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MULTIPLE SOLUTIONS FOR NEUMANN SYSTEMS IN AN ORLICZ-SOBOLEV SPACE SETTING

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Abstract. In this paper, the authors improve some results on the existence of at least three weak solutions for non-homogeneous systems. The proof of the main result relies on a recent variational principle due to Ricceri.

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1. INTRODUCTION

This paper is motivated by phenomena that are described by non-homogeneous Neumann double eigenvalue systems of the type

$$\begin{cases} -\mathrm{div}\big(\alpha_i(|\nabla u_i|)\nabla u_i\big)+\alpha_i(|u_i|)u_i=\lambda F_{u_i}(x,u_1,\dots,u_n) \\ \qquad\qquad +\mu G_{u_i}(x,u_1,\dots,u_n), \qquad \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu}=0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for $1 \le i \le n$, where Ω is a bounded domain in $\mathbb{R}^N$ ($N \ge 3$) with a smooth boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, λ and μ are positive parameters, $F, G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ are measurable functions with respect to $x \in \Omega$ for every $(t_1,\dots,t_n) \in \mathbb{R}^n$ and are C^1 with respect to $(t_1,\dots,t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$. Here, F_{u_i} and G_{u_i} denote the partial derivative of F and G with respect to u_i, respectively, and the functions $\alpha_i(t) : \mathbb{R} \to \mathbb{R}$ for $1 \le i \le n$ will be specified later.

For any function $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$, we denote by (H0) the following assumption on H:

(H0) H is measurable with respect to $x \in \Omega$ for every $(t_1,\dots,t_n) \in \mathbb{R}^n$, and for every $M > 0$ and for every $1 \le i \le n$,

$$\sup_{|(t_1,\dots,t_n)| \le M} |H_{u_i}(x,t_1,\dots,t_n)| \in L^1(\Omega).$$

Our approach in this paper relies on variational methods in Orlicz-Sobolev spaces. Such spaces originated with Nakano [26] and were further developed by Musielak and Orlicz [25]. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces, mainly by Dankert [11], Donaldson and Trudinger [14], and O'Neill [27] (see also [1] for an excellent account of those works). Orlicz-Sobolev spaces have been used in the last two decades to model various phenomena. These kind of spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, nonlinear potential theory, quasiconformal mappings, differential geometry, geometric function theory, and probability theory. Due to this, several authors have widely studied the existence of solutions for eigenvalue problems involving non-homogeneous operators (see, for example, [2, 3, 5–10, 15, 17, 18, 20–22, 33]).

Notice that if $i = 1$ and $\mu = 0$, then problem (1.1) becomes

$$\begin{cases} -\operatorname{div}\big(\alpha(|\nabla u|)\nabla u\big) + \alpha(|u|)u = \lambda f(x,u), \ \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Bonanno, Molica Bisci, and Rădulescu [5] used a recent critical points result in order to prove the existence of an open interval of positive eigenvalues for which the problem (1.2) admits at least three weak solutions. The same authors in [6] established an open interval of positive parameters for which the problem (1.2) admits infinitely many weak solutions that strongly converges to zero.

In [8] Cammaroto and Vilasi proved the existence of three weak solutions to the nonhomogeneous boundary value problem:

$$\begin{cases} -\operatorname{div}\big(\alpha(|\nabla u|)\nabla u\big) = \lambda f(x,u) + \mu g(x,u), \ \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Note that if we take $\alpha(t) = |t|^{p-2}$ $(p > 1)$ in problems (1.2) and (1.3), the operator on the left hand side reduces to the classical p-Laplacian; these types of problems have been widely investigated and a large variety of existence results can be found in the literature.

The classical three critical point theorem of Pucci and Serrin [28, 29] asserts that if $h : Y \to \mathbb{R}$ is of class C^1 (Y is a Banach space), satisfies the Palais-Smale condition, and has two local minima, then h has a third critical point. The general variational principle of Ricceri [31, 32] extends the Pucci-Serrin theorem and provides alternatives for the multiplicity of critical points of certain functionals depending on a parameter. We refer to [19] and the monograph of Graef and Kong [16] for several applications of Ricceri's variational principles.

In this paper, motivated by the above facts and the recent paper of Molica Bisci and Rădulescu [23], under appropriate conditions on F and G, we present sufficient conditions for the system (1.1) to possesses three weak solutions in an Orlicz-Sobolev space.

Our paper is organized as follows. In Section 2, some preliminaries and the abstract Orlicz-Sobolev space settings are presented. In Section 3, we discuss the existence of three weak solutions for the system (1.1). We also point out special cases of the results and we illustrate the results by presenting an example.

2. Functional setting

We now recall the following three critical points theorem that follows from a combination of [4, Theorem 3.6] and [32, Theorem 1].

Theorem 1. *Let X be a reflexive real Banach space, $J : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functional that is bounded on bounded subsets of X and whose Gâteaux derivative admits a continuous inverse on the dual space X^*, and let $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and satisfies*

$$J(0) = I(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$ with $r < J(\bar{x})$ such that

(j) $\sup_{J(x)\leq r} I(x) < rI(\bar{x})/J(\bar{x})$,

(jj) *for each λ in*

$$\Lambda_r := \left(\frac{J(\bar{x})}{I(\bar{x})}, \frac{r}{\sup_{J(x)\leq r} I(x)} \right),$$

the functional $J - \lambda I$ is coercive.

Then, for each compact interval $[a,b] \in \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $\Gamma : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation $J'(x) - \lambda I'(x) - \mu\Gamma'(x) = 0$ has at least three solutions in X whose norms are less than ρ.

In order to study the problem (1.1), let us introduce the function spaces settings to be used. We will give just a brief review of some basic concepts and facts of the theory of Orlicz-Sobolev spaces that will be useful in what follows; for more details we refer the reader to Adams [1], Diening [12], Musielak [24], and Rao and Ren [30].

We begin by recalling some facts from the theory of Orlicz-Sobolev spaces that will be used in the present paper. Suppose that the function $\alpha : \mathbb{R} \to \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for} \quad t \neq 0, \\ 0, & \text{for} \quad t = 0, \end{cases}$$

is an odd and strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Consider the functions

$$\Phi(t) = \int_0^t \varphi(s)ds \quad \text{and} \quad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)ds \quad \text{for} \quad t \in \mathbb{R}.$$

We observe that Φ is a Young's function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty,$$

Φ is called an N–function. The function $\Phi^\star$ is called the complementary function of Φ and satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s) : \ s \geq 0\} \quad \text{for all} \quad t \geq 0.$$

We note that $\Phi^\star$ is also an N–function and the following Young's inequality holds:

$$st \leq \Phi(s) + \Phi^\star(t), \quad \text{for all} \quad s,t \geq 0.$$

Assume that Φ satisfies the structural hypotheses

$$(\Phi 0) \qquad 1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^+ := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty$$

and

$$(\Phi 1) \qquad N < \varphi^- := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}.$$

We also assume that the function

$$(\Phi 2) \qquad t \to \Phi(\sqrt{t}) \quad \text{for} \quad t \in [0,\infty)$$

is convex. To illustrate these notions, we have the following example.

Example 1. Let $p > N + 1$. Define

$$\varphi(t) = \frac{|t|^{p-2}}{\log(1+|t|)} t \quad \text{for} \quad t \neq 0, \quad \varphi(0) = 0,$$

and

$$\Phi(t) = \int_0^t \varphi(s)\, ds.$$

By [9, Example 3, p. 243], we have

$$\varphi^- = p - 1 < \varphi^+ = p = \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions $(\Phi 0)$ and $(\Phi 1)$ are satisfied. Furthermore, by a direct computation, we see that the function $t \mapsto \Phi(\sqrt{t})$ for $t \in [0,\infty)$ is convex, so $(\Phi 2)$ is also satisfied.

The Orlicz space $L_\Phi(\Omega)$ defined by the N–function Φ (see for instance [1] and [13]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{ \left| \int_\Omega u(x)v(x)\, dx \right| : \int_\Omega \Phi^\star(|v(x)|)\, dx \le 1 \right\} < \infty .$$

The space $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_{1,\Phi} := \inf\left\{ k > 0 : \int_\Omega \Phi\left(\frac{u(x)}{k}\right) dx \le 1 \right\} .$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz-Sobolev space for problem (1.1) defined by

$$W^1L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega) : \frac{\partial u}{\partial x_j} \in L_\Phi(\Omega),\ j = 1, ..., N \right\} .$$

This is a Banach space with respect to the norm

$$\|u\|_{2,\Phi} = \||\nabla u|\|_{1,\Phi} + \|u\|_{1,\Phi};$$

see [1, 10, 15].

The following two lemmas can be found in [18].

Lemma 1. ([18, Lemma 2.2]) *In $W^1L_\Phi(\Omega)$, the norms*

$$\|u\|_{2,\Phi} = \||\nabla u|\|_{1,\Phi} + \|u\|_{1,\Phi},$$

$$\|u\|_{3,\Phi} = \max\{\||\nabla u|\|_{1,\Phi}, \|u\|_{1,\Phi}\},$$

and

$$\|u\|_\Phi = \inf\left\{ \mu > 0;\ \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right) \right] dx \le 1 \right\}$$

are equivalent. More precisely, for every $u \in W^1L_\Phi(\Omega)$ we have

$$\|u\|_\Phi \le 2\|u\|_{3,\Phi} \le 2\|u\|_{2,\Phi} \le 4\|u\|_\Phi .$$

Lemma 2. ([18, Lemma 2.3]) *Let $u \in W^1L_\Phi(\Omega)$. Then:*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\, dx \ge \|u\|_\Phi^{\varphi^-}, \quad \text{if } \|u\|_\Phi > 1;$$

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\, dx \ge \|u\|_\Phi^{\varphi^+}, \quad \text{if } \|u\|_\Phi < 1.$$

These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto |t|^p/p$ is assumed by a more general convex function $\Phi(t)$.

Proposition 1. (See [1, p. 241 and p. 247]) *The spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable and reflexive Banach spaces.*

Remark 1. In view of condition ($\Phi 1$), from [10, Lemma D.2] it follows that $W^1L_\Phi(\Omega)$ is continuously embedded in $W^{1,\varphi^-}(\Omega)$. On the other hand, since we assumed that $\varphi^- > N$, we can conclude that $W^{1,\varphi^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, we have that $W^1L_\Phi(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$.

Throughout this paper, the functions $\alpha_i : \mathbb{R} \to \mathbb{R}$ are such that the mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_i(t) = \begin{cases} \alpha_i(|t|)t, & \text{for} \quad t \neq 0, \\ 0, & \text{for} \quad t = 0, \end{cases}$$

for $1 \leq i \leq n$ are odd and strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Also, assume that the functions

$$\Phi_i(t) = \int_0^t \varphi_i(s)\, ds \quad \text{for} \quad t \in \mathbb{R}$$

for $1 \leq i \leq n$ satisfying the structural hypotheses ($\Phi 0$), ($\Phi 1$), and ($\Phi 2$).

We let E be the Cartesian product of n Orlicz-Sobolev spaces $W^1L_{\Phi_i}(\Omega)$ for $1 \leq i \leq n$, i.e., $E = \prod_{i=1}^n W^1L_{\Phi_i}(\Omega)$ endowed with the norm

$$\|u\| := \sum_{i=1}^n \|u\|_{\Phi_i}, \qquad u = (u_1, \ldots, u_n) \in E.$$

In the space $C^0(\overline{\Omega})$, we consider the norm $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$. Set

$$m := \max\left\{ \sup_{u \in W^1L_{\Phi_i}(\Omega)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|_{\Phi_i}} : \text{for}\, 1 \leq i \leq n \right\}. \tag{2.1}$$

Since $N < \varphi_i^-$ for $1 \leq i \leq n$, from Remark 1, the embedding $E \hookrightarrow C^0(\overline{\Omega}) \times \ldots \times C^0(\overline{\Omega})$ is compact, so that $m < +\infty$.

We say that $u = (u_1, \ldots, u_n) \in E$ is a *weak solution* of the problem (1.1) if

$$\int_\Omega \sum_{i=1}^n \big(\alpha_i(|\nabla u_i(x)|)\nabla u_i(x)\nabla v_i(x) + \alpha_i(|u_i(x)|)u_i(x)v_i(x)\big)\, dx$$

$$-\lambda \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)\, dx$$

$$-\mu \int_\Omega \sum_{i=1}^n G_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)\, dx = 0$$

for every $v = (v_1, \ldots, v_n) \in E$.

3. MAIN RESULTS

We begin this section with our main existence result.

Theorem 2. *Assume that $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies* (H0) *and the following conditions:*

(F1) *there exist $d \in L^1(\Omega)$ and n positive constants β_i, with $\beta_i < \varphi_i^-$ for $1 \le i \le n$, such that*

$$0 \le F(x, t_1, \dots, t_n) \le d(x)\left(1 + \sum_{i=1}^{n} |t_i|^{\beta_i}\right)$$

for a.e. $x \in \Omega$ and all $(t_1, \dots, t_n) \in \mathbb{R}^n$;

(F2) *$F(x, 0, \dots, 0) = 0$ for a.e $x \in \Omega$;*

(F3) *there exist $0 < c_i < m$ and $s_i \in \mathbb{R}$ for $1 \le i \le n$ with*

$$|\Omega| \sum_{i=1}^{n} \Phi_i(|s_i|) > \min\left\{ \left(\frac{c_i}{m}\right)^{\varphi_i^+} : 1 \le i \le n \right\},$$

such that

$$\int_\Omega \sup_{|t_1| \le c_1, \dots, |t_n| \le c_n} F(x, t_1, \dots, t_n)\, dx < \frac{\min\left\{ \left(\frac{c_i}{m}\right)^{\varphi_i^+} : 1 \le i \le n \right\}}{|\Omega| \sum_{i=1}^{n} \Phi_i(|s_i|)} \int_\Omega F(x, s_1, \dots, s_n)\, dx,$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω.

Then, setting

$$\Lambda := \left(\frac{|\Omega| \sum_{i=1}^{n} \Phi_i(|s_i|)}{\int_\Omega F(x, s_1, \dots, s_n)\, dx}, \frac{\min\left\{ \left(\frac{c_i}{m}\right)^{\varphi_i^+} : 1 \le i \le n \right\}}{\int_\Omega \sup_{|t_1| \le c_1, \dots, |t_n| \le c_n} F(x, t_1, \dots, t_n)\, dx} \right),$$

for each compact interval $[a, b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and for every function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying condition (H0)*, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system* (1.1) *admits at least three weak solutions in E whose norms are less than ρ.*

Proof. In order to prove the theorem, we will apply Theorem 1. For each $(u_1, \dots, u_n) \in E$, let the functionals $J, I : E \to \mathbb{R}$ be defined by

$$J(u) := \int_\Omega \sum_{i=1}^{n} \left(\Phi_i(|\nabla u_i(x)|) + \Phi_i(|u_i(x)|) \right) dx$$

and

$$I(u) := \int_\Omega F(x, u_1(x) \dots, u_n(x))\, dx.$$

Clearly, J is bounded on each bounded subset of E and similar arguments as those used in [21, Lemma 4.2] imply that J and I are continuously Gâteaux differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_n) \in E$ are the functional $J'(u)$ and $I'(u)$ given by

$$J'(u)(v) = \int_\Omega \sum_{i=1}^n \big(\alpha_i(|\nabla u_i(x)|)\nabla u_i(x)\nabla v_i(x) + \alpha_i(|u_i(x)|)u_i(x)v_i(x)\big)\, dx$$

and

$$I'(u)(v) = \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x) \dots, u_n(x))v_i(x)\, dx$$

for every $v = (v_1, \dots, v_n) \in E$. Moreover, since Φ_i for $1 \le i \le n$ are convex, it follows that J is a convex functional, and hence it is sequentially weakly lower semicontinuous. By Lemma 2, we see that J is coercive, and arguing as in the proof of [18, Lemma 3.2], we have that $J' : E \to E^*$ is a uniformly monotone operator in E. By applying the Minty-Browder theorem (Theorem 26.A of [34]), J' admits a continuous inverse on E^*.

We claim that $I' : E \to E^*$ is a compact operator. To this end, it suffices to show that I' is strongly continuous on E, so for fixed $(u_1, \dots, u_n) \in E$, let $(u_{1m}, \dots, u_{nm}) \to (u_1, \dots, u_n)$ weakly in E as $m \to +\infty$. Then we have $(u_{1m}, \dots, u_{nm})$ converges uniformly to $(u_1, \dots, u_n)$ on Ω as $m \to +\infty$ (see [34]). Since $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$, the partial derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \Omega$, so for $1 \le i \le n$, $F_{u_i}(x, u_{1m}, \dots, u_{nm}) \to F_{u_i}(x, u_1, \dots, u_n)$ strongly as $m \to +\infty$. By the Lebesgue dominated convergence theorem, $I'(u_{1m}, \dots, u_{nm}) \to I'(u_1, \dots, u_n)$ strongly as $m \to +\infty$. Thus, I' is strongly continuous on E, which implies that I' is a compact operator by [34, Proposition 26.2]. Hence, our claim is true.

From the definitions of J and I and from (F2), we have

$$J(0) = I(0) = 0.$$

Next, set $w(x) := (s_1, \dots, s_n)$ for any $x \in \Omega$, and

$$r := \min\left\{ \left(\frac{c_i}{m}\right)^{\varphi_i^+} : 1 \le i \le n \right\}.$$

Clearly, $w \in E$, and we have

$$J(w) = |\Omega| \sum_{i=1}^n \Phi_i(|s_i|) > r > 0.$$

Let $J(u) \le r$ for $u = (u_1, \dots, u_n) \in E$. By Lemma 2, for $1 \le i \le n$, we have

$$\|u\|_{\Phi_i} \le \max\{r^{1/\varphi_i^-}, r^{1/\varphi_i^+}\}.$$

Then, for $1 \le i \le n$,

$$\|u\|_{\Phi_i} \le \frac{c_i}{m}.$$

Taking into account that for each $u \in W^1 L_{\Phi_i}(\Omega)$,

$$\|u\|_\infty \le m \|u\|_{\Phi_i}$$

for $1 \le i \le n$ (see (2.1)), we obtain

$$\|u_i\|_\infty \le c_i \quad \text{for } 1 \le i \le n.$$

Therefore, for every $u = (u_1, \dots, u_n) \in E$,

$$\begin{aligned} \sup_{u \in J^{-1}((-\infty, r))} I(u) &= \int_\Omega F(x, u_1(x), \dots, u_n(x))\, dx \\ &\le \int_\Omega \sup_{|t_1| \le c_1, \dots, |t_n| \le c_n} F(x, t_1, \dots, t_n)\, dx. \end{aligned}$$

Condition (F3) implies

$$\begin{aligned} \frac{\sup_{u \in J^{-1}((-\infty, r))} I(u)}{r} &\le \frac{\int_\Omega \sup_{|t_1| \le c_1, \dots, |t_n| \le c_n} F(x, t_1, \dots, t_n)\, dx}{\min\left\{ \left(\frac{c_i}{m}\right)^{\varphi_i^+} : 1 \le i \le n \right\}} \\ &< \frac{\int_\Omega F(x, s_1, \dots, s_n)\, dx}{|\Omega| \sum_{i=1}^n \Phi_i(|s_i|)} \\ &= \frac{I(w)}{J(w)}. \end{aligned}$$

Thus, assumption (j) of Theorem 1 is satisfied.

In view of (F1), the functional $J - \lambda I$ is coercive for every positive parameter λ, in particular, for every

$$\lambda \in \Lambda \subseteq \left(\frac{J(w)}{I(w)}, \frac{r}{\sup_{J(u) \le r} I(u)} \right),$$

so condition (jj) of Theorem 1 holds. In addition, for every function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying condition (H0), the functional

$$\Gamma(u) := \int_\Omega G(x, u_1(x) \dots, u_n(x))\, dx$$

is well defined and continuously Gâteaux differentiable on E with a compact derivative given by

$$\Gamma'(u)(v) = \int_\Omega \sum_{i=1}^{n} G_{u_i}(x, u_1(x)\dots,u_n(x))v_i(x)\,dx$$

for every $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_n) \in E$. Thus, all the conditions of Theorem 1 are satisfied. Also, note that the solutions of the equation

$$J'(u) - \lambda I'(u) - \mu\Gamma'(u) = 0$$

are exactly the weak solutions of (1.1). So, the conclusion follows from Theorem 1. □

Next, we consider the special case of Theorem 2 where F does not depend on $x \in \Omega$.

Theorem 3. *Assume that $F : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function satisfying*

(F4) *there exist $n+1$ positive constants η and β_i with $\beta_i < \varphi_i^-$ for $1 \le i \le n$, such that*

$$0 \le F(t_1,\dots,t_n) \le \eta\left(1 + \sum_{i=1}^{n} |t_i|^{\beta_i}\right)$$

for all $(t_1,\dots,t_n) \in \mathbb{R}^n$;

(F5) $F(0,\dots,0) = 0$*;*

(F6) *there exist $0 < c_i < m$ and $s_i \in \mathbb{R}$ for $1 \le i \le n$ with*

$$|\Omega| \sum_{i=1}^{n} \Phi_i(|s_i|) > \min\left\{ (\frac{c_i}{m})^{\varphi_i^+} : 1 \le i \le n \right\},$$

such that

$$\sup_{|t_1|\le c_1,\dots,|t_n|\le c_n} F(t_1,\dots,t_n) < \frac{\min\left\{ (\frac{c_i}{m})^{\varphi_i^+} : 1 \le i \le n \right\}}{|\Omega| \sum_{i=1}^{n} \Phi_i(|s_i|)} F(s_1,\dots,s_n).$$

Then, setting

$$\Lambda := \left(\frac{\sum_{i=1}^{n} \Phi_i(|s_i|)}{F(s_1,\dots,s_n)}, \frac{\min\left\{ (\frac{c_i}{m})^{\varphi_i^+} : 1 \le i \le n \right\}}{|\Omega| \sup_{|t_1|\le c_1,\dots,|t_n|\le c_n} F(t_1,\dots,t_n)} \right),$$

for each compact interval $[a,b] \subset \Lambda$, *there exists* $\rho > 0$ *with the following property: for every* $\lambda \in [a,b]$ *and for every function* $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *satisfying condition* (H0), *there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the system*

$$\begin{cases} -\operatorname{div}\big(\alpha_i(|\nabla u_i|)\nabla u_i\big) + \alpha_i(|u_i|)u_i = \lambda F_{u_i}(u_1,\dots,u_n) \\ \qquad\qquad +\mu G_{u_i}(x,u_1,\dots,u_n), \qquad \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for $1 \le i \le n$, *admits at least three weak solutions in* E *whose norms are less than* ρ.

Now, consider the problem

$$\begin{cases} -\operatorname{div}\big(\alpha(|\nabla u|)\nabla u\big) + \alpha(|u|)u = \lambda f(u) + \mu g(x,u) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where λ and μ are positive parameters, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, and α is as in Section 2.

Let

$$k := \sup_{u \in W^1 L_\Phi(\Omega)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|_\Phi}$$

and

$$F(t) := \int_0^t f(\xi)d\xi \quad \text{for} \quad t \in \mathbb{R}. \tag{3.3}$$

The following result is a special case of Theorem 3.

Theorem 4. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a non-negative continuous function, and assume that the function* F *in* (3.3) *satisfies:*

(F7) *there exist two positive constants* η *and* $\beta < \varphi^-$ *such that*

$$F(t) \le \eta\left(1 + |t|^\beta\right) \quad \textit{for} \quad t \in \mathbb{R};$$

(F8) *there exist* $0 < c < k$ *and* $s \in \mathbb{R}$ *with*

$$|\Omega|\Phi(|s|) > \left(\frac{c}{k}\right)^{\varphi^+}$$

such that

$$\sup_{|t| \le c} F(t) < \frac{\left(\frac{c}{k}\right)^{\varphi^+}}{|\Omega|\Phi(|s|)} F(s).$$

Then, setting

$$\Lambda := \left(\frac{\Phi(|s|)}{F(s)}, \frac{\left(\frac{c}{k}\right)^{\varphi^+}}{|\Omega| \sup_{|t|\le c} F(t)} \right),$$

for each compact interval $[a,b] \subset \Lambda$, *there exists* $\rho > 0$ *with the following property: for every* $\lambda \in [a,b]$ *and for every* L^1*-Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$, *there* *exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the system (3.2) admits at least three weak solutions in $W^1L_\Phi(\Omega)$ whose norms are less than ρ.*

As an example of our main result, we state a special case of Theorem 4.

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative (not identically zero) continuous function such that*

$$(\ell_0) \qquad \lim_{t\to 0^+} \frac{f(t)}{t^{\varphi^+-1}} = 0.$$

Assume that

$$F(t) \le \eta\left(1+|t|^\beta\right) \quad for \quad t \in \mathbb{R},$$

and set

$$\Lambda := \left(|\Omega| \inf_{s\in S} \frac{\Phi(s)}{F(s)}, +\infty\right),$$

where

$$S := \{s > 0 : F(s) > 0\}.$$

Then, for each compact interval $[a,b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and for every L^1-Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the system (3.2) admits at least three weak solutions in $W^1L_\Phi(\Omega)$ whose norms are less than ρ.

Proof. Fix $\lambda > |\Omega| \inf_{s\in S} \frac{\Phi(s)}{F(s)}$. Then, there exists $\overline{s}$ such that $F(\overline{s}) > 0$ and

$$\lambda > |\Omega| \frac{\Phi(\overline{s})}{F(\overline{s})}.$$

From condition (ℓ_0), we have

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^{\varphi^+}} = 0.$$

Therefore, we can find a positive constant $\overline{c}$ such that

$$\overline{c} < k \min\left\{1, \left(|\Omega|\Phi(|s|)\right)^{1/\varphi^+}\right\},$$

and

$$\frac{F(\overline{c})}{\overline{c}^{\varphi^+}} < \frac{1}{k^{\varphi^+}} \min\left\{\frac{F(\overline{s})}{|\Omega|\Phi(|s|)}, \frac{1}{\lambda|\Omega|}\right\}.$$

Hence,

$$\lambda \in \left(\frac{\Phi(|s|)}{F(s)}, \frac{\left(\frac{c}{k}\right)^{\varphi^+}}{|\Omega|F(c)}\right).$$

The hypotheses of Theorem 4 are satisfied, so the conclusion follows. □

Next, we give a concrete application of Corollary 1.

Example 2. Let Ω be a non-empty bounded open subset of the Euclidean Space $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, and let Φ be a Young's function that satisfies hypotheses $(\Phi 0)$–$(\Phi 2)$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) := \begin{cases} 0, & \text{if } t < 0, \\ t^{\varphi^+}, & \text{if } 0 \leq t \leq 1, \\ t^s, & \text{if } t > 1, \end{cases}$$

where $s \in (0, \varphi^- - 1)$. Setting

$$\Lambda := \left(\frac{\Phi(|s|)}{F(s)}, \frac{\left(\frac{c}{k}\right)^{\varphi^+}}{|\Omega| \sup_{|t| \leq c} F(t)} \right),$$

from Corollary 1 for each compact interval $[a,b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the system (3.2) admits at least three weak solutions in $W^1 L_\Phi(\Omega)$ whose norms are less than ρ.

In particular, let $\Omega \subset \mathbb{R}^3$ with $|\Omega| = 1$. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) := \begin{cases} 0, & \text{if } t < 0, \\ t^5, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{if } t > 1. \end{cases}$$

Set $\Phi(s) := \displaystyle\int_0^s \frac{t|t|^3}{\log(1+|t|)}\, dt$. Then, for every $s > 0$,

$$\frac{\Phi(s)}{F(s)} := \begin{cases} 6\dfrac{\displaystyle\int_0^s \frac{t|t|^3}{\log(1+|t|)}\, dt}{s^6}, & \text{if } 0 \leq s \leq 1, \\[2ex] 6\dfrac{\displaystyle\int_0^s \frac{t|t|^3}{\log(1+|t|)}\, dt}{2s^3 - 1}, & \text{if } s > 1. \end{cases}$$

Moreover, by direct computations, since the function $\dfrac{\Phi(s)}{F(s)}$ attains its minimum at $s_0 \approx 1.189089126$, it follows that

$$\inf_{s>0} \frac{\Phi(s)}{F(s)} \approx 1.804670144.$$

So, for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the non-homogeneous Neumann problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|\nabla u|^3}{\log(1+|\nabla u|)}\nabla u\Big) + \dfrac{|u|^3}{\log(1+|u|)}u = 2f(u) + \mu g(x,u), \quad in \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad on \quad \partial\Omega, \end{cases}$$

admits at least three weak solutions in $W^1 L_\Phi(\Omega)$.

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