(1-\beta)}{6|k_3|}; \frac{m(1-\beta)(1+m(1-\beta))}{6|k_3|}; \frac{m(1-\beta)}{6|k_3|} \left(1 + \frac{m(1-\beta)|3k_3 - k_2^2|}{|k_2|^2} \right) \right\}.$$

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