



SOME REMARKS ON THE BILOCAL PROBLEM

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Abstract. Using a method proposed by T.A. Burton in [6] and Banach’s Contraction Principle, we study the existence, uniqueness and approximation of the solution for a bilocal problem. We compare our results with the classical similarly results given for the Fredholm integral operator associated. The present paper extends and complements the results from [14].

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1. INTRODUCTION

It is well-known (see [1, 3, 9, 20]) that the existence, uniqueness and the approximation of the solution for the bilocal problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) \\ x(a) = x_0, x(b) = x_1 \end{cases}, \quad t \in [a, b] \quad (1.1)$$

can be proved using the Contraction Principle for the Fredholm integral operator

$$F : C^1[a, b] \rightarrow C^1[a, b], \quad (1.2)$$

$$F(x)(t) = \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t, s) f(s, x(s), x'(s)) ds,$$

where

$$G(t, s) = \begin{cases} \frac{(b-t)(s-a)}{b-a}, & \text{if } a \leq s \leq t \\ \frac{(t-a)(b-s)}{b-a}, & \text{if } t \leq s \leq b \end{cases} \quad (1.3)$$

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is the corresponding Green function. Therefore, a function $x \in C^1[a, b]$ is a solution of (1.1) if $x(t)$ is a fixed point of the Fredholm integral operator (1.2) (see [1, 9, 18, 21, 24]).

In the last thirty years, it has been given a particular interest in studying the properties of the solutions for a large class of problems from nonlinear analysis. For example, the study of the periodic solutions of two-point boundary value problems using the method of successive approximations has been made in [19] and [22]. Moreover, in [17], Precup studies the existence, uniqueness and iterative approximation of solutions to a boundary value problem for second-order differential equation on bounded sets in Banach spaces. The tools used here are an extension of Granas' continuation principle for contractive mappings to spaces endowed with two metrics. Other existence results that were obtained for these type of problems can be found in [16]. Then, in [12] the problem of existence of solutions for Fredholm and Volterra integral equations are investigated. Using a monotone method, are given sufficient conditions when integral equations have their solutions in a corresponding sector. Here, some delay differential and integral inequalities are also discussed. Returning to the method of successive approximations, recently, in [5], are presented some convergence and numerical stability properties of the method applied to two-point boundary value problems for nonlinear second order neutral differential equations in Banach spaces. The considered algorithm has a practical stopping criterion and its accuracy is illustrated by examples. Other discussions about the properties of the successive approximation sequence and the numerical methods built using them can be found in [2, 4, 7, 8, 10, 11, 13, 23].

Following a method proposed by T.A. Burton in [6], in this paper we study the bilocal problem using the Contraction Principle for another operator, namely:

$$B : C[a, b] \rightarrow C[a, b],$$

$$B(u)(t) = f \left(t, \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t, s)u(s)ds, \right. \quad (1.4)$$

$$\left. \frac{x_1 - x_0}{b-a} - \int_a^b H(t, s)u(s)ds \right),$$

where

$$H(t, s) = \begin{cases} \frac{a-s}{b-a}, & \text{if } a \leq s \leq t \\ \frac{b-s}{b-a}, & \text{if } t \leq s \leq b \end{cases}. \quad (1.5)$$

More exactly, we construct an approximation scheme for the solution of the bilocal problem (1.1) based on the method of successive approximations. Since the constant obtained for the so-called *Burton operator* defined by (1.4) is less than the constant

of the Fredholm operator, the successive approximations sequence offers a better approximation for the solution of problem (1.1).

Burton's approach was also exemplified in [14] and the main purpose was to approximate the solution of the Cauchy problem and its derivative using two operators: Volterra operator, in the classical case, and Burton operator, after the method proposed in [6]. Then, in [15], the same method was used to give existence and uniqueness results for first order differential equations and systems.

Throughout the paper, the norms on $C^1[a, b]$ and $C[a, b]$ are given by

$$\|x\|^* = \sup_{t \in [a, b]} |x(t)| + \sup_{t \in [a, b]} |x'(t)| \quad (1.6)$$

and

$$\|u\| = \sup_{t \in [a, b]} |u(t)|, \quad (1.7)$$

respectively.

We also recall an important existence and uniqueness result that we shall use in our framework (see [1]). Here, the solution of the bilocal problem is given as the fixed point of the Fredholm operator (1.2), where the considered norm is given by (1.6).

Theorem 1. *If there exist two positive constants L_1 and L_2 such that the function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the Lipschitz condition*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq L_1 |u - \bar{u}| + L_2 |v - \bar{v}|, \quad (1.8)$$

for all $t \in [a, b]$, $u, v, \bar{u}, \bar{v} \in \mathbb{R}$, and

$$\alpha_F = \max\{L_1, L_2\} \left(\frac{(b-a)^2}{8} + \frac{b-a}{2} \right) < 1 \quad (1.9)$$

holds, then:

(i) problem (1.1) has a unique solution x^* , the unique fixed point of the Fredholm operator (1.2);

(ii) x^* can be approximated by the sequence $x_0, x_1 = F(x_0), x_2 = F(x_1), \dots, x_n = F(x_{n-1}), \dots$ and we have

$$\|x_n - x^*\|^* \leq \frac{\alpha_F^n}{1 - \alpha_F} \|x_1 - x_0\|^*, \quad (1.10)$$

for all $n \geq 1$.

The paper is organized as follows: in Section 2 using Burton's method, we obtain an existence and uniqueness results for the solution of the bilocal problem (1.1) and then, in Section 3 we present the main results in approximating this solution with the sequence of successive approximations of the two operators defined by (1.2) and (1.4). As we have already mentioned, since Banach's Contraction Principle offers an

approximation of the fixed point, we compare these approximations in the last section.

2. EXISTENCE AND UNIQUENESS FOR BURTON OPERATOR CASE

In order to obtain the fixed point problem for the Burton operator (1.4), which is equivalent with the considered bilocal problem, we start with

$$x'(t) = \int_a^t x''(s) ds + x'(a),$$

which gives

$$\begin{aligned} x(t) &= \int_a^t x'(s) ds + x(a) \\ &= \int_a^t \left(\int_a^s x''(u) du \right) ds + x'(a)(t-a) + x_0 \\ &= \int_a^t \int_u^t x''(u) ds du + x'(a)(t-a) + x_0 \\ &= \int_a^t (t-s)x''(s) ds + x'(a)(t-a) + x_0. \end{aligned}$$

Since $x_1 = x(b)$ we have that

$$x_1 = \int_a^b (b-s)x''(s) ds + x'(a)(b-a) + x_0,$$

obtaining the formula for $x'(a)$,

$$x'(a) = \frac{x_1 - x_0}{b-a} - \int_a^b \frac{b-s}{b-a} x''(s) ds.$$

Using this formula, we obtain that

$$x'(t) = \frac{x_1 - x_0}{b-a} - \int_a^b H(t,s)x''(s) ds \quad (2.1)$$

and

$$x(t) = \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)x''(s)ds, \quad (2.2)$$

where $G(t,s)$, $H(t,s)$ are given by (1.3) and (1.5), respectively.

Replacing (2.1) and (2.2) in (1.1), the bilocal problem (1.1) is equivalent with

$$x''(t) = f \left(t, \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)x''(s)ds, \right. \\ \left. \frac{x_1-x_0}{b-a} - \int_a^b H(t,s)x''(s)ds \right).$$

At this point, Burton's approach suggests to find solutions of (1.1) by using the fixed points of the operator (1.4). This appears if problem (1.1) is rewritten as

$$x''(t) = f \left(t, \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)x''(s)ds, \right. \\ \left. \frac{x_1-x_0}{b-a} - \int_a^b H(t,s)x''(s)ds \right) \equiv B(x'')(t).$$

In order to obtain the equivalent integral form of the bilocal problem, denote

$$x''(t) = u(t).$$

Hence, the solution of the bilocal problem is the solution of the Burton operator defined by (1.4).

In what follows, we give our main existence and uniqueness result. For this, we consider the problem (1.4) defined on the Banach space $C[a,b]$, with the norm given by (1.7).

Theorem 2. *If function $f : [a,b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the Lipschitz condition (1.8) and*

$$\alpha_B = L_1 \cdot \frac{(b-a)^2}{8} + L_2 \cdot \frac{b-a}{2} < 1 \quad (2.3)$$

holds, then:

(i) *problem (1.1) has a unique solution x^* which can be determined with u^* , the unique fixed point of the Burton operator (1.4);*

(ii) *u^* can be approximated by the sequence $u_0, u_1 = B(u_0), u_2 = B(u_1), \dots, u_n =$*

$B(u_{n-1}), \dots$ and we have

$$\|u_n - u^*\| \leq \frac{\alpha_B^n}{1 - \alpha_B} \|u_1 - u_0\|, \quad (2.4)$$

for all $n \geq 1$.

Proof. (i) For any $u, \bar{u} \in C[a, b]$ we have

$$\begin{aligned} & |B(u)(t) - B(\bar{u})(t)| \\ & \leq L_1 \cdot \int_a^b |G(t, s)| |u(s) - \bar{u}(s)| ds + L_2 \cdot \int_a^b |H(t, s)| |u(s) - \bar{u}(s)| ds \\ & \leq \left(L_1 \cdot \sup \int_a^b |G(t, s)| ds + L_2 \cdot \sup \int_a^b |H(t, s)| ds \right) \|u - \bar{u}\| \\ & \leq \left(L_1 \cdot \frac{(b-a)^2}{8} + L_2 \cdot \frac{b-a}{2} \right) \|u - \bar{u}\|. \end{aligned}$$

Taking the supremum, we have that

$$\|B(u)(t) - B(\bar{u})(t)\| \leq \alpha_B \cdot \|u - \bar{u}\|.$$

Since $\alpha_B < 1$ we have that the operator B is a α_B -contraction and hence, the existence and uniqueness of the fixed point of (1.4). Now, applying Banach's Contraction Principle, we obtain (ii). Moreover, the unique solution of the considered bilocal problem (1.1) is given by

$$x^*(t) = \frac{b-t}{b-a} x_0 + \frac{t-a}{b-a} x_1 - \int_a^b G(t, s) u^*(s) ds,$$

where $G(t, s)$ is the corresponding Green function. \square

3. COMPARISON OF THE APPROXIMATIONS

In the previous sections we have given two existence and uniqueness results for the solution of the bilocal problem (1.1). But, in the classical result given in Theorem 1 we have that the fixed point of the Fredholm operator (1.2) is the solution $x^*(t)$ for the "trajectory" $x(t)$; also, from Theorem 2, the fixed point of the Burton operator (1.4) is the solution $u^*(t)$ for the "acceleration" $x''(t)$. In both cases we have the approximation of the solution with the successive approximations sequences $(F(x_n))_{n \in \mathbb{N}}$ and $(B(u_n))_{n \in \mathbb{N}}$, respectively. In what follows we study in what manner we can approximate $x^*(t)$ by the sequence $(B(u_n))_{n \in \mathbb{N}}$ and also, $u^*(t)$ by the sequence $(F(x_n))_{n \in \mathbb{N}}$.

Proposition 1. *If function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the Lipschitz condition (1.8) and there exist some constants α_F given by (1.9) and α_B given by (2.3) such that $\alpha_B \leq \alpha_F$, then:*

(i) *For any $x_0 \in C^1[a, b]$, the sequence $(F(x_n))_{n \in \mathbb{N}}$ has the property*

$$\|(F(x_n))'' - u^*\| \leq (\alpha_B)^{n-1} \|(F(x_0))'' - u^*\|; \quad (3.1)$$

(ii) *For any $u_0 \in C[a, b]$, the sequence $(B(u_n))_{n \in \mathbb{N}}$ has the property*

$$\|B(u_n) - x^*\|^* \leq (\alpha_F)^{n-1} \|B(u_0) - x^*\|^*. \quad (3.2)$$

Proof. First, we observe that

$$(x^*)'' = u^*,$$

$$x^*(t) = \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)u^*(s)ds,$$

and, since $\alpha_B \leq \alpha_F$, we obtain that $\alpha_B < 1$. Hence, both *Theorems 2* and *1* can be applied.

(i) Consider the sequence $(x_n)_{n \in \mathbb{N}}$ with

$$\begin{aligned} x_n(t) &= F(x_{n-1})(t) = \\ &= \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)f(s, x_{n-1}(s), x'_{n-1}(s))ds, \\ x'_n(t) &= \frac{-1}{b-a}x_0 + \frac{1}{b-a}x_1 - \int_a^b H(t,s)f(s, x_{n-1}(s), x'_{n-1}(s))ds \end{aligned}$$

and

$$x''_n(t) = f(t, x_{n-1}(t), x'_{n-1}(t)).$$

We have that

$$\begin{aligned} &|x''_n(t) - u^*(t)| \\ &= |f(t, x_{n-1}(t), x'_{n-1}(t)) - \\ &\quad - f\left(t, \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)u^*(s)ds, \right. \\ &\quad \left. \frac{-1}{b-a}x_0 + \frac{1}{b-a}x_1 - \int_a^b H(t,s)u^*(s)ds\right) \end{aligned}$$

$$\begin{aligned}
&\leq L_1 \left| x_{n-1}(t) - \frac{b-t}{b-a}x_0 - \frac{t-a}{b-a}x_1 + \int_a^b G(t,s)u^*(s)ds \right| \\
&+ L_2 \left| x'_{n-1}(t) - \frac{x_1-x_0}{b-a} + \int_a^b H(t,s)u^*(s)ds \right| \\
&\leq L_1 \int_a^b |G(t,s)| \cdot |x''_{n-1}(s) - u^*(s)| ds \\
&+ L_2 \int_a^b |H(t,s)| \cdot |x''_{n-1}(s) - u^*(s)| ds \\
&\leq \left(L_1 \cdot \sup \int_a^b |G(t,s)| ds + L_2 \cdot \sup \int_a^b |H(t,s)| ds \right) \|x''_{n-1}(s) - u^*(s)\| \\
&\leq \left(L_1 \cdot \frac{(b-a)^2}{8} + L_2 \cdot \frac{b-a}{2} \right) \|x''_{n-1}(s) - u^*(s)\|.
\end{aligned}$$

From the calculus above, we conclude that the approximation for u^* is given by:

$$\|x''_n - u^*\| \leq (\alpha_B)^{n-1} \|x''_1 - u^*\|.$$

(ii) Considering

$$\begin{aligned}
y_n(t) &= B(u_n)(t) = \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)u_n(s)ds, \\
y'_n(t) &= \frac{x_1-x_0}{b-a} + \int_a^b H(t,s)u_n(s)ds
\end{aligned}$$

and

$$\begin{aligned}
x^*(t) &= \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)f(s, x^*(s), (x^*)'(s))ds, \\
(x^*)'(t) &= \frac{x_1-x_0}{b-a} + \int_a^b H(t,s)f(s, x^*(s), (x^*)'(s))ds,
\end{aligned}$$

respectively, we obtain

$$|y_n(t) - x^*(t)|$$

$$\begin{aligned}
&= \left| \frac{b-t}{b-a}x_0 + \frac{t-a}{b-a}x_1 - \int_a^b G(t,s)u_n(s)ds \right. \\
&\quad \left. - \frac{b-t}{b-a}x_0 - \frac{t-a}{b-a}x_1 + \int_a^b G(t,s)f\left(s, x^*(s), (x^*)'(s)\right)ds \right| \\
&\leq \int_a^b |G(t,s)| \cdot \left| u_n(s) - f\left(s, x^*(s), (x^*)'(s)\right) \right| ds \\
&= \int_a^b |G(t,s)| \cdot \left| f\left(s, \frac{b-s}{b-a}x_0 + \frac{s-a}{b-a}x_1 - \int_a^b G(t,s)u_{n-1}(s)ds, \right. \right. \\
&\quad \left. \left. \frac{x_1-x_0}{b-a} - \int_a^b H(t,s)u_{n-1}(s)ds\right) - f\left(s, x^*(s), (x^*)'(s)\right) \right| ds \\
&\leq \int_a^b |G(t,s)| L_1 \cdot |y_{n-1}(s) - x^*(s)| + L_2 \cdot |y'_{n-1}(s) - (x^*)'(s)| ds \\
&\leq \max\{L_1, L_2\} \cdot \sup \int_a^b |G(t,s)| ds \|y_{n-1} - x^*\|^* \\
&= \max\{L_1, L_2\} \cdot \frac{(b-a)^2}{8} \cdot \|y_{n-1} - x^*\|^*.
\end{aligned}$$

We also have that

$$\begin{aligned}
&\left| y'_n(t) - (x^*)'(t) \right| \\
&= \left| \frac{x_1-x_0}{b-a} - \int_a^b H(t,s)u_n(s)ds - \frac{x_1-x_0}{b-a} + \int_a^b H(t,s)f\left(s, x^*(s), (x^*)'(s)\right)ds \right| \\
&\leq \int_a^b |H(t,s)| \cdot \left| u_n(s) - f\left(s, x^*(s), (x^*)'(s)\right) \right| ds \\
&= \int_a^b |H(t,s)| \cdot \left| f\left(s, y_{n-1}(s), y'_{n-1}(s)\right) - f\left(s, x^*(s), (x^*)'(s)\right) \right| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_a^b |H(t, s)| \left(L_1 \cdot |y_{n-1}(s) - x^*(s)| + L_2 \cdot |y'_{n-1}(s) - (x^*)'(s)| \right) ds \\
&\leq \max\{L_1, L_2\} \cdot \sup \int_a^b |G(t, s)| ds \|y_n - x^*\| \\
&= \max\{L_1, L_2\} \cdot \frac{(b-a)^2}{8} \cdot \|y_n - x^*\|.
\end{aligned}$$

Hence, we conclude that the approximation for x^* is given by

$$\|y_n - x^*\|^* \leq (\alpha_F)^{n-1} \|y_1 - x^*\|^*.$$

□

The novelty of this paper is given by the results obtained in approximating the solutions of problem (1.1) using the two operators defined on different spaces and with different norms. A similar work was done in paper [14], but for the Cauchy problem.

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