ON POLYNOMIAL IDENTITIES FOR RECURSIVE SEQUENCES

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Abstract. In this paper we extend the notion of Melham sum to the Pell and Pell-Lucas sequences. While the proofs of general statements rely on the binomial theorem, we prove some spacial cases by the known Pell identities. We also give extensions of obtained expressions to the other recursive sequences.

1. INTRODUCTION

The Pell sequence $(P_n)_{n \geq 0}$ and the Pell-Lucas sequence $(Q_n)_{n \geq 0}$ are defined as the second order recurrences,

$$P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0, \quad P_1 = 1 \tag{1.1}$$

$$Q_{n+2} = 2Q_{n+1} + Q_n, \quad Q_0 = 2, \quad Q_1 = 2. \tag{1.2}$$

Equivalently, these sequences can be defined as the solutions of Diophantine equations

$$x^2 - dy^2 = \pm 1$$

for $d = 2$. More precisely, the pairs $(Q_n/2, P_n)$ are all solutions of these equations. The n-th term of the Pell sequence can also be expressed by the closed form equation. The Pell-Lucas sequence is sometimes called *companion Pell sequence* and there is also similar closed form for this sequence. We let γ denote the silver ratio, $\gamma := 1 + \sqrt{2}$ and we set $\delta := 1 - \sqrt{2}$. Then the closed formula for Pell sequence can be written as

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \tag{1.3}$$

while for the companion Pell numbers we have $Q_n = \gamma^n - \delta^n$.

Both the Pell sequence and Pell equation are the subject of numerous papers. Among the most remarkable theoretical number properties let us mention the facts that

- P_{2n+1} divides the sum $\sum_{k=0}^{2n} P_{2k+1}$,

- P_{2n} divides the sum $\sum_{k=1}^{2n} P_{2k-1}$,

and sum of the first $4n + 1$ Pell numbers $\sum_{k=1}^{4n+1} P_k$ is a perfect square [7]. Furthermore, in [6] Dujella found quadruples of the Pell and Pell-Lucas numbers that have the *property of Diophantus* of a certain order. In particular, the quadruples

$$\{P_{2n}, P_{2n+2}, 2P_{2n}, Q_{2n}P_{2n+1}Q_{2n+1}\},$$
$$\{P_{2n}, P_{2n+2}, 2P_{2n+2}, P_{2n+1}Q_{2n+1}Q_{2n+2}\}$$

have the property of Diophantus of order 1, meaning that $a_i a_j + 1$, is a perfect square where $a_i, a_j, i \neq j$ are the elements of a quadruple.

Some recent surveys on the Pell equation one can find in [1, 3].

There are also many known combinatorial properties and identities for Pell and Pell-Lucas sequences [4, 5, 12]. This includes several identities encountering both of the sequences,

$$Q_n = P_{n-1} + P_{n+1} \tag{1.4}$$

being the basic one. Recall that the Cassini identity [11] for Pell numbers has form

$$P_{n-1}P_{n+1} - P_n^2 = (-1)^n. \tag{1.5}$$

An elegant proof is based on the fact that

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} P_{n-1} & P_n \\ P_n & P_{n+1} \end{pmatrix},$$

which can be proved by induction. When applying the Cauchy-Binet theorem for determinants, the statement follows immediately. We will also use relation

$$P_{m+n} = P_{m-1}P_n + P_m P_{n+1}, \tag{1.6}$$

for the purpose to prove some polynomial identities for Pell numbers. Identity (1.6) can be proved by induction.

This paper aims at finding Pell identities and polynomial representation for the Pell numbers. In what follows, firstly we prove that $(2m + 1)n$-th Pell number is represented as a polynomial in P_n. Then we extend the notion of Melham sum [10] to the Pell and Pell-Lucas sequences and find related expansions into the power series of P_n, where exponents are odd. Finally, we give extensions of the obtained identities for a certain, more general class of recursive sequences.

2. THE $(2m + 1)n$-TH PELL NUMBER AS A POLYNOMIAL IN P_n

Proposition 1. *For the Pell sequence* $(P_n)_{n \geq 0}$ *we have*

$$i) \qquad P_{3n} = 8P_n^3 + 3(-1)^n P_n \tag{2.1}$$

$$ii) \qquad P_{5n} = 64P_n^5 + 40(-1)^n P_n^3 + 5P_n. \tag{2.2}$$

Proof. According to relations (1.5) and (1.6) we get

$$P_{3n} = P_{2n+n} = P_{2n-1}P_n + P_{2n}P_{n+1}$$
$$= P_{n-1}^2 P_n + P_n^3 + (P_n - 1P_n + P_n P_{n+1})(2P_n + P_{n-1})$$
$$= P_n(P_{n-1}^2 + 2P_n P_{n-1} + P_n^2 + 2P_n P_{n+1} + P_{n-1}^2 + P_n^2 + (-1)^n)$$

which finally gives

$$P_{3n} = P_n(3P_n^2 + 2(-1)^2 + 2P_n P_{n+1} + P_{n-1}^2)$$
$$= P_n(3P_n^2 + 2(-1)^2 + 4P_n^2 + P_{n-1}(2P_n + P_{n-1})) \quad = P_n(8P_n^2 + 3(-1)^n).$$

Application of the same relations also proves identities for P_{5n}. $\square$

Furthermore, for the next instance when n is odd we have

$$P_{7n} = 512 P_n{}^7 - 448 P_n{}^5 + 112 P_n{}^3 - 7P_n \tag{2.3}$$

while all coefficients are positive when n is even.

Theorem 1. *For the Pell sequence* $(P_n)_{n\geq 0}$

$$P_{(2m+1)n} = \sum_{i=0}^{m} (-1)^{n(m+i)} 2^{3i} \frac{2m+1}{2i+1} \binom{m+i}{2i} P_n^{2i+1}. \tag{2.4}$$

Proof. We use equalities (2.5) and (2.6), which are results of D. Jennings available in [9] and which can be proved by induction.

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1$$
$$= \sum_{i=0}^{m} \frac{2m+1}{m+i+1} \binom{m+i+1}{2i+1} \left(x - \frac{1}{x}\right)^{2i} \tag{2.5}$$

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1}\left(x^2 + \frac{1}{x^2}\right) + (-1)^m$$
$$= \sum_{i=0}^{m} (-1)^{m+i} \frac{2m+1}{m+i+1} \binom{m+i+1}{2i+1} \left(x + \frac{1}{x}\right)^{2i} \tag{2.6}$$

Having in mind Binet formula for the Pell numbers (1.3) and the fact that

$$\gamma \cdot \delta = -1 \tag{2.7}$$
$$\gamma - \delta = 2\sqrt{2} \tag{2.8}$$

we have

$$\frac{P_{pn}}{P_n} = \frac{\gamma^{pn} - \delta^{pn}}{\gamma^n - \delta^n} = x^{p-1} + x^{p-2}y + \cdots + xy^{p-2} + y^{p-1}, \tag{2.9}$$

where $x = \gamma^n$ and $y = \delta^n = \frac{(-1)^n}{x}$. When p is odd, the r.h.s. of (2.9) reduces to

$$\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + \cdots + \left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n \tag{2.10}$$

when $p \equiv 3 \pmod 4$ or to

$$\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + \cdots + (-1)^n\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + 1 \tag{2.11}$$

when $p \equiv 1 \pmod 4$. Now, we have

$$x + \frac{1}{x} = \gamma^n + \frac{1}{\gamma^n} = \gamma^n + (-1)^n \delta^n$$

which gives

$$x + \frac{1}{x} = (\gamma - \delta) P_n, \quad n \equiv 1 \pmod 2 \tag{2.12}$$

$$x - \frac{1}{x} = (\gamma - \delta) P_n, \quad n \equiv 0 \pmod 2 \tag{2.13}$$

and furthermore

$$\left(x + \frac{1}{x}\right)^2 = 8P_n^2, \quad n \equiv 1 \pmod 2 \tag{2.14}$$

$$\left(x - \frac{1}{x}\right)^2 = 8P_n^2, \quad n \equiv 0 \pmod 2 \tag{2.15}$$

Since we get expression (2.10) assuming that p is odd we now substitute $p = 2m + 1$. Now, when n is even we obtain all positive terms in (2.10) and then r.h.s. of (2.9) is equal to the l.h.s. of equality (2.5),

$$P_{(2m+1)n} = \sum_{i=0}^{m} (-1)^{n(m+i)} 2^{3i} \frac{2m+1}{m+i+1} \binom{m+i+1}{2i+1} P_n^{2i+1}. \tag{2.16}$$

Analogue reasoning when n is odd gives the same relation, thus (2.16) holds true for any natural number n. Finally, a simple manipulation with (2.16) leads to the final form of the theorem. $\square$

One can easily see that relations (2.1), (2.2) and (2.3) appear from Theorem 1 for $m = 1, 2$ and 3, respectively. When $m = 4$ Theorem 1 gives

$$P_{9n} = 2^{12} P_n{}^9 - 9 \cdot 2^9 P_n{}^7 + 1728 P_n{}^5 - 240 P_n{}^3 + 9 P_n \tag{2.17}$$

when n is odd while all coefficients are positive otherwise. Note that the leading coefficient in (2.4) is always a power of 2, 2^{3m}, while the absolute value of the coefficient in the term of the smallest degree is $2m + 1$.

3. MELHAM SUM FOR THE PELL AND PELL-LUCAS SEQUENCE

Proposition 2. *Twice the sum of the Pell numbers having even indexes from 2 to n is equal to the (2n+1)-st Pell number diminished by 1,*

$$1 + 2 \sum_{k=1}^{n} P_{2k} = P_{2n+1}. \tag{3.1}$$

Proof. The statement follows immediately from defining properties of Pell sequence,

$$\begin{aligned}
P_{2n+1} &= 2P_{2n} + P_{2n-1} \\
&= 2P_{2n} + 2P_{2n-2} + P_{2n-3} \\
&= 2P_{2n} + 2P_{2n-2} + \cdots + 2P_2 + P_1.
\end{aligned}$$

$\square$

Note that relation (3.1) can be seen as the expansion of the expression $Q_1 \sum_{k=1}^{n} P_{2k}$ into polynomial in P_{2n+1},

$$Q_1 \sum_{k=1}^{n} P_{2k} = P_{2n+1} - 1.$$

In what follows we extend this idea to full generality. The expression

$$Q_1 Q_2 \cdots Q_{2m+1} \sum_{k=1}^{n} P_{2k}^{2m+1},$$

we shall call the *Melham sum for Pell and Pell-Lucas sequences*, because there is analogy with established term for Fibonacci and Lucas sequences. More on the Fibonacci sequence one can find in the classic book by S. Vajda. Introduction to Fibonacci polynomials one can find in [8], and some recent development in [2].

Lemma 1. *For the sequences $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$ and $m \in \mathbb{N}$*

$$Q_m \sum_{k=1}^{n} P_{2mk} = P_{m(2n+1)} - P_m. \tag{3.2}$$

Proof. By relation (1.6) we have

$$\begin{aligned}
P_{m+n} &= (P_{m-n} - P_n P_{n-1}(-1)^n)(-1)^{n+1} + P_m P_{n-1} \\
&= P_{m-1}(-1)^{n+1} + P_m P_{n-1} + P_m P_{n+1} \\
&= P_{m-1}(-1)^{n+1} + P_m(P_{n-1} + P_{n+1}) \\
&= P_m Q_n - (-1)^n P_{m-n}.
\end{aligned}$$

Now we prove the statement of lemma by induction where this result is used in a step of induction. Thus, from the fact that the statement holds true for $n = 1$ we have to derive equality $Q_n \sum_{k=1}^{n+1} P_{2mk} = P_{m(2n+3)} - P_m$. We have

$$
\begin{aligned}
Q_n \sum_{k=1}^{n+1} P_{2mk} &= Q_n \Big(\sum_{k=1}^{n} P_{2mk} + P_{2m(n+1)} \Big) \\
&= P_{m(2n+1)} - P_m + Q_n P_{2m(n+1)} \\
&= P_{m(2n+1)} + Q P_{2m(n+1)} - P_m \\
&= P_{2m(n+1)+m} - P_m \\
&= P_{m(2n+3)} - P_m,
\end{aligned}
$$

which completes the statement of the lemma. $\qquad\square$

Lemma 2. *For the sequences* $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$ *and* $m \in \mathbb{N}$

$$
P_n{}^{2m+1} = \frac{1}{2^{3m}} \sum_{j=0}^{m} (-1)^{j(n+1)} \binom{2m+1}{j} P_{(2m+1-2j)n}. \tag{3.3}
$$

Proof. By means of binomial theorem and using (2.7) as well as (2.8) we have

$$
\begin{aligned}
P_n{}^{2m+1} &= \Big(\frac{\gamma^n - \delta^n}{\gamma - \delta} \Big)^{2m+1} \\
&= \frac{1}{(\gamma - \delta)^{2m+1}} \sum_{j=0}^{2m+1} (-1)^{j+1} \binom{2m+1}{j} \gamma^{jn} \delta^{(2m+1-j)n} \\
&= \frac{1}{8^m (\gamma - \delta)} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \big(\gamma^{(2m+1-j)n} \delta^{jn} - \gamma^{jn} \delta^{(2m+1-j)n} \big) \\
&= \frac{1}{2^{3m}} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \gamma^{jn} \delta^{jn} \Big(\frac{\gamma^{(2m+1-2j)n} - \delta^{(2m+1-2j)n}}{\gamma - \delta} \Big) \\
&= \frac{1}{2^{3m}} \sum_{j=0}^{m} (-1)^{j(n+m)} P_{(2m+1-2j)n}
\end{aligned}
$$

which completes the statement of lemma. $\qquad\square$

Theorem 2. *For* $m \in \mathbb{N}$ *and the sequences* $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$

$$
\sum_{k=1}^{n} P_{2k}{}^{2m+1} = \frac{1}{2^{3m}} \sum_{j=0}^{m} \frac{(-1)^{j}}{Q_{2m+1-2j}} \binom{2m+1}{j} \big(P_{(2m+1-2j)(2n+1)} - P_{2m+1-2j} \big). \tag{3.4}
$$

Proof. In Lemma 2 we substitute $n = 2k$ and then sum both sides of equality from $k = 1$ through n. It follows

$$P_{2k}{}^{2m+1} = \frac{1}{2^{3m}} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \sum_{k=1}^{n} P_{(2m+1-2j)2k}.$$

When we substitute $\sum_{k=1}^{n} P_{(2m+1-2j)2k}$ by the expression in Lemma 1, the proof is completed. □

Theorem 3. *For $m \in \mathbb{N}$ and sequences $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$*

$$\sum_{k=1}^{n} P_{2k}^{2m+1} = \sum_{i=0}^{m} P_{2n+1}^{2i+1} \sum_{j=0}^{m-i} \frac{(-1)^{m+i} 2^{3(i-m)}(2m-2j+1)}{Q_{2m+1-2j}(2i+1)} \binom{2m+1}{j} \binom{m-j+i}{2i}$$

$$+ \sum_{j=0}^{m} \frac{(-1)^{j+1} P_{2m+1-2j}}{2^{3m} Q_{2m+1-2j}} \binom{2m+1}{j}. \tag{3.5}$$

Proof. When substitute m with $m - j$ and n with $2n + 1$ in Theorem 1 one get

$$P_{(2m+1-2j)(2n+1)} = \sum_{i=0}^{m-j} (-1)^{(2n+1)(m-j+i)} 2^{3i} \frac{2m-2j+1}{2i+1} \binom{m-j+i}{2i} P_{2n+1}^{2i+1}.$$

We substitute this expression in Theorem 2 and the statement follows immediately. □

Now we consider some particular cases of Theorem 3. When $m = 1$ we obtain

$$\sum_{k=1}^{n} P_{2k}^3 = \frac{1}{14} \left(P_{2n+1}^3 - 3 P_{2n+1} + 2 \right).$$

When multiply this relation with $Q_1 Q_3$ we get polynomial identity for the Melham sum in case $m = 1$

$$Q_1 Q_3 \sum_{k=1}^{n} P_{2k}^3 = 2 P_{2n+1}^3 - 6 P_{2n+1} + 4. \tag{3.6}$$

The next case, when $m = 2$ gives

$$Q_1 Q_3 Q_5 \sum_{k=1}^{n} P_{2k}^5 = 28 P_{2n+1}^5 - 120 P_{2n+1}^3 + 220 P_{2n+1} - 128. \tag{3.7}$$

4. FURTHER EXTENSIONS

Given $s, t \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define the second order recurrence with the relation

$$a_{n+2} = s a_{n-1} + t a_n \tag{4.1}$$

and initial values a_0 and a_1. We say that a sequence $(a_n)_{n \geq 0}$ is a solution of (4.1) if its terms satisfies this recurrence. Here we consider a class of (4.1) defined by $t = 1$

and initial terms $a_0 = 0$, $a_1 = 1$. We let $(A_n)_{n \geq 0}$ denote the sequence defined by this class. It is worth mentioning that two notable representatives of this class are Fibonacci and Pell numbers.

Proposition 3. *For the sequence of numbers $(A_n)_{n \geq 0}$ we have*

$$i) \qquad A_{3n} = (s^2 + 4)A_n^3 + 3(-1)^n A_n \tag{4.2}$$

$$ii) \qquad A_{5n} = (s^2 + 4)^2 A_n^5 + 5(s^2 + 4)(-1)^n A_n^3 + 5A_n. \tag{4.3}$$

Proof. By induction we prove that

$$A_{n-1} A_{n+1} - A^2 = (-1)^2 \tag{4.4}$$

and also

$$A_{m+n} = A_{m-1} A_n + A_m A_{n+1}. \tag{4.5}$$

Now we employ (4.5) to get

$$\begin{aligned} A_{3n} = A_{2n+n} &= A_{2n-1} A_n + A_{2n} A_{n+1} \\ &= A_{n-1}^2 A_n + A_n^3 + (A_{n-1} A_n + A_n A_{n+1})(A_2 A_n + A_{n-1}) \\ &= A_n(A_{n-1}^2 + A_n^2 + sA_{n-1}A_n + sA_n A_{n+1} + A_{n-1}^2 + A_{n-1}A_{n+1}). \end{aligned}$$

Having in mind that

$$A_{n-1}^2 + sA_{n-1}A_n = A_n^2 + (-1)^n$$

by (4.4), we obtain

$$A_{3n} = A_n(2A_n^2 + 2(-1)^n + A_n^2 + sA_n A_{n+1} + A_{n-1}^2).$$

When applying again (4.4) to the terms $sA_n A_{n+1}$ and A_{n-1}^2 we finally have

$$\begin{aligned} A_{3n} &= A_n[4A_n^2 + s^2 A_n^2 + 3(-1)^n] \\ &= A_n\big[(s^2 + 4)A_n^2 + 3(-1)^n\big]. \end{aligned}$$

The second relation can be proved by analogue calculation. $\qquad\square$

Clearly, further identities can be proved in the same fashion as Proposition 3 was proved. Instead, we give a more elegant family of identities (4.6) that generalize Proposition 3. It follows as a corollary of Theorem 1.

Corollary 1. *For $m \in \mathbb{N}$ and the sequence of numbers $(A_n)_{n \geq 0}$ we have*

$$A_{(2m+1)n} = \sum_{i=0}^{m} (-1)^{n(m+i)} (s^2 + 4)^i \frac{2m+1}{2i+1} \binom{m+i}{2i} A_n^{2i+1}. \tag{4.6}$$

In order to prove Corollary 1 we use the fact that the closed form relation for the terms of sequence $(A_n)_{n \geq 0}$ is

$$A_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where

$$\alpha = \frac{1}{2}(s + \sqrt{s^2 + 4}), \ \ \beta = \frac{1}{2}(s - \sqrt{s^2 + 4}).$$

Furthermore, it holds $\alpha \cdot \beta = -1$, $\alpha - \beta = \sqrt{s^2 + 4}$ which generalize relations (2.7) and (2.8) in the proof of Theorem 1. This completes the statement of Corollary 1.

Further generalizations and extensions of expressions presented in this work are also possible.

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