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SOME RESULTS ON THE JOINT HIGHER NUMERICAL RANGES AND RADII OF MATRICES

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Abstract. In this paper, some algebraic properties of the joint $k-$numerical radius, joint $k-$radius and the joint $k-$norm of matrices are investigated. Moreover, using the joint higher numerical ranges of diagonal matrices which are convex polyhedrons, a description for the shape of the higher numerical ranges of matrix polynomials is given.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{M}_{n\times m}$ be the vector space of all $n\times m$ complex matrices. For the case $n=m$, $\mathbb{M}_{n\times n}$ is denoted by $\mathbb{M}_n$; namely, the algebra of all $n\times n$ complex matrices. Throughout the paper, k,m and n are considered as positive integers, and $k\leq n$. Moreover, I_k denotes the $k\times k$ identity matrix. The set of all $n\times k$ isometry matrices is denoted by $\mathcal{X}_{n\times k}$; i.e., $\mathcal{X}_{n\times k}=\{X\in\mathbb{M}_{n\times k} : X^*X=I_k\}$. Also, the group of $n\times n$ unitary matrices is denoted by $\mathcal{U}_n$; namely,

$$\mathcal{U}_n=\{U\in\mathbb{M}_n\ :\ U^*U=I_n\}=\mathcal{X}_{n\times n}.$$

The notion of $k-$numerical range of $A\in\mathbb{M}_n$, which was first introduced by P. R. Halmos in [8], is defined and denoted by

$$W_k(A)=\{\frac{1}{k}tr(X^*AX)\ :\ X\in\mathcal{X}_{n\times k}\},$$

where $tr(.)$ denotes the trace. The sets $W_k(A)$, where $k\in\{1,2,\ldots,n\}$, are generally called the *higher numerical ranges* of A. When $k=1$, we have the classical numerical range $W_1(A)=W(A)=\{x^*Ax\ :\ x\in\mathbb{C}^n,\ x^*x=1\}$, which has been studied extensively; see for example [7] and [9, Chapter 1]. Motivation of our study comes from finite-dimensional quantum systems. In quantum physics, e.g., see [6], the quantum states are represented by density matrices, i.e., positive semidefinite matrices with trace one. If a quantum state $D\in\mathbb{M}_n$ has rank one, i.e., $D=xx^*$

for some $x \in \mathbb{C}^n$ with $x^*x = 1$, then D is called a pure quantum state; otherwise, D is said to be a mixed quantum state which can be written as a convex combination of pure quantum states. So, for $A \in \mathbb{M}_n$, we have $W(A) = \{tr(AD) \ : \ D \in \mathbb{M}_n \ is \ a \ pure \ quantum \ state\}$. It is known that A is Hermitian if and only if $W(A) \subseteq \mathbb{R}$, and also A is a positive semidefinite matrix if and only if $W(A) \subseteq [0, \infty]$. Moreover, by the fact that the convex hull of the set

$$\{\frac{1}{k}P \ : P \in \mathbb{M}_n \text{ is Hermitian, } P^2 = P, \text{ and } tr(P) = k\}$$

equals to the set $\mathcal{S}_k$ of density matrices $D \in \mathbb{M}_n$ satisfying $\frac{1}{k}I_n - D$ is positive semidefinite, we have

$$\begin{aligned} W_k(A) &= \{\frac{1}{k}tr(AP) \ : \ P \in \mathbb{M}_n, \ P^2 = P = P^*, \ tr(P) = k\} \\ &= \{tr(AD) \ : \ D \in \mathcal{S}_k\}. \end{aligned}$$

Let $A \in \mathbb{M}_n$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, counting multiplicities. The set of all $k-$averages of eigenvalues of A is denoted by $\sigma^{(k)}(A)$; namely,

$$\sigma^{(k)}(A) = \{\frac{1}{k}(\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}) \ : \ 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$

Notice that if $k = 1$, then $\sigma^{(1)}(A) = \sigma(A)$, i.e., the spectrum of A. Next, we list some properties of the $k-$numerical range of matrices which will be useful in our discussion. For more details, see [8, 13] and their references.

Proposition 1. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*

(i) *$W_k(A)$ is a compact and convex set in $\mathbb{C}$;*
(ii) *$conv(\sigma^{(k)}(A)) \subseteq W_k(A)$, where $conv(S)$ denotes the convex hull of a set $S \subseteq \mathbb{C}$. The equality holds if A is normal;*
(iii) *$\{\frac{1}{n}tr(A)\} = W_n(A) \subseteq W_{n-1}(A) \subseteq \cdots \subseteq W_2(A) \subseteq W_1(A) = W(A)$;*
(iv) *If $V \in \mathcal{X}_{n\times s}$, where $k \le s \le n$, then $W_k(V^*AV) \subseteq W_k(A)$. The equality holds if $s = n$, i.e., $W_k(U^*AU) = W_k(A)$, where $U \in \mathcal{U}_n$;*
(v) *For any $\alpha, \beta \in \mathbb{C}$, $W_k(\alpha A + \beta I_n) = \alpha W_k(A) + \beta$, and for the case $k < n$, $W_k(A) = \{\alpha\}$ if and only if $A = \alpha I_n$.*

Let $A \in \mathbb{M}_n$. The *k-numerical radius*, *k-spectral radius* and the *k-spectral norm of A* are defined and denoted, respectively, by

$$r_k(A) := max\{|z| \ : \ z \in W_k(A)\}, \tag{1.1}$$

$$\rho_k(A) := max\{|z| \ : \ z \in \sigma^{(k)}(A)\}, \tag{1.2}$$

and

$$\|A\|_{(k)} := max\{|\frac{1}{k}tr(X^*AY)| \ : \ X, Y \in \mathcal{X}_{n\times k}\}. \tag{1.3}$$

It is clear that $r_1(A) = r(A)$, $\rho_1(A) = \rho(A)$, and $\|A\|_{(1)} = \|A\|$, which are the numerical radius, the spectral radius, and the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm) of A, respectively. Also, $r_k(.)$ for the case $k<n$, and $\|.\|_{(k)}$ are vector norms on $\mathbb{M}_n$. Now, in the following proposition, we state some other properties of $\rho_k(.)$, $r_k(.)$ and $\|.\|_{(k)}$, and their relations. For more details, see [10, 12].

Proposition 2. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*

(i) $r_k(V^*AV) \le r_k(A)$ *and* $\|V^*AV\|_{(k)} \le \|A\|_{(k)}$ *for all* $V \in \mathcal{X}_{n\times s}$, *where* $k \le s \le n$. *The equality holds if* $s = n$;

(ii) $\|A\|_{(k)} = \frac{1}{k}\sum_{i=1}^{k} s_i(A)$, *where* $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A)$ *are the singular values of* A;

(iii) $\rho_k(A) \le r_k(A) \le \|A\|_{(k)}$;

(iv) $r_k(A) = \|A\|_{(k)}$ *if and only if there exists a number* $\theta \in \mathbb{R}$ *such that* $\alpha_j = a_j e^{i\theta}$ *for all* $j = 1,2,\ldots,k$, *where* $a_1 \ge a_2 \ge \cdots \ge a_n$ *are the singular values and* $\alpha_1,\alpha_2,\ldots,\alpha_n$, *where* $|\alpha_1| \ge |\alpha_2| \ge \cdots \ge |\alpha_n|$, *are the eigenvalues of* A;

(v) *If* $k < n$, *then* $\frac{1}{2(2k-1)}\|A\| \le r_k(A) \le \|A\|$;

(vi) $\|A\|_{(n)} \le \|A\|_{(n-1)} \le \cdots \le \|A\|_{(1)} = \|A\|$.

In Section 2, we study some algebraic properties of the joint $k-$numerical radius, joint $k-$spectral radius and the joint $k-$norm of matrices. In Section 3, using the joint higher numerical ranges of diagonal matrices which are convex polyhedrons, we give a description for the shape of the higher numerical ranges of matrix polynomials.

2. Main results

We begin this section by introducing the notion of the joint $k-$numerical radius of matrices.

Definition 1. Let $(A_1,A_2,\ldots,A_m) \in \mathbb{M}_n^m$. The *joint k-numerical radius* of $(A_1,A_2,\ldots,A_m)$ is defined and denoted by

$$r_k(A_1,A_2,\ldots,A_m) := \sup\{l_2(a_1,a_2,\ldots,a_m) \\ : (a_1,a_2,\ldots,a_m) \in W_k(A_1,A_2,\ldots,A_m)\},$$

where

$$l_2(a_1,a_2,\ldots,a_m) = \left(\sum_{i=1}^{m}|a_i|^2\right)^{\frac{1}{2}}, \text{ and}$$

$$W_k(A_1,A_2,\ldots,A_m) = \left\{\left(\frac{1}{k}tr(X^*A_1X),\ldots,\frac{1}{k}tr(X^*A_mX)\right) : X \in \mathcal{X}_{n\times k}\right\}$$

is the joint $k-$numerical range of $(A_1,A_2,\ldots,A_m)$.

Next, we are going to state a new description of $r_k(A_1,A_2,\ldots,A_m)$ which is one of the our main results in this section. For this, we need the following lemma.

Lemma 1. *Let* $(a_1, a_2, \dots, a_m) \in \mathbb{C}^m$. *Then*

$$l_2(a_1, a_2, \dots, a_m) = \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} |\sum_{j=1}^{m} \lambda_j a_j|,$$

where $S^1 = \{(z_1, z_2, \dots, z_m) \in \mathbb{C}^m \ : \ \sum_{i=1}^{m} |z_i|^2 = 1\}$.

Proof. To avoid of trivial case, we assume that $(a_1, a_2, \dots, a_m) \neq (0, 0, \dots, 0)$. Let $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1$ be arbitrary. By the Cauchy-Schwartz inequality, we have

$$|\sum_{i=1}^{m} \lambda_i a_i| \leq \left(\sum_{i=1}^{m} |\lambda_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} |a_i|^2\right)^{\frac{1}{2}} = l_2(a_1, a_2, \dots, a_m).$$

So,

$$\sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} |\sum_{j=1}^{m} \lambda_j a_j| \leq l_2(a_1, a_2, \dots, a_m).$$

By setting $\lambda_i = \frac{\overline{a_i}}{l_2(a_1, \dots, a_m)}$, where $i = 1, 2, \dots, m$, we have $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1$. Moreover,

$$|\sum_{j=1}^{m} \lambda_j a_j| = \frac{\sum_{j=1}^{m} |a_j|^2}{l_2(a_1, a_2, \dots, a_m)} = l_2(a_1, a_2, \dots, a_m).$$

Therefore, the result holds. □

Theorem 1. *If* $(A_1, A_2, \dots, A_m) \in \mathbb{M}_n^m$, *then*

$$r_k(A_1, A_2, \dots, A_m) = \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} r_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m).$$

Proof. For every $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1$, we have

$$W_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m) = \{\frac{1}{k} tr(X^*(\sum_{j=1}^{m} \lambda_j A_j) X) \ : \ X \in \mathcal{X}_{n \times k}\}$$

$$= \{\sum_{j=1}^{m} \lambda_j a_j \ : \ (a_1, a_2, \dots, a_m) \in W_k(A_1, A_2, \dots, A_m)\}.$$

Thus,

$$r_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m)$$
$$= \sup\{|\sum_{j=1}^{m} \lambda_j a_j| \ : \ (a_1, a_2, \dots, a_m) \; in W_k(A_1, A_2, \dots, A_m)\}.$$

Now, using Definition 1 and Lemma 1, we have:

$$\begin{aligned} r_k(A_1,A_2,\dots,A_m) &= \sup_{(a_1,a_2,\dots,a_m)\in W_k(A_1,A_2,\dots,A_m)} l_2(a_1,a_2,\dots,a_m) \\ &= \sup_{(a_1,a_2,\dots,a_m)\in W_k(A_1,A_2,\dots,A_m)} \ \sup_{(\lambda_1,\lambda_2,\dots,\lambda_m)\in S^1} \Big|\sum_{j=1}^m \lambda_j a_j\Big| \\ &= \sup_{(\lambda_1,\lambda_2,\dots,\lambda_m)\in S^1} \ \sup_{(a_1,a_2,\dots,a_m)\in W_k(A_1,A_2,\dots,A_m)} \Big|\sum_{j=1}^m \lambda_j a_j\Big| \\ &= \sup_{(\lambda_1,\lambda_2,\dots,\lambda_m)\in S^1} r_k(\lambda_1 A_1+\lambda_2 A_2+\cdots+\lambda_m A_m). \end{aligned}$$

So, the proof is complete. □

In the following proposition which follows from Theorem 1 and Propositions 1 and 2, we state some basic properties of the joint k-numerical radius of matrices.

Proposition 3. *Let $(A_1,A_2,\dots,A_m)\in \mathbb{M}_n^m$. Then the following assertions are true:*

(i) *If $k<n$, then $r_k(A_1,A_2,\dots,A_m)=0$ if and only if $A_1=A_2=\cdots=A_m=0$;*
(ii) *$r_k(\lambda A_1,\lambda A_2,\dots,\lambda A_m)=|\lambda| r_k(A_1,A_2,\dots,A_m)$ for all $\lambda\in\mathbb{C}$;*
(iii) *$r_k(A_1+A_1',\dots,A_m+A_m')\le r_k(A_1,\dots,A_m)+r_k(A_1',\dots,A_m')$, where $A_1',A_2',\dots,A_m'\in\mathbb{M}_n$;*
(iv) *$r_k(S^*A_1S,S^*A_2S,\dots,S^*A_mS)\le r_k(A_1,A_2,\dots,A_m)$ for all $S\in\mathcal{X}_{n\times t}$, where $k\le t\le n$. The equality holds if $t=n$. Consequently, for every $U\in\mathcal{U}_n$,*

$$r_k(U^*A_1U,U^*A_2U,\dots,U^*A_mU)=r_k(A_1,A_2,\dots,A_m).$$

In view of Proposition 3, $r_k(.,\dots,.)$, where $k<n$, is a vector norm on $\mathbb{M}_n^m$. Moreover, Theorem 1 leads us to introduce the notion of the joint $k-$norm of matrices.

Definition 2. Let $(A_1,\dots,A_m)\in\mathbb{M}_n^m$. The joint $k-$norm of $(A_1,\dots,A_m)$ is defined and denoted by

$$\|(A_1,A_2,\dots,A_m)\|_{(k)} := \sup_{(\lambda_1,\lambda_2,\dots,\lambda_m)\in S^1} \|\lambda_1A_1+\lambda_2A_2+\cdots+\lambda_mA_m\|_{(k)}.$$

Using Definition 2 and Proposition 2, we can show that the joint $k-$norm of matrices satisfies in the following basic properties.

Proposition 4. *Let $(A_1,A_2,\dots,A_m)\in\mathbb{M}_n^m$. Then the following assertions are true:*

(i) *If $k<n$, then $\|(A_1,A_2,\dots,A_m)\|_{(k)}=0$ if and only if $A_1=A_2=\cdots=A_m=0$;*

(ii) $\|(\lambda A_1, \lambda A_2, \ldots, \lambda A_m)\|_{(k)} = |\lambda| \|(A_1, A_2, \ldots, A_m)\|_{(k)}$ *for all* $\lambda \in \mathbb{C}$;

(iii) $\|(A_1 + A'_1, \ldots, A_m + A'_m)\|_{(k)} \leq \|(A_1, \ldots, A_m)\|_{(k)} + \|(A'_1, \ldots, A'_m)\|_{(k)}$, *where* $(A'_1, A'_2, \ldots, A'_m) \in \mathbb{M}_n^m$;

(iv) $\|(S^* A_1 S, S^* A_2 S, \ldots, S^* A_m S)\|_{(k)} \leq \|(A_1, A_2, \ldots, A_m)\|_{(k)}$ *for all* $S \in \mathcal{X}_{n \times t}$, *where* $k \leq t \leq n$. *The equality is holds if* $t = n$. *Consequently, for every* $U \in \mathcal{U}_n$,

$$\|(U^* A_1 U, U^* A_2 U, \ldots, U^* A_m U)\|_{(k)} = \|(A_1, A_2, \ldots, A_m)\|_{(k)}.$$

In view of Proposition 4, for the case $k < n$, the joint $k-$norm is a vector norm on $\mathbb{M}_n^m$. In the final result of this section, we are going to study the relations between the joint $k-$numerical radius and the joint $k-$norm of matrices. For this, we need the following lemma.

Lemma 2. *Let* $A \in \mathbb{M}_n$. *Then*

$$\frac{1}{2(2k-1)} \|A\|_{(k)} \leq r_k(A) \leq \|A\|_{(k)}.$$

Proof. The first inequality follows from Proposition 2((v) and (vi)). Also, by (1.1) and (1.3), it is clear that $r_k(A) \leq \|A\|_{(k)}$, and so, the proof is complete. □

Theorem 2. *Let* $(A_1, A_2, \ldots, A_m) \in \mathbb{M}_n^m$. *Then*

$$\begin{aligned}\frac{1}{2(2k-1)} \|(A_1, \ldots, A_m)\|_{(k)} &\leq r_k(A_1, \ldots, A_m) \\ &\leq min\{\|(A_1, \ldots, A_m)\|_{(k)}, l_2(r_k(A_1), \ldots, r_k(A_m))\}.\end{aligned}$$

Moreover, the right inequality is sharp.

Proof. Applying Lemma 2 to the matrix $A := \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m$, where $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in S^1$, and using Definition 2 and Theorem 1, we have

$$\frac{1}{2(2k-1)} \|(A_1, \ldots, A_m)\|_{(k)} \leq r_k(A_1, \ldots, A_m) \leq \|(A_1, \ldots, A_m)\|_{(k)}.$$

Moreover, it is clear that $W_k(A_1, A_2, \ldots, A_m) \subseteq W_k(A_1) \times W_k(A_2) \times \cdots \times W_k(A_m)$. Then,

$$\begin{aligned} r_k(A_1, \ldots, A_m) &= \sup\{l_2(a_1, \ldots, a_m) \; : \; (a_1, \ldots, a_m) \in W_k(A_1, \ldots, A_m)\} \\ &\leq l_2(r_k(A_1), \ldots, r_k(A_m)). \end{aligned}$$

Therefore, the right inequality also holds.

To prove that the right inequality is sharp, we consider the matrix $A \in \mathbb{M}_n$ with singular values $a_1 \geq a_2 \geq \cdots \geq a_n$ and eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|$. Moreover, we assume that there exist a $\theta \in \mathbb{R}$ such that $\alpha_j = a_j e^{i\theta}$ for all $j = 1, 2, \ldots, n$. Then by Proposition 2(v), we have $r_k(A) = \|A\|_{(k)}$. Now, suppose that $A_j = t_j A$, where $t_j \in \mathbb{C}$ for all $j = 1, 2, \ldots, m$. Then by Theorem 1, Proposition 4 and Definition 2, we have:

$$\begin{aligned} r_k(A_1, A_2, \ldots, A_m) &= \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_m)\in S^1} r_k(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m) \\ &= \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_m)\in S^1} |\sum_{j=1}^{m} t_j \lambda_j| r_k(A) \\ &= \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_m)\in S^1} |\sum_{j=1}^{m} t_j \lambda_j| \|A\|_{(k)} \\ &= \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_m)\in S^1} \|\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m\|_{(k)} \\ &= \|(A_1, A_2, \ldots, A_m)\|_{(k)}. \end{aligned}$$

Now, if we choose $t_j \geq 0$, then by Lemma 1 and the above inequalities, we have:

$$\|(A_1, A_1, \ldots, A_m)\|_{(k)} = l_2(r_k(A_1), r_k(A_2), \ldots, r_k(A_m)).$$

Hence, the right inequality changes to equality. This completes the proof. □

3. Additional results

In this section, by using the joint higher numerical ranges of diagonal matrices, we find an approximation for the higher numerical ranges of matrix polynomials. For this, suppose that

$$P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + A_0 \tag{3.1}$$

is a matrix polynomial, where $A_i \in \mathbb{M}_n$ $(i = 0, 1, \ldots, m)$, $A_m \neq 0$ and λ is a complex variable. The numbers m and n are referred as the *degree* and the *order* of $P(\lambda)$, respectively. The matrix polynomial $P(\lambda)$ is said to be a *diagonal matrix polynomial* if all the coefficients A_i are diagonal matrices. A scalar $\lambda_0 \in \mathbb{C}$ is called an *eigenvalue* of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is said to be the *spectrum* of $P(\lambda)$; namely,

$$\sigma[P(\lambda)] = \{\mu \in \mathbb{C} \,:\, det(P(\mu)) = 0\}.$$

The *(classical) numerical range* of $P(\lambda)$ is defined and denoted by

$$W[P(\lambda)] := \{\mu \in \mathbb{C} \,:\, x^* P(\mu) x = 0 \ for\ some\ nonzero\ x \in \mathbb{C}^n\},$$

which is closed and contains $\sigma[P(\lambda)]$. The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with finite number of degrees of freedom, and it is also related to the stability theory; e.g., see [5, 11] and its references. Notice that the notion of $W[P(\lambda)]$ is generalization of the *classical numerical range* of a matrix $A \in \mathbb{M}_n$; namely, $W[\lambda I - A] = W(A)$. In the last few years, the generalization of the numerical range of matrices and matrix polynomials has attracted much attention and many interesting results have been obtained; e.g., see [1–4, 15]. One of these generalizations is the notion of higher numerical ranges. The *k-numerical range* of $P(\lambda)$ is defined and denoted, e.g., see [2], by

$$W_k[P(\lambda)] = \{\mu \in \mathbb{C} \,:\, tr(X^*P(\mu)X) = 0 \ for\ some\ X \in \mathcal{X}_{n\times k}\}. \tag{3.2}$$

Also, the $k-$ *spectrum* of $P(\lambda)$ is defined as

$$\sigma^{(k)}[P(\lambda)] = \{\mu \in \mathbb{C} \,:\, 0 \in \sigma^{(k)}(P(\mu))\}. \tag{3.3}$$

It is clear that $\sigma^{(k)}[P(\lambda)] \subseteq W_k[P(\lambda)] = \{\mu \in \mathbb{C} \,:\, 0 \in W_k(P(\mu))\}$, $\sigma^{(1)}[P(\lambda)] = \sigma[P(\lambda)]$, and $W_1[P(\lambda)] = W[P(\lambda)]$. Moreover, if $P(\lambda) = \lambda I - A$, where $A \in \mathbb{M}_n$, then $W_k[P(\lambda)] = W_k(A)$ and $\sigma^{(k)}[P(\lambda)] = \sigma^{(k)}(A)$. The sets $W_k[P(\lambda)]$, where $k \in \{1,2\dots,n\}$, are generally called the *higher numerical ranges* of $P(\lambda)$. The *joint $k-$numerical range* of $P(\lambda)$ is defined as the joint $k-$numerical range of its coefficients; namely, $JW_k[P(\lambda)] := W_k(A_0, A_1, \dots, A_m)$. It is known, e.g., see [2, Theorem 2.2(iii)], that:

$$\begin{aligned} W_k[P(\lambda)] &= \{\mu \in \mathbb{C} \,:\, c_m\mu^m + \dots + c_0 = 0,\ (c_0,\dots,c_m) \in JW_k[P(\lambda)]\} \\ &= \{\mu \in \mathbb{C} \,:\, c_m\mu^m + \dots + c_0 = 0,\ (c_0,\dots,c_m) \in conv(JW_k[P(\lambda)])\}. \end{aligned} \tag{3.4}$$

So, if $Q(\lambda)$ is a matrix polynomial of degree m and arbitrary order such that $JW_k[P(\lambda)] \subseteq JW_k[Q(\lambda)]$ or if $conv\,(JW_k[P(\lambda)]) \subseteq conv\,(JW_k[Q(\lambda)])$, then we have $W_k[P(\lambda)] \subseteq W_k[Q(\lambda)]$. In the following proposition, we characterize the joint $k-$numerical range of a diagonal matrix polynomial.

Proposition 5. *Let $P(\lambda)$, as in (3.1), be a diagonal matrix polynomial. Then $JW_k[P(\lambda)]$ is a convex polyhedron. Conversely, every convex polyhedron $H \subseteq \mathbb{C}^{m+1}$ is the joint $k-$numerical range of a diagonal matrix polynomial of degree m.*

Proof. Suppose that $A_i = diag(a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)})$ for $i = 0,1,\dots,m$ and $S_1, S_2, \dots, S_{c(n,k)}$ are all the subsets of $\{1,2,\dots,n\}$ with k elements. By considering $q_{S_i}^{(j)} = \frac{\sum_{t\in S_i} a_t^{(j)}}{k}$ for $j = 0,1,\dots,m$, we have

$$JW_k[P(\lambda)] = conv(\{(q_{S_i}^{(1)}, q_{S_i}^{(2)}, \dots, q_{S_i}^{(m)}) \,:\, i = 1,2,\dots,c(n,k)\}).$$

Conversely, suppose that $H \subseteq \mathbb{C}^{m+1}$ is a convex polyhedron with n vertices $v_1, v_2, \dots, v_n$. Now, we can find the points $(c_{i,0}, c_{i,1}, \dots, c_{i,m}) \in \mathbb{C}^{m+1}$, $i = 1,2,\dots,q$ with $q \geq n$ such that their $k-$averages are $v_1, v_2, \dots, v_n$. Then $H = JW_k[D(\lambda)]$, where

$$D(\lambda) = diag(c_{1,m},\dots,c_{q,m})\lambda^m + \dots + diag(c_{1,1},\dots,c_{q,1})\lambda + diag(c_{1,0},\dots,c_{q,0}).$$

So, the proof is complete. □

The following theorem is a generalization of [14, Theorem 2.4].

Theorem 3. *Let $P(\lambda)$ be a matrix polynomial as in (3.1). Then*

$$\bigcup W_k[D_1(\lambda)] = W_k[P(\lambda)] = \bigcap W_k[D_2(\lambda)],$$

where the union [intersection] is taken over all diagonal matrix polynomials $D_1(\lambda)$ $[D_2(\lambda)]$ of degree m for which $JW_k[D_1(\lambda)] \subseteq JW_k[P(\lambda)] \subseteq JW_k[D_2(\lambda)]$.

Proof. By [1, Theorem 3.1(ii)], the left equality holds.
For the right equality, note that for every diagonal matrix polynomial $D_2(\lambda)$ of degree m for which $JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]$, we have $W_k[D_2(\lambda)] \supseteq W_k[P(\lambda)]$. Thus

$$W_k[P(\lambda)] \subseteq \bigcap_{JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]} W_k[D_2(\lambda)].$$

Conversely, let

$$\lambda_0 \in \bigcap_{JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]} W_k[D_2(\lambda)].$$

By considering $T_1(\lambda_0) = \{(c_0, c_1, \ldots, c_m) \in \mathbb{C}^{m+1} \ : \ c_m\lambda_0^m + \cdots + c_1\lambda_0 + c_0 = 0\}$, we have $T_1(\lambda_0) \cap JW_k[D_2(\lambda)] \neq \varnothing$ for every diagonal matrix polynomial $D_2(\lambda)$ satisfying $JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]$. Thus, by [14, lemma 4.1] and Proposition 5, we have

$$T_1(\lambda_0) \cap conv(JW_k[P(\lambda)]) \neq \varnothing,.$$

So, by (3.4), $\lambda_0 \in W_k[P(\lambda)]$. Therefore,

$$W_k[P(\lambda)] \supseteq \bigcap_{JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]} W_k[D_2(\lambda)],$$

and hence, the proof is complete. □

Remark 1. In view of Proposition 5 and Theorem 3, one can estimate the shape of the $k-$numerical range of matrix polynomials by convex polyhedrons.

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