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A PEROV TYPE THEOREM FOR CYCLIC CONTRACTIONS AND APPLICATIONS TO SYSTEMS OF INTEGRAL EQUATIONS

ADRIAN MAGDAŞ

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Abstract. In this paper we will prove a fixed point theorem of Perov type for cyclic contractions on complete generalized metric spaces. Then, as an application, we will study the existence, uniqueness and approximation of the solution for a system of integral equations.

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1. Preliminaries

We begin the considerations with some notions and results which will be useful further in this paper.

Let (X,d) be a metric space. We denote:

$$P(X) := \{Y \subset X \mid Y \text{ is nonempty}\};$$

$$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\};$$

If $T : Y \subseteq X \to X$ is a single-valued operator, then the symbol

$$\mathrm{F_T} := \{\mathrm{x} \in \mathrm{Y} \mid \mathrm{x} \in \mathrm{Tx}\}$$

denotes the fixed point set of T.

Definition 1. A matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ is called a matrix convergent to zero if $S^k \to 0$ as $k \to +\infty$.

Theorem 1 ([5], [4])**.** *Let $S \in \mathcal{M}_p(\mathbb{R}_+)$. The following statements are equivalent:*

(i) *S is a matrix convergent to zero;*
(ii) *$S^k x \to 0$ as $k \to +\infty$, $\forall\, x \in \mathbb{R}^p$;*
(iii) *$I_p - S$ is non-singular and*

$$(I_p - S)^{-1} = I_p + S + S^2 + \dots \tag{1.1}$$

(iv) *$I_p - S$ is non-singular and $(I_p - S)^{-1}$ has nonnegative elements;*
(v) *$\lambda \in \mathbb{C}$, $\det(S - \lambda I_p) = 0$ imply $|\lambda| < 1$.*

The matrices convergent to zero were used by A.I. Perov [2] to generalize the contraction principle in the case of metric spaces with a vector-valued distance.

Definition 2 ([5]). Let (X,d) be a metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance and $T : X \to X$. The operator T is called an S-contraction if there exists a matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ such that:

(i) S is a matrix convergent to zero;
(ii) $d(T(x),T(y)) \le Sd(x,y),\ \forall\ x,y \in X$.

Theorem 2 (Perov, [2]). *Let (X,d) be a complete metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance and $T : X \to X$ be an S-contraction. Then:*

(i) *T has a unique fixed point $x^* \in X$;*
(ii) *$T^k x \xrightarrow{d} x^*$ as $k \to +\infty$, for all $x \in X$;*
(iii) *$d(T^k x, x^*) \le S^k (I_p - S)^{-1} d(x,Tx)$, for all $x \in X$ and $k \in \mathbb{N}$;*
(iv) *$d(x,x^*) \le (I_p - S)^{-1} d(x,Tx)$ for all $x \in X$.*

Another consistent generalization of the contraction principle was given by Kirk, Srinivasan and Veeramani, using the concept of cyclic operator.

Theorem 3 ([1]). *Let $\{A_i\}_{i=1}^m$ be nonempty subsets of a complete metric space, and suppose $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ satisfies the following conditions (where $A_{m+1} = A_1$):*

(1) *$TA_i \subseteq A_{i+1}$ for $1 \le i \le m$;*
(2) *$\exists\ k \in (0,1)$ such that $d(Tx,Ty) \le kd(x,y),\ \forall\ x \in A_i,\ y \in A_{i+1}$, for $1 \le i \le m$.*

Then T has a unique fixed point.

This theorem suggested the introduction of the following

Definition 3 ([3]). Let X be a nonempty set, m a positive integer and $T : X \to X$ an operator. By definition, $\bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T if:

(i) $X = \bigcup_{i=1}^m A_i$, with $A_i \in P(X)$, for $1 \le i \le m$;
(ii) $TA_i \subseteq A_{i+1}$, for $1 \le i \le m$, where $A_{m+1} = A_1$.

2. Main results

Definition 4. Let (X,d) be a metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance, $A_1, \dots, A_m \in P_{cl}(X)$ and $T : X \to X$ be an operator. If:

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T;

(ii) there exists a matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ convergent to zero such that

$$d(Tx,Ty) \leq S \cdot d(x,y), \text{ for any } x \in A_i,\ y \in A_{i+1}, \text{ where } A_{m+1} = A_1,$$

then, by definition, we say that T is a cyclic S-contraction.

Theorem 4. *Let (X,d) be a complete metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance, $A_1, A_2, \ldots, A_m \in P_{cl}(X)$. If $T : X \to X$ is a cyclic S-contraction then the following statements hold:*

(1) *T has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $\{x_n\}_{n\geq 0}$ given by*

$$x_n = Tx_{n-1},\ n \geq 1,$$

converges to x^* for any starting point $x_0 \in X$;

(2) *the following estimates hold:*

$$d(x_n,x^*) \leq S^n (I_p - S)^{-1} d(x_0,x_1),\ n \geq 1; \tag{2.1}$$

$$d(x_n,x^*) \leq (I_p - S)^{-1} d(x_n,x_{n+1}),\ n \geq 1; \tag{2.2}$$

(3) *for any $x \in X$,*

$$d(x,x^*) \leq (I_p - S)^{-1} d(x,Tx). \tag{2.3}$$

Proof. (1)

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \leq Sd(x_{n-1},x_n) \\ &\leq \ldots \leq S^n d(x_0,x_1) \end{aligned}$$

For $k \geq 1$ we have

$$\begin{aligned} d(x_n,x_{n+k}) &\leq S^n d(x_0,x_1) + S^{n+1} d(x_0,x_1) + \ldots + S^{n+k-1} d(x_0,x_1) \\ &= S^n (I_p + S + S^2 + \ldots + S^{k-1}) d(x_0,x_1) \\ &\leq S^n (I_p + S + S^2 + \ldots) d(x_0,x_1) \to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.4}$$

which means that $(x_n)_{n\geq 0}$ is a Cauchy sequence.

(X,d) is a complete metric space, so the sequence $(x_n)_{n\geq 0}$ is convergent to a $q \in X$.

The sequence $(x_n)_{n\geq 0}$ has an infinite number of terms in each A_i, $i = \overline{1,m}$, so from each A_i one we can extract a subsequence of $(x_n)_{n\geq 0}$ which converges to $q = \lim_{n\to\infty} x_n$.

Because A_i are closed, $q \in \bigcap_{i=1}^{m} A_i$, so $\bigcap_{i=1}^{m} A_i \neq \varnothing$.

Let be the restriction $T\Big|_{\bigcap\limits_{i=1}^{m} A_i} : \bigcap\limits_{i=1}^{m} A_i \to \bigcap\limits_{i=1}^{m} A_i$.

$\bigcap\limits_{i=1}^{m} A_i$ is also complete. Applying Perov's theorem, $T\Big|_{\bigcap\limits_{i=1}^{m} A_i}$ has a unique fixed point, which can be obtained by means of the Picard iteration starting from any initial point. It remains to prove that the Picard iteration converges to x^*, for any initial guess $x \in X$.

$$\begin{aligned} d(x_{n+1}, x^*) &= d(Tx_n, Tx^*) \le S d(x_n, x^*) \\ &\le \ldots \le S^n d(x_0, x^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

(2) $$\begin{aligned} d(x_n, x_{n+k}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\le d(x_n, x_{n+1}) + S d(x_n, x_{n+1}) + \ldots + S^{k-1} d(x_n, x_{n+1}) \\ &= (I_p + S + \ldots + S^{k-1}) d(x_n, x_{n+1}), \text{ for any } n \in \mathbb{N},\ k \ge 1. \end{aligned} \tag{2.5}$$

Using the statement (iii) from Theorem 1, by letting $k \to \infty$ in (2.4) and (2.5) we obtain the estimates (2.1) and (2.2).

(3) Let $x \in X$. For $n = 0$, $x_0 := x$, the a posteriori estimate (2.2) becomes

$$d(x, x^*) \le (I_p - S)^{-1} d(x, Tx).$$

□

Theorem 5. (Data dependence theorem) *Let $T : X \to X$ be as in Theorem 4 with $F_T = \{x_T^*\}$. Let $U : X \to X$ be an operator such that:*

(i) *U has at least one fixed point x_U^*;*

(ii) *there exists $\eta > 0$ such that*

$$d(Tx, Ux) \le \eta, \textit{ for any } x \in X.$$

Then $d(x_T^*, x_U^*) \le \eta (I_p - S)^{-1}$.

Proof. By letting $x := x_U^*$ in the inequality (2.3), we have

$$\begin{aligned} d(x_U^*, x_T^*) &\le (I_p - S)^{-1} d(x_U^*, T x_U^*) = (I_p - S)^{-1} d(U x_U^*, T x_U^*) \\ &\le (I_p - S)^{-1} \eta. \end{aligned}$$

□

Theorem 6. *Let $T : X \to X$ be as in Theorem 4. Then the fixed point problem for T is well posed, that is, assuming there exist $z_n \in X$, $n \in \mathbb{N}$ such that $d(z_n, Tz_n) \to 0$, as $n \to \infty$, this implies that $z_n \to x^*$, as $n \to \infty$, where $F_T = \{x^*\}$.*

Proof. By letting $x := z_n$ in the inequality (2.3), we have

$$d(z_n, x^*) \leq (I_p - S)^{-1} d(z_n, Tz_n),\ n \in \mathbb{N}$$

and letting $n \to \infty$ we obtain $d(z_n, x^*) \to 0$, $n \to \infty$. □

3. An application to a system of integral equations

We apply the results given by Theorem 2.1 to study the existence and the uniqueness of solutions of the following system of integral equations:

$$\begin{cases} x_1(t) = \int_a^b G_1(t,s) f_1(s, x_1(s), x_2(s)) ds \\ x_2(t) = \int_a^b G_2(t,s) f_2(s, x_1(s), x_2(s)) ds \end{cases},\ t \in [a,b] \tag{3.1}$$

where $a, b \in \mathbb{R}$, $a < b$,

$$G_1, G_2 \in C([a,b] \times [a,b], [0,\infty)),$$

$$f_1, f_2 \in C([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}).$$

Theorem 7. *We suppose that:*

(i) *there exist $\alpha_k, \beta_k \in C([a,b], \mathbb{R})$, $m_k, M_k \in \mathbb{R}$ with $m_k \leq \alpha_k(t) \leq \beta_k(t) \leq M_k$, for any $t \in [a,b]$, such that*

$$\begin{cases} \alpha_k(t) \leq \int_a^b G_k(t,s) f_k(s, \beta_1(s), \beta_2(s)) ds \\ \beta_k(t) \geq \int_a^b G_k(t,s) f_k(s, \alpha_1(s), \alpha_2(s)) ds \end{cases} \quad \text{for } k \in \{1,2\} \tag{3.2}$$

(ii) *there exist $a_1, b_1, a_2, b_2 \in \mathbb{R}_+$ such that*

$$\begin{aligned} |f_1(s,u_1,u_2) - f_1(s,v_1,v_2)| &\leq a_1|u_1 - v_1| + a_2|u_2 - v_2|, \\ |f_2(s,u_1,u_2) - f_2(s,v_1,v_2)| &\leq b_1|u_1 - v_1| + b_2|u_2 - v_2|, \end{aligned} \tag{3.3}$$

for any $s \in [a,b]$ and $u_k, v_k \in \mathbb{R}$, with

$$\begin{cases} u_k \leq M_k \\ v_k \geq m_k \end{cases} \quad or \quad \begin{cases} u_k \geq m_k \\ v_k \leq M_k \end{cases} \quad \text{for } k \in \{1,2\};$$

(iii) $\sup_{t \in [a,b]} \int_a^b G_k(t,s) ds \leq 1$ *for $k \in \{1,2\}$;*

(iv) *f_k is decreasing in each of the last two variables, that is,*

$$u_1, u_2, v_1, v_2 \in \mathbb{R},\ u_1 \leq v_1,\ u_2 \leq v_2 \Rightarrow f_k(s,u,v) \geq f_k(s,u_2,v_2),$$

for any $s \in [a,b]$, and $k \in \{1,2\}$;

(v) *the matrix* $S = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ *converges to zero.*

Then the system (3.1) has a unique solution $x^* = (x_1^*, x_2^*) \in C([a,b], \mathbb{R}^2)$*, with* $\alpha_k \le x_k^* \le \beta_k$*, for* $k \in \{1,2\}$*.*
This solution can be obtained by the successive approximations method, starting at any element $x^0 \in C([a,b], \mathbb{R}^2)$*. Moreover, if* x^n *is the* n^{th} *successive approximation, then we have the following estimation:*

$$\|x^* - x^n\| \le S^n (I_2 - S)^{-1} \|x^0 - x^1\|,$$

where

$$\|x\| = \begin{pmatrix} |x_1|_\infty \\ |x_2|_\infty \end{pmatrix} \quad \text{and} \quad |x|_\infty = \max_{t\in[a,b]} |x(t)|.$$

Proof. Let us denote

$$X := (C([a,b], \mathbb{R}), |\cdot|_\infty), \; Z = X \times X \tag{3.4}$$

$$\|\cdot\| : Z \to \mathbb{R}^2, \; \|x\| = \|(x_1, x_2)\| = \begin{pmatrix} |x_1|_\infty \\ |x_2|_\infty \end{pmatrix},$$

where $|x_k|_\infty = \max\limits_{t\in[a,b]} |x_k(t)|$ is the Chebyshev norm.

Then $(Z, \|\cdot\|)$ is a generalized Banach space.

We consider the following closed subsets of X:

$$A_1 = \{(x_1, x_2) \in Z \mid x_k \le \beta_k, \; k \in \{1,2\}\},$$

$$A_2 = \{(x_1, x_2) \in Z \mid x_k \ge \alpha_k, \; k \in \{1,2\}\},$$

and the operator $T : Z \to Z$,

$$(x_1, x_2) = x \mapsto Tx = (T_1 x, T_2 x),$$

$$T_k x(t) := \int_a^b G_k(t,s) f_k(s, x_1(s), x_2(s)) ds, \; \text{for } k \in \{1,2\}. \tag{3.5}$$

The system (3.1) is equivalent with the equation $Tx = x$.
We will prove that $A_1 \cup A_2$ is a cyclic representation of Z with respect to T.
Let $x = (x_1, x_2) \in A_1 \Rightarrow x_k(s) \le \beta_k(s), \; \forall\, s \in [a,b]$, for $k \in \{1,2\}$.
Using the monotonicity of f_k we have

$$G_k(t,s) f_k(s, x_1(s), x_2(s)) \ge G_k(t,s) f_k(s, \beta_1(s), \beta_2(s)), \; \text{for } k \in \{1,2\}$$

and from (i), by integration,

$$\int_a^b G_k(t,s) f_k(s, x_1(s), x_2(s)) ds \ge \alpha_k(t),$$

which means that

$$T_k x(t) \ge \alpha_k(t), \; \forall\, t \in [a,b], \; \text{for } k \in \{1,2\} \Rightarrow Tx \in A_2.$$

So $TA_1 \subseteq A_2$. In a similar way we have $TA_2 \subseteq A_1$.

Using the conditions (ii) and (iii) we have

$$\begin{aligned}
|T_k x(t) - T_k y(t)| &\leq \int_a^b G_k(t,s)|f_k(s,x_1(s),x_2(s)) - f_k(s,y_1(s),y_2(s))|ds \\
&\leq \int_a^b G_k(t,s)(a_k|x_1(s)-y_1(s)| + b_k|x_2(s)-y_2(s)|)ds \\
&\leq \int_a^b G_k(t,s)(a_k|x_1-y_1|_\infty + b_k|x_2-y_2|_\infty) \\
&\leq a_k|x_1-y_1|_\infty + b_k|x_2-y_2|_\infty,\ \forall\ t \in [a,b] \\
&\Rightarrow |T_k x - T_k y|_\infty \leq a_k|x_1-y_1|_\infty + b_k|x_2-y_2|_\infty \\
&\Rightarrow \begin{pmatrix} |T_1 x - T_1 y|_\infty \\ |T_2 x - T_2 y|_\infty \end{pmatrix} \leq S \begin{pmatrix} |x_1-y_1|_\infty \\ |x_2-y_2|_\infty \end{pmatrix},
\end{aligned}$$

so we have

$$\|Tx - Ty\| \leq S\|x-y\|, \text{ for any } (x,y) \in A_1 \times A_2,$$

and by the condition (v) it results that the operator T is a cyclic S-contraction.

All the conditions of Theorem 4 are satisfied, so T has a unique fixed point

$$x^* = (x_1^*, x_2^*) \in A_1 \cap A_2, \text{ with } \alpha_k \leq x_k^* \leq \beta_k, \text{ for } k \in \{1,2\}.$$

This finishes the proof.

□

Further on, we will study the continuous dependence phenomenon for the system (3.1).

We consider the perturbed system of integral equations

$$\begin{cases} y_1(t) = \displaystyle\int_a^b H_1(t,s)g_1(s,y_1(s),y_2(s))ds \\ y_2(t) = \displaystyle\int_a^b H_2(t,s)g_2(s,y_1(s),y_2(s))ds \end{cases} \tag{3.6}$$

where

$$H_1, H_2 \in C([a,b]\times[a,b],[0,\infty)), \quad g_1, g_2 \in C([a,b]\times\mathbb{R}\times\mathbb{R},\mathbb{R}).$$

Theorem 8. *We suppose that the conditions of Theorem 7 are satisfied and we denote by x^* the unique solution of the system of integral equations (3.1).*

If $y^* \in C([a,b],\mathbb{R}^2)$ is a solution of the perturbed system of integral equations (3.6), and

$$\sup_{t\in[a,b]} \int_a^b H_k(t,s)ds \leq 1,$$

then we have the following estimation:

$$\|x^* - y^*\|_{\mathbb{R}^2} \le (I_2 - S)^{-1}(\eta + \tau), \tag{3.7}$$

where $\eta = (\eta_1, \eta_2)$, $\tau = (\tau_1, \tau_2)$ *and*

$$\begin{cases} \eta_k = \sup\{|f_k(s,u,v)| \mid s \in [a,b],\ u,v \in \mathbb{R}\}, \\ \tau_k = \sup\{|g_k(s,u,v)| \mid s \in [a,b],\ u,v \in \mathbb{R}\}, \end{cases} \textit{for } k \in \{1,2\}.$$

Proof. We consider the operator $T : Z \to Z$ attached to the system (3.1), defined by the relation (3.5).

Let $U : Z \to Z$ be an operator attached to the perturbed system (3.6) and defined by the relation:

$$(y_1, y_2) = y \mapsto Uy = (U_1 y, U_2 y),$$

$$U_k y(t) := \int_a^b H_k(t,s) g_k(s, y_1(s), y_2(s)) ds, \text{ for } k \in \{1,2\}.$$

We have

$$\begin{aligned} |T_k x(t) - U_k x(t)| &\le \int_a^b G_k(t,s) |f_k(s, x_1(s), x_2(s)) ds \\ &\quad + \int_a^b H_k(t,s) |g_k(s, x_1(s), x_2(s))| ds \\ &\le \eta_k \int_a^b G_k(t,s) ds + \tau_k \int_a^b H_k(t,s) ds \\ &\le \eta_k + \tau_k,\ \forall\, t \in [a,b], \text{ for } k \in \{1,2\} \\ &\Rightarrow |T_k x - U_k x|_\infty \le \eta_k + \tau_k \\ &\Rightarrow \|Tx - Ux\| \le \eta + \tau,\ \forall\, x \in Z. \end{aligned}$$

The conditions of Theorem 6 are satisfied, so estimation (3.7) is proved.

□

Remark 1. A similar approach can be achieved for a system of Volterra type integral equations using, instead of the supremum norm, the Bielecki type norm approach.

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Author's address

Adrian Magdaş

Babeş-Bolyai University, Faculty of Mathematics and Computer Science, Kogălniceanu Street, No. 1, 400084 Cluj-Napoca, Romania

E-mail address: amagdas@yahoo.com