Miskolc Mathematical Notes
Vol. 18 (2017), No. 1, pp. 199–202

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1947

WALKS IN PATH GRAPH ON FOUR VERTICES AND FIBONACCI SEQUENCE

RADIM HOŠEK

Received 05 April, 2016

Abstract. Using elementary knowledge of graph theory, we show that a path graph on four vertices exhibits *Fibonacci structure*. For arbitrary start and end vertices, the number of walks of any length is given by a Fibonacci number.

2010 *Mathematics Subject Classification:* 05C38; 05C50; 11B39

Keywords: path graph, walk in graph, Fibonacci sequence, Fibonacci structure

The Fibonacci sequence is a well known mathematical object, having interesting connections to different features in nature. It is given by a simple recurrence formula.

Definition 1. A sequence $\Phi : \mathbb{N} \to \mathbb{N}$ defined with

$$\Phi(n+1) = \Phi(n) + \Phi(n-1), \quad \forall n \in \mathbb{N}, \tag{0.1}$$

with initial values $\Phi(0) = 0$, $\Phi(1) = 1$ is called the *Fibonacci sequence*.

In this short comment we reveal the link between Fibonacci numbers and number of walks in a path graph on four vertices, i.e. of length three, see graph P_3 in Figure 1. Note that the Fibonacci structure is linked with P_3 and cannot be generalized to P_N.

P_3

FIGURE 1. Path-graph of length three.

To avoid any confusion, we recall the definition of a walk in a graph, see [2, p. 9]. A *walk* of length k in graph G is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$.

We denote with

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{0.2}$$

the *adjacency* matrix of P_3. An elementary assertion of graph theory (see [1, Lemma 2.5]) gives the *number of walks of length k between vertices i and j* as the appropriate element of k-th power of the adjacency matrix B of the graph, i.e.

$$w_{k,i,j} := \{B^k\}_{i,j}. \tag{0.3}$$

The following lemma enables us to compute powers of adjacency matrix of a path graph (regardless the number of vertices).

Lemma 1. *Let $B \in \mathbb{R}^{N\times N}$ be an adjacency matrix of a path graph of length $N-1$. Then for any $k \in \mathbb{N}$ it holds that*

$$\{B^{k+1}\}_{i,j} = \{B^k\}_{i-1,j} + \{B^k\}_{i+1,j}, \tag{0.4}$$

and

$$\{B^{k+1}\}_{i,j} = \{B^k\}_{i,j-1} + \{B^k\}_{i,j+1}, \tag{0.5}$$

where we set $\{B^p\}_{l,m} = 0$ for any $p \in \mathbb{N}$ and $(l,m) \notin \{1,\dots,N\}^2$.

Proof. The claim immediately follows from graph theory interpretation of (0.3): Any walk of length $k+1$ between vertices i and j contains a unique subwalk of length k from either $i-1$ or $i+1$ (if such vertex exists) to j. Hence we can conclude (0.4), similar argument proves also (0.5). □

Note that any path-graph is a bipartite graph, which implies non-existence of walk of odd length between vertices of the same parity (and walk of even length between vertices of different parity), i.e. $\{B^k\}_{i,j} = 0$ if the parity of k and $i+j$ differs. Using this property we define the following split of k-th power of the adjacency matrix of P_3, to matrices of corner, edge and middle elements, as they represent three qualitatively distinct cases.

We set matrices C_k, E_k, M_k for $k \in \mathbb{N}$ with

$$\{C_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i,j) \in \{1,4\}\times\{1,4\} \; and \; (i+j+k) \; even, \\ 0 & otherwise, \end{cases} \tag{0.6}$$

$$\{E_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i,j) \in \big(\{2,3\}\times\{1,4\}\big) \cup \big(\{1,4\}\times\{2,3\}\big) \\ & \; and \; (i+j+k) \; even, \\ 0 & otherwise, \end{cases} \tag{0.7}$$

$$\{M_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i,j) \in \{2,3\}\times\{2,3\} \; and \; (i+j+k) \; even, \\ 0 & otherwise. \end{cases} \tag{0.8}$$

They create a *complete split* of A^k, i.e. $A^k = C_k + E_k + M_k$ and maximally one of them has a non-zero contribution for each element $\{A^k\}_{i,j}$, as it is illustrated in the following scheme.

$$A^k = \begin{pmatrix} 0 & E_k & 0 & C_k \\ E_k & 0 & M_k & 0 \\ 0 & M_k & 0 & E_k \\ C_k & 0 & E_k & 0 \end{pmatrix}, \qquad A^l = \begin{pmatrix} C_l & 0 & E_l & 0 \\ 0 & M_l & 0 & E_l \\ E_l & 0 & M_l & 0 \\ 0 & E_l & 0 & C_l \end{pmatrix},$$

for $k = 2s-1, l = 2s, s \in \mathbb{N}$.

We call elements defined by first lines in (0.6), (0.7) or (0.8) *structural non-zeros*. Note that C_1 is a zero matrix, the only one whose structural non-zeros attain zero value.

Matrix C_k of corner elements represents walks between vertices of the degree 1 (not necessarily distinct), E_k represents the walks connecting vertices with different degrees and finally M_k represents the walks between vertices with degree equal to 2.

We use the previously established split in the next Theorem.

Theorem 1. *For given $k \in \mathbb{N}$, the* structural non-zeros *of matrix C_k (E_k, M_k) are equal to a constant, which we denote by c_k (e_k, m_k). Moreover, it holds that*

$$c_k = \Phi(k-1), \qquad e_k = \Phi(k), \qquad m_k = \Phi(k+1). \tag{0.9}$$

Proof. Applying (0.5) to all structural non-zeros of C_{k+1} we get

$$c_{k+1} = e_k. \tag{0.10}$$

Applying (0.4) and (0.5) to structural non-zeros of E_{k+1} gives two relations

$$e_{k+1} = m_k, \tag{0.11}$$

$$e_{k+1} = c_k + e_k. \tag{0.12}$$

Combining (0.10) and (0.12) one gets

$$c_{k+2} = e_{k+1} = c_k + e_k = c_{k+1} + c_k.$$

From (0.2) we deduce initial values $c_1 = 0$ and, with the help of (0.10), $c_2 = e_1 = 1$. Hence we conclude $c_k = \Phi(k-1)$ for any $k \in \mathbb{N}$. Then (0.10) and (0.11) imply $e_k = \Phi(k)$ and $m_k = \Phi(k+1)$, for $k \in \mathbb{N}$. □

We can conclude from Theorem 1, that any non-zero entry in A^k is a Fibonacci number. In other words, number of k-walks in P_3 from i to j is represented by a Fibonacci number $\Phi(n) = \Phi(n(i,j,k))$, where $n(i,j,k) \in \{k-1, k, k+1\}$ according to (0.6)–(0.8) and Theorem 1.

Remark 1. The link between Fibonacci sequence and walks in path graph was revealed during the research on a generalization of bistable equation with non-smooth minima, see [3] or [4]. When substituting the double-well potential with a *quadruple-well* one, then the nontrivial and qualitatively distinct stationary solutions can be represented by walks in graph P_3, provided additional geometrical constraints are met by the potential to ensure the transitions between minima have the equal length.

ACKNOWLEDGEMENT

The author was supported by the Grant 13-00863S of the Grant Agency of the Czech Republic.

REFERENCES

[1] N. Biggs, *Algebraic graph theory. 2nd ed.*, 2nd ed. Cambridge: Cambridge University Press, 1994. doi: 10.1017/CBO9780511608704.

[2] R. Diestel, *Graph theory. 3rd revised and updated ed.*, 3rd ed. Berlin: Springer, 2005.

[3] P. Drábek and R. Hošek, "Properties of solution diagrams for bistable equations." *Electron. J. Differ. Equ.*, vol. 2015, no. 156, pp. 1–19, 2015.

[4] P. Drábek, R. F. Manásevich, and P. Takáč, "Manifolds of critical points in a quasilinear model for phase transitions." in *Nonlinear elliptic partial differential equations. Workshop in celebration of Jean-Pierre Gossez's 65th birthday, Bruxelles, Belgium, September 2–4, 2009.* Providence, RI: American Mathematical Society (AMS), 2011, pp. 95–134.

Author's address

Radim Hošek

R.H., Department of Mathematics and NTIS, University of West Bohemia, Univerzitní 22, 30614 Plzeň, Czech Republic,

E-mail address: radhost@kma.zcu.cz