INITIAL VALUE PROBLEMS FOR FUNCTIONAL AND NEUTRAL FUNCTIONAL HADAMARD TYPE FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. This paper is concerned with the existence of solutions for initial value problems of functional and neutral functional Hadamard type fractional differential inclusions. We apply appropriate fixed point theorems for multivalued maps to obtain the existence results for the given problems covering convex as well as non-convex cases. Some illustrative examples are also presented.

1. INTRODUCTION

In this paper we study the existence of solutions for initial value problems of functional and neutral functional Hadamard type fractional differential inclusions given by

$$D^\alpha y(t) \in F(t, y_t), \text{ for each } t \in J := [1, b], \quad 0 < \alpha < 1, \tag{1.1}$$

$$y(t) = \vartheta(t), \quad t \in [1 - r, 1], \tag{1.2}$$

and

$$D^\alpha[y(t) - g(t, y_t)] \in F(t, y_t), \ t \in J, \tag{1.3}$$

$$y(t) = \vartheta(t), \ t \in [1 - r, 1], \tag{1.4}$$

where D^α is the Hadamard fractional derivative, $F : J \times C([-r, 0], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$) is a given function and $\vartheta \in C([1 - r, 1], \mathbb{R})$ with $\vartheta(1) = 0$ and $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function such that $g(1, \vartheta) = 0$. For any function y defined on $[1 - r, b]$ and any $t \in J$, let y_t denote the element of $C([-r, 0], \mathbb{R})$ defined by $y_t(\chi) = y(t + \chi)$, $\chi \in [-r, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t.

Functional and neutral functional differential equations arise in the mathematical modelling of biological, physical, and engineering problemss, see, for example, the texts [19], [17] and the references cited therein.

Fractional order operators are found to be valuable tools in describing several phenomena occurring in applied sciences and engineering. Examples include viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [18, 21–23]).

Fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have extensively been studied by several researchers [1–4, 6, 7, 15, 24, 25]. On the other hand, Hadamard type fractional differential equations are not studied in depth and many aspects of these equations need further attention. It is imperative to note that the definition of Hadamard fractional derivative contains a logarithmic function of arbitrary exponent in contrast to aforementioned derivatives. For details, we refer the reader to the papers [9–11] and references cited therein.

The IVPs (1.1)–(1.2) and (1.3)–(1.4) involving Riemann-Liouville fractional derivative and infinite delay were studied in [8] while the case of functional fractional Hadamard type differential equations has recently been discussed in [5]. Here we extend the problems addressed in [5] to the multi-valued case.

The purpose of this paper is to establish some existence results for the problems (1.1)–(1.2) and (1.3)-(1.4) for convex and non-convex values of the multivalued maps involved in the problems. Our main results rely on the nonlinear alternative of Leray-Schauder type and a fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

The paper is organized as follows. Section 2 contains some preliminaries needed for the sequel. In Section 3, we establish the existence results for the problem (1.1)–(1.2) while the results for the problem (1.3)-(1.4) are presented in Section 4. The paper concludes with illustrating examples.

2. PRELIMINARIES

Here we fix notations and state some definitions, and preliminary facts that we need in the sequel.

The space $C(J, \mathbb{R})$ denote the Banach space of all continuous functions from J into $\mathbb{R}$ equipped with the norm $\|y\|_\infty := \sup\{|y(t)| : t \in J\}$ and $C([-r, 0], \mathbb{R})$ is endowed with norm $\|\cdot\|_C$ defined by $\|\vartheta\|_C := \sup\{|\vartheta(\chi)| : -r \leq \chi \leq 0\}$.

Definition 1 ([18])**.** The Hadamard derivative of fractional order q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\log \frac{t}{s} \right)^{n-q-1} \frac{g(s)}{s} ds, \ n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2 ([18])**.** The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{g(s)}{s} ds, \ q > 0,$$

provided the integral exists.

For the convenience of the reader, we briefly describe some basic definitions.

Let $(\mathcal{D}, \|\cdot\|)$ denote a normed space and $\mathcal{P}_1 = \mathcal{P}(\mathcal{D}) := \{A \subset \mathcal{D} : A \neq \varnothing\}$. Then we define $\mathcal{P}_{cl}(\mathcal{D}) = \{\mathcal{D}_1 \in \mathcal{P}_1 : \mathcal{D}_1 \text{ is closed}\}$, $\mathcal{P}_b(\mathcal{D}) = \{\mathcal{D}_1 \in \mathcal{P}_1 : \mathcal{D}_1 \text{ is bounded}\}$, $\mathcal{P}_{cp}(\mathcal{D}) = \{\mathcal{D}_1 \in \mathcal{P}_1 : \mathcal{D}_1 \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(\mathcal{D}) = \{\mathcal{D}_1 \in \mathcal{P}_1 : \mathcal{D}_1 \text{ is compact and convex}\}$.

Definition 3. A multi-valued map $\mathcal{G} : \mathcal{D} \to \mathcal{P}_1$ is

(a) convex (closed) valued if $\mathcal{G}(a)$ is convex (closed) for all $a \in \mathcal{D}$;
(b) bounded on bounded sets if $\mathcal{G}(\mathbb{B}) = \cup_{x \in \mathbb{B}} \mathcal{G}(x)$ is bounded in $\mathcal{D}$ for all $\mathbb{B} \in \mathcal{P}_b(\mathcal{D})$ (i.e. $\sup_{x \in \mathbb{B}} \{\sup\{|y| : y \in G(x)\}\} < \infty$);
(c) upper semi-continuous (u.s.c.) on $\mathcal{D}$ if for each $a_0 \in \mathcal{D}$, the set $\mathcal{G}(a_0)$ is a nonempty closed subset of $\mathcal{D}$, and if for each open set $\mathcal{N}$ of $\mathcal{D}$ containing $\mathcal{G}(a_0)$, there exists an open neighborhood $\mathcal{N}_0$ of a_0 such that $\mathcal{G}(\mathcal{N}_0) \subseteq \mathcal{N}$;
(d) completely continuous if $\mathcal{G}(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(\mathcal{D})$.

Definition 4. The map $\mathcal{G}$ has a fixed point if there is $a \in \mathcal{D}$ such that $a \in \mathcal{G}(a)$. The fixed point set of the multivalued operator $\mathcal{G}$ will be denoted by *Fix$\mathcal{G}$*.

Definition 5. The map $\mathcal{G} : J \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $b \in \mathbb{R}$, the function $t \longmapsto d(b, \mathcal{G}(t)) = \inf\{|b - c| : c \in \mathcal{G}(t)\}$ is measurable.

Definition 6. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t, y)$ is measurable for each $x \in \mathbb{R}$;
(ii) $y \longmapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \leq \varphi_\rho(t)$$

for all $\|y\| \leq \rho$ and for a.e. $t \in J$.

3. Existence results for the problem (1.1)–(1.2)

In this section, we establish the existence criteria for the problem (1.1)–(1.2). Before presenting the main results, we outline some auxiliary material.

For each $y \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y_t) \text{ for a.e. } t \in J\}.$$

Definition 7. A function $x \in C^1([1-r, b], \mathbb{R})$ is called a solution of problem (1.1)–(1.2) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y_t)$, a.e. J such that $D^\alpha y(t) = v(t)$ for a.e. $t \in J$ and $y(t) = \vartheta(t), t \in [1-r, 1]$.

To discuss the existence of the solutions for the initial value problem (1.1)–(1.2) when the right-hand side is convex valued (upper semicontinuous case), we need the following lemmas.

Lemma 1 ([16]). *(Nonlinear alternative for Kakutani maps.) Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathscr{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

   (i) *F has a fixed point in $\overline{U}$, or*
   (ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Lemma 2 ([20]). *Let X be a Banach space. Let $F : J \times X \to \mathscr{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathscr{P}_{cp,c}(C(J,X)), \ \ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Theorem 1. *Assume that*

(H_1) *$F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*
(H_2) *there exists a continuous nondecreasing function $\beta : [0,\infty) \to (0,\infty)$ and a function $\zeta \in C(J,\mathbb{R}^+)$ such that*

$$\|F(t,y)\|_{\mathscr{P}} := \sup\{|v| : v \in F(t,y)\} \le \zeta(t)\beta(\|y\|_C),$$

     for each $(t,y) \in J \times C([-r,0],\mathbb{R})$;
(H_3) *there exists a constant $\sigma > 0$ such that*

$$\frac{\sigma}{\beta(\sigma)\|\zeta\|_\infty \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}} > 1.$$

Then the initial value problem (1.1)–(1.2) has at least one solution on $[1-r,b]$.

Proof. Define an operator $\Omega_F : C([1-r,b],\mathbb{R}) \to \mathscr{P}(C([1-r,b],\mathbb{R}))$ by

$$\Omega_F(y) = \left\{ \begin{array}{l} h \in C([1-r,b],\mathbb{R}) : \\[2mm] h(t) = \left\{ \begin{array}{ll} \vartheta(t), & \text{if } t \in [1-r,1], \\[4mm] \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \dfrac{t}{s}\right)^{\alpha-1} \dfrac{v(s)}{s} ds, & \text{if } t \in [1,b], \end{array} \right. \end{array} \right\}$$
(3.1)

for $v \in S_{F,y}$. It will be shown that the operator Ω_F satisfies the assumptions of Lemma 1. Firstly we observe that Ω_F is convex for each $x \in C([1-r,b],\mathbb{R})$ since $S_{F,x}$ is convex (F has convex values). Next, we show that Ω_F maps bounded sets into bounded sets in $C([1-r,b],\mathbb{R})$. For a positive number r, let $B_r = \{y \in C([1-$

$r, b], \mathbb{R}) : \|y\|_{[1-r,b]} \leq r\}$ be a bounded ball in $C([1-r,b], \mathbb{R})$. Then, for each $h \in \Omega_F(y), y \in B_r$, there exists $v \in S_{F,y}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Then for $t \in J$, we have

$$\begin{aligned}
|h(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |v(s)| \frac{ds}{s} \\
&\leq \frac{\beta(\|y\|_{[1-r,b]}) \|\zeta\|_\infty}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{\beta(\|y\|_{[1-r,b]}) \|\zeta\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha.
\end{aligned}$$

Thus

$$\|h\| \leq \frac{\beta(r) \|\zeta\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha := \tilde{\ell}.$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $C([1-r,b], \mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. For each $h \in \Omega_F(y)$, we obtain

$$\begin{aligned}
|h(t_2) - h(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\alpha-1} - \left(\log \frac{t_1}{s} \right)^{\alpha-1} \right] f(s, y_s) \frac{ds}{s} \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} f(s, y_s) \frac{ds}{s} \right| \\
&\leq \frac{\beta(r) \|\zeta\|_\infty}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\alpha-1} - \left(\log \frac{t_1}{s} \right)^{\alpha-1} \right] \frac{ds}{s} \\
&\quad + \frac{\beta(r) \|\zeta\|_\infty}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} \frac{ds}{s}.
\end{aligned}$$

Clearly the right hand side of the above inequality tends to zero independent of $x \in B_r$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, it follows by the Ascoli-Arzelá theorem that $\Omega_F : C([1-r,b], \mathbb{R}) \to \mathcal{P}(C([1-r,b], \mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F is upper semicontinuous. It is known [14, Proposition 1.2] that Ω_F will be upper semicontinuous if we establish that it has a closed graph, since Ω_F is already shown to be completely continuous. Thus we will prove that Ω_F *has a closed graph*. Let $y_n \to y_*, h_n \in \Omega_F(y_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(y_*)$. Associated with $h_n \in \Omega_F(y_n)$, there exists

$v_n \in S_{F,y_n}$ such that for each $t \in J$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}.$$

Thus it suffices to show that there exists $v_* \in S_{F,y_*}$ such that for each $t \in J$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s}.$$

Let us consider the linear operator $\Theta : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ given by

$$v \mapsto \Theta(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s}.$$

Notice that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (v_n(s) - v_*(s)) \frac{ds}{s} \right\| \to 0, \quad \text{as } n \to \infty.$$

Thus, it follows by Lemma 2 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,y_n})$. Since $y_n \to y_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s},$$

for some $v_* \in S_{F,y_*}$.

Finally, we show there exists an open set $U \subseteq C(J, \mathbb{R})$ with $y \notin \Omega_F(y)$ for any $\lambda \in (0,1)$ and all $y \in \partial U$. Let $\lambda \in (0,1)$ and $y \in \lambda \Omega_F(y)$. Then there exists $v \in L^1(J, \mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in J$, we have

$$y(t) = \lambda \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} \right).$$

By the assumption (H_2), for each $t \in J$, we get

$$|y(t)| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \zeta(s) \beta(\|y_s\|) \frac{ds}{s}$$

$$\leq \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)} (\log b)^\alpha,$$

which can be expressed as

$$\frac{\|y\|_{[1-r,b]}}{\beta(\|y\|_{[1-r,b]}) \|\zeta\|_\infty \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}} \leq 1.$$

In view of (H_3), there exists σ such that $\|y\|_{[1-r,b]} \neq \sigma$. Let us set

$$U = \{ y \in C([1-r,b], \mathbb{R}) : \|y\|_{[1-r,b]} < \sigma \}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}C([1-r,b],\mathbb{R})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda \Omega_F(y)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 1), we deduce that Ω_F has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.1)–(1.2). This completes the proof. $\square$

Next we prove the existence of solutions for the problem (1.1)–(1.2) with a non-convex valued right hand side (Lipschitz case) by applying a fixed point theorem for multivalued map due to Covitz and Nadler. Let us first introduce terminology.

Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a; b)$ and $d(a, B) = \inf_{b \in B} d(a; b)$.

Definition 8. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called:

(a) γ−Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

(b) a contraction if and only if it is γ−Lipschitz with $\gamma < 1$.

Lemma 3 ([13])**.** *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \varnothing$.*

Theorem 2. *Assume that:*

(H_4) *$F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, y) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$.*

(H_5) *$H_d(F(t, y), F(t, \bar{y})) \leq \ell(t)\|y - \bar{y}\|_C$ for almost all $t \in J$ and $y, \bar{y} \in C([-r, 0], \mathbb{R})$ with $\ell \in C(J, \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq \ell(t)$ for almost all $t \in J$.*

Then, if $\dfrac{(\log b)^\alpha}{\Gamma(\alpha + 1)}\|\ell\|_\infty < 1$, the initial value problem (1.1)–(1.2) has at least one solution on $[1-r, b]$.

Proof. Observe that the set $S_{F,y}$ is nonempty for each $x \in C([1-r, b], \mathbb{R})$ by the assumption (H_4), so F has a measurable selection (see Theorem III.6 [12]). Now we show that the operator Ω_F, defined by (3.1), satisfies the hypothesis of Lemma 3. To show that $\Omega_F(y) \in \mathcal{P}_{cl}(C([1-r, b], \mathbb{R}))$ for each $y \in C([1-r, b], \mathbb{R})$, let $\{u_n\}_{n \geq 0} \in \Omega_F(y)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([1-r, b], \mathbb{R})$. Then $u \in C([1-r, b], \mathbb{R})$ and there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$u_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}.$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J, \mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in J$, we have

$$u_n(t) \to u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} v(s) \frac{ds}{s}.$$

Hence, $u \in \Omega(y)$.

Next we show that there exists $\delta < 1$ $(\delta := \frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty)$ such that

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \delta \|y - \bar{y}\|_C \quad \text{for each} \quad y, \bar{y} \in C([1-r, b], \mathbb{R}).$$

Let $y, \bar{y} \in C([1-r, b], \mathbb{R})$ and $h_1 \in \Omega_F(y)$. Then there exists $v_1(t) \in F(t, y_t)$ such that, for each $t \in J$,

$$h_1(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} v_1(s) \frac{ds}{s}.$$

By (H_5), we have

$$H_d(F(t, y), F(t, \bar{y})) \leq \ell(t) \|y - \bar{y}\|_C.$$

So, there exists $w \in F(t, \bar{y}_t)$ such that

$$|v_1(t) - w| \leq \ell(t) \|y - \bar{y}\|_C, \quad t \in J.$$

Define $U : J \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq \ell(t) \|y - \bar{y}\|_C\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{y}_t)$ is measurable (Proposition III.4 [12]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{y}_t)$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq \ell(t) \|y - \bar{y}\|_C$.

For each $t \in J$, let us define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} v_2(s) \frac{ds}{s}.$$

Thus,

$$
\begin{aligned}
|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \ell(s) \|y - \bar{y}\|_C \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^b \left(\log \frac{t}{s} \right)^{\alpha-1} \ell(s) \|y - \bar{y}\|_{[1-r, b]} \frac{ds}{s} \\
&\leq \frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r, b]}.
\end{aligned}
$$

Hence,

$$\|h_1 - h_2\| \leq \frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r,b]}.$$

Analogously, interchanging the roles of y and $\overline{y}$, we obtain

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \frac{1}{\Gamma(\alpha+1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r,b]}.$$

Since Ω_F is a contraction, it follows by Lemma 3 that Ω_F has a fixed point x which is a solution of (1.1)–(1.2). This completes the proof. $\square$

4. Existence results for the problem (1.3)–(1.4)

This section is devoted to some results dealing with the existence of solutions for the problem (1.3)–(1.4).

Definition 9. A function $y \in C^1([1-r,b], \mathbb{R})$, is said to be a solution of (1.3)–(1.4) if there exists a function $v \in L^1([1,b], \mathbb{R})$ with $v(t) \in F(t, y_t)$, a.e. $[1,b]$ such that $D^\alpha[y(t) - g(t, y_t)] = v(t)$ on J, and the condition $y(t) = \vartheta(t)$ on $[1-r,1]$.

Theorem 3. *Assume that (H_1)–(H_2) hold. Further we suppose that*

(H_6) *the function g is continuous and completely continuous, and for any bounded set B in $C([1-r,b], \mathbb{R})$, the set $\{t \to g(t, y_t) : y \in B\}$ is equicontinuous in $C([1,b], \mathbb{R})$, and there exist constants $0 \leq d_1 < 1$, $d_2 \geq 0$ such that*

$$|g(t,u)| \leq d_1 \|u\|_C + d_2, \quad t \in [1,b], \ u \in C([-r,0], \mathbb{R}).$$

(H_7) *there exists a constant $M > 0$ such that*

$$\frac{(1-d_1)M}{d_2 + \dfrac{\|\zeta\|_\infty \beta(M)}{\Gamma(\alpha+1)}(\log b)^\alpha} > 1.$$

Then the IVP (1.3)–(1.4) has at least one solution on $[1-r,b]$.

Proof. Define the operator $N : C([1-r,b], \mathbb{R}) \to \mathscr{P}C([1-r,b], \mathbb{R})$ by

$$N(y) = \left\{ \begin{array}{l} h \in C([1-r,b], \mathbb{R}) : \\ h(t) = \left\{ \begin{array}{ll} \vartheta(t), & \text{if } t \in [1-r,1], \\ g(t, x_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \dfrac{t}{s}\right)^{\alpha-1} \dfrac{v(s)}{s} ds, & \text{if } t \in [1,b], \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,y}$.

Using (H_6), it suffices to show that the operator $N_1 : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by

$$
N_1(x) = \left\{
\begin{array}{l}
h \in C([1-r,b],\mathbb{R}) : \\[4pt]
h(t) = \left\{
\begin{array}{ll}
\vartheta(t), & \text{if } t \in [1-r,1], \\[10pt]
\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log\dfrac{t}{s}\right)^{\alpha-1} \dfrac{v(s)}{s} ds, & \text{if } t \in [1,b],
\end{array}
\right.
\end{array}
\right\}
$$

for $v \in S_{F,y}$, is continuous and completely continuous. The proof is similar to that of Theorem 1. So we omit the details.

Next we now show that there exists an open set $U \subseteq C([1-r,b],\mathbb{R})$ with $y \neq \lambda N(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$.

Let $y \in C([1-r,b],\mathbb{R})$ be such that $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [1,b]$, we have

$$
y(t) = \lambda \left(g(t,y_t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right).
$$

For each $t \in J$, it follows by (H_2) and (H_4) that

$$
|y(t)| \le d_1 \|y_t\|_C + d_2 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \zeta(s)\beta(\|y_s\|_C) \frac{ds}{s}
$$

$$
\le d_1 \|y_t\|_C + d_2 + \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)} (\log b)^\alpha,
$$

which yields

$$
(1-d_1)\|y\|_{[1-r,b]} \le d_2 + \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)} (\log b)^\alpha.
$$

In consequence, we get

$$
\frac{(1-d_1)\|y\|_{[1-r,b]}}{d_2 + \dfrac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha} \le 1.
$$

In view of (H_7), there exists M such that $\|y\|_{[1-r,b]} \neq M$. Let us set

$$
U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.
$$

Note that the operator $N_1 : \overline{U} \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N_1 y$ for some $\lambda \in (0,1)$. Thus, by the nonlinear alternative of Leray-Schauder type (Lemma 1), we deduce that N_1 has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.3)-(1.4). This completes the proof. □

Theorem 4. *Assume that $(H_4),(H_5)$ hold. In addition we suppose that:*

(H_8) *There exists a constant $L > 0$ such that*

$$|g(t,x) - g(t,y)| \leq L\|x - y\|_C, \text{ for all } t \in [1,b] \text{ and } x, y \in C([-r,0],\mathbb{R}).$$

Then, if $L + \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}\|\ell\|_\infty < 1$, the IVP (1.3)–(1.4) has at least one solution on $[1-r,b]$.

Proof. Since the proof is similar to that of Theorem 2, so it is omitted. $\qquad\square$

5. EXAMPLES

Example 1. For any function $\vartheta \in C([1-r,1],\mathbb{R})$ with $\vartheta(1) = 0$, consider the problem

$$D^\alpha y(t) \in F(t,y_t), \text{ for each } t \in J := [1,e], \quad 0 < \alpha < 1, \qquad (5.1)$$

$$y(t) = \vartheta(t), \quad t \in [1-r,1], \qquad (5.2)$$

where

$$F(t,y_t) = \left[\frac{1}{4+e-t}\left(\frac{|y|_t}{2(1+|y|_t)} + \frac{1}{4}\right), \frac{1}{16}(1+e^{-t})\right].$$

Clearly

$$\|F(t,y_t)\|_{\mathcal{P}} := \sup\{|u| : u \in F(t,y_t)\} \leq \frac{1}{4}\left(\frac{3}{4}\right), \quad y_t \in \mathbb{R}.$$

With $\zeta(t) = 1/4$, $\beta(\|y_t\|) = 3/4$, by the condition (H_3), we find that

$$M > \frac{3}{16\Gamma(\alpha+1)}, 0 < \alpha < 1.$$

Hence, by Theorem 1, the problem (5.1)-(5.2) has a solution on $[1-r,e]$.

Example 2. Let us consider the problem (5.1)-(5.2) with

$$F(t,y_t) = \left[\frac{1}{16}, \frac{1}{\pi\sqrt{t+3}}\tan^{-1}(y_t) + \frac{1}{12}\right]. \qquad (5.3)$$

Observe that

$$H_d(F(t,y_t), F(t,\bar{y}_t)) \leq \frac{1}{\pi\sqrt{t+3}}\|y - \bar{y}\|_C.$$

Letting $\ell(t) = \dfrac{1}{\pi\sqrt{t+3}}$, we find that $d(0, F(t,0)) \leq \ell(t)$ for almost all $t \in J$ and $\dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}\|\ell\|_\infty = \dfrac{1}{2\pi\Gamma(\alpha+1)} < 1$, for $0 < \alpha < 1$. Thus all the conditions of Theorem 2 are satisfied. Hence, by the conclusion of Theorem 2, the problem (5.1)-(5.2) with (5.3) has a solution on $[1-r,e]$.

Remark 1. Illustrating examples for Theorems 3 and 4 can be constructed like Examples 1 and 2.

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