



q -ANALOGUE OF MITTAG-LEFFLER OPERATORS

GÜRHAN İÇÖZ AND BAYRAM ÇEKİM

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Abstract. In this paper, we study q -extension of Mittag-Leffler operators. We establish moments of these operators and estimate convergence results with the help of classical modulus of continuity. Also, we give weighted approximation property and A -statistically convergence of the operators $L_{n,q}^{(\beta)}$.

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1. INTRODUCTION

We first mention some notations of q -calculus. Throughout the present paper, assume that q is a real number satisfying the inequality $0 < q \leq 1$. For $n \in \mathbb{N}$, we have

$$[n]_q = [n] := \begin{cases} (1 - q^n) / (1 - q) & , \quad q \neq 1 \\ n & , \quad q = 1 \end{cases} ,$$

$$[n]_q! = [n]! := \begin{cases} [n][n-1] \dots [1] & , \quad n \geq 1 \\ 1 & , \quad n = 0 \end{cases} ,$$

and

$$(1+x)_q^n := \begin{cases} \prod_{j=0}^{n-1} (1+q^j x) & , \quad n = 1, 2, \dots \\ 1 & , \quad n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} .$$

The q -analogue of integration, introduced by Jackson [7] in the interval $[0, a]$, is defined by

$$\int_0^a f(x) d_q x := a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad 0 < q < 1 \text{ and } a > 0.$$

The q -improper integral used in the present paper is defined as

$$\int_0^{\infty/A} f(x) d_q x := (1-q) \sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0,$$

provided the sum converges absolutely. The two q -Gamma functions are defined as

$$\Gamma_q(x) = \int_0^{1/1-q} t^{x-1} E_q(-qt) d_q t \quad \text{and} \quad \gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1} e_q(-t) d_q t. \quad (1.1)$$

There are two q -analogues of the exponential function e^x , see [8],

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]!} = (1 + (1-q)x)_q^{\infty}, \quad |q| < 1.$$

By Jackson [7], it was shown that the q -Beta function defined in the usual formula

$$B_q(t, s) = \frac{\Gamma_q(s) \Gamma_q(t)}{\Gamma_q(s+t)}.$$

The q -integral representation of q -Beta function, which is a q -analogue of Euler's formula, is

$$B_q(t, s) = \int_0^1 x^{t-1} (1-qx)_q^{s-1} d_q x, \quad t, s > 0. \quad (1.2)$$

In 1903, G.M. Mittag-Leffler [11] defined the Mittag-Leffler function by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}; \quad (z \in \mathbb{C}, \quad R(\alpha) > 0).$$

In 1905, A. Wiman [15] gave the two-index Mittag-Leffler function definition by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}; \quad (z, \beta \in \mathbb{C}, \quad R(\alpha) > 0).$$

Note that, for $\beta = 1$ then we get $E_{\alpha, 1}(z) = E_{\alpha}(z)$.

M.A. Özarslan [12] investigated properties of the following Mittag-Leffler operators

$$L_n^{(\beta)}(f; x) = \frac{1}{E_{1, \beta}\left(\frac{nx}{b_n}\right)} \sum_{k=0}^{\infty} f\left(\frac{kb_n}{n}\right) \frac{(nx)^k}{b_n^k \Gamma(k + \beta)}, \quad (1.3)$$

M.H. Annaby and Z.S. Mansour [1] defined q -analogues of the Mittag-Leffler functions by

$$e_{\alpha,\beta}(z; q) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k + \beta)}, \quad |z(1-q)^\alpha| < 1,$$

and

$$E_{\alpha,\beta}(z; q) = \sum_{k=0}^{\infty} \frac{q^{\alpha \frac{k(k-1)}{2}} z^k}{\Gamma_q(\alpha k + \beta)}, \quad z \in \mathbb{C}$$

where $\beta \in \mathbb{C}$ and $R(\alpha) > 0$. Here we will use q -Mittag-Leffler function for $\alpha = 1$ as

$$E_{1,\beta} \left(\frac{nz}{b_n}; q \right) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nz}{b_n} \right)^k.$$

Now we introduce the following operators

$$L_{n,q}^{(\beta)}(f; x) = \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n}; q \right)} \sum_{k=0}^{\infty} f \left(\frac{[k] b_n}{n} \right) \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n} \right)^k \tag{1.4}$$

where (b_n) is a sequence of positive real numbers, $\beta > 1$ is fixed, $n \in \mathbb{N}$, and

$$f \in E := \left\{ f \in C[0, \infty) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ is finite} \right\},$$

where $C[0, \infty)$ is the space of continuous functions defined on $[0, \infty)$. The norm of Banach lattice E is given by

$$\|f\|_* := \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}.$$

It is obvious that the operators $L_{n,q}^{(\beta)}(f; x)$ are linear and positive.

When $q = 1$, we have the Mittag-Leffler operators given by Özarslan [12]. When $q = 1$ and $\beta = 1$, we get the modified Szász-Mirakjan operators.

Lemma 1. For each $0 < q < 1$, $x \geq 0$ and $n \in \mathbb{N}$, we have

$$L_{n,q}^{(\beta)}(1; x) = 1, \tag{1.5}$$

$$\left| L_{n,q}^{(\beta)}(t; x) - x \right| \leq \frac{[\beta - 1] b_n}{n}, \tag{1.6}$$

$$\left| L_{n,q}^{(\beta)}(t^2; x) - x^2 \right| \leq \frac{b_n}{n} \left(x + \frac{[\beta - 1] b_n}{n} \right) + 2x \frac{b_n [\beta - 1]}{n}, \tag{1.7}$$

and

$$L_{n,q}^{(\beta)}((t-x)^2; x) \leq \frac{b_n}{n} (1 + 4[\beta - 1])x + \left(\frac{b_n}{n} \right)^2 [\beta - 1]. \tag{1.8}$$

Proof. From definition of the operators $L_{n,q}^{(\beta)}$ given by (1.4), we get

$$L_{n,q}^{(\beta)}(1; x) = 1.$$

Using $[k] = [k + \beta - 1] - q^k[\beta - 1]$ and from (1.4), we have

$$\begin{aligned} L_{n,q}^{(\beta)}(t; x) &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{[k] b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\ &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=1}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta - 1)} \left(\frac{nx}{b_n}\right)^k \\ &\quad - \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}} q^k}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\ &\leq \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k+1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^{k+1} \\ &\quad + \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}} q^k}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k. \end{aligned}$$

For $q^k \leq 1$, we get

$$\begin{aligned} L_{n,q}^{(\beta)}(t; x) &\leq \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^{k+1} \\ &\quad + \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\ &\leq x + \frac{[\beta - 1]b_n}{n}. \end{aligned}$$

So we can write

$$\left| L_{n,q}^{(\beta)}(t; x) - x \right| \leq \frac{[\beta - 1]b_n}{n}.$$

Using definition of the operators $L_{n,q}^{(\beta)}$, we have

$$L_{n,q}^{(\beta)}(t^2; x) = \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \left(\frac{[k] b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k.$$

Using the fact that $[k] = q[k - 1] + 1$ and $[k] = [k + \beta - 1] - q^k[\beta - 1]$, we obtain

$$\begin{aligned}
 L_{n,q}^{(\beta)}(t^2; x) &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \left(\frac{b_n}{n}\right)^2 (q[k - 1][k] + [k]) \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k - 1][k] \frac{q^{\frac{k(k-1)}{2} + 1}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad + \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \left(\frac{b_n}{n}\right)^2 [k] \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k - 1] ([k + \beta - 1] - q^k[\beta - 1]) \frac{q^{\frac{k(k-1)}{2} + 1}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad + \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k - 1] \frac{q^{\frac{k(k-1)}{2} + 1}}{\Gamma_q(k + \beta - 1)} \left(\frac{nx}{b_n}\right)^k + \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) \\
 &\quad - \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=1}^{\infty} \left(\frac{b_n}{n}\right)^2 [k - 1] \frac{q^{\frac{k(k-1)}{2} + 1 + k}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 ([k + \beta - 2] - q^{k-1}[\beta - 1]) \frac{q^{\frac{k(k-1)}{2} + 1}}{\Gamma_q(k + \beta - 1)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad - \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k - 1] \frac{q^{\frac{k(k-1)}{2} + 1 + k}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k + \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) \\
 &= \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) + \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2} + 1}}{\Gamma_q(k + \beta - 2)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad - \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2} + k}}{\Gamma_q(k + \beta - 1)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad - \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=1}^{\infty} \left(\frac{b_n}{n}\right)^2 [k - 1] \frac{q^{\frac{k(k-1)}{2} + 1 + k}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) + \frac{x^2}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} q^{3k+3}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k \\ &\quad + \frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2}+k}}{\Gamma_q(k+\beta-1)} \left(1 + \frac{q[k-1]}{[k+\beta-1]}\right) \left(\frac{nx}{b_n}\right)^k. \end{aligned}$$

For $q^k \leq 1$ and $\frac{[k-1]}{[k+\beta-1]} \leq 1$, we have

$$L_{n,q}^{(\beta)}(t^2; x) \leq x^2 + \frac{b_n}{n} \left(x + \frac{[\beta-1]b_n}{n}\right) + 2x \frac{b_n[\beta-1]}{n}. \quad (1.9)$$

So we can write

$$\left|L_{n,q}^{(\beta)}(t^2; x) - x^2\right| \leq \frac{b_n}{n} \left(x + \frac{[\beta-1]b_n}{n}\right) + 2x \frac{b_n[\beta-1]}{n}.$$

Finally, using the (1.6) and (1.7), we acquire that

$$\begin{aligned} L_{n,q}^{(\beta)}((t-x)^2; x) &= L_{n,q}^{(\beta)}(t^2; x) - 2xL_{n,q}^{(\beta)}(t; x) + x^2L_{n,q}^{(\beta)}(1; x) \\ &\leq \left|L_{n,q}^{(\beta)}(t^2; x) - x^2\right| + 2x \left|L_{n,q}^{(\beta)}(t; x) - x\right| \\ &\leq \frac{b_n}{n} \left(x + \frac{[\beta-1]b_n}{n}\right) + 2x \frac{b_n[\beta-1]}{n} + \frac{2[\beta-1]b_n}{n} x \\ &= \frac{b_n}{n} (1 + 4[\beta-1])x + \left(\frac{b_n}{n}\right)^2 [\beta-1]. \end{aligned}$$

□

2. WEIGHTED APPROXIMATION

Here, we give weighted approximation theorem for the operator $L_{n,q}^{(\beta)}$.

Theorem 1. Let $q = q_n \in (0, 1)$ such that $q_n \rightarrow 1$ as $n \rightarrow \infty$, then for each $f \in E$ and $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$, we get

$$\lim_{n \rightarrow \infty} \left\|L_{n,q_n}^{(\beta)}(f; x) - f(x)\right\|_* = 0$$

where $\beta > 1$.

Proof. From (1.5), we know that $L_{n,q}^{(\beta)}(1; x) = 1$, so

$$\left\|L_{n,q_n}^{(\beta)}(1; x) - 1\right\|_* = 0.$$

From (1.6), for $n > 1$, we get

$$\left\|L_{n,q_n}^{(\beta)}(t; x) - x\right\|_* = \sup_{x \in [0, \infty)} \frac{\left|L_{n,q_n}^{(\beta)}(t; x) - x\right|}{1 + x^2}$$

$$\begin{aligned} &\leq \frac{[\beta - 1]b_n}{n} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \\ &\leq \frac{[\beta - 1]b_n}{n}. \end{aligned}$$

So we can write

$$\lim_{n \rightarrow \infty} \left\| L_{n, q_n}^{(\beta)}(t; x) - x \right\|_* = 0.$$

Finally, for (1.7), we get

$$\begin{aligned} \left\| L_{n, q_n}^{(\beta)}(t^2; x) - x^2 \right\|_* &= \sup_{x \in [0, \infty)} \frac{\left| L_{n, q_n}^{(\beta)}(t^2; x) - x^2 \right|}{1 + x^2} \\ &\leq (1 + 2[\beta - 1]) \frac{b_n}{n} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} \\ &\quad + \left(\frac{b_n}{n} \right)^2 [\beta - 1] \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \\ &\leq (1 + 2[\beta - 1]) \frac{b_n}{2n} + \left(\frac{b_n}{n} \right)^2 [\beta - 1]. \end{aligned}$$

Then, we acquire that

$$\left\| L_{n, q_n}^{(\beta)}(t^2; x) - x^2 \right\|_* \rightarrow 0 \text{ as } n \rightarrow \infty.$$

□

3. RATE OF CONVERGENCE

Lemma 2. For all $x \in [0, \infty)$ and $n \in \mathbb{N}$, we get

$$\omega(x) L_{n, q}^{(\beta)} \left(\frac{1}{\omega}; x \right) \leq \mu(\beta)$$

where $\left(\frac{b_n}{n} \right)$ is a bounded sequence of positive numbers and $\beta > 1$ is fixed. Also $\mu(\beta)$ is a constant and $\omega(x) = \frac{1}{1+x^2}$. Additionally for all $f \in E$ and $q \in (0, 1)$, we can obtain

$$\left\| L_{n, q}^{(\beta)}(f) \right\|_* \leq \mu(\beta) \|f\|_*.$$

Proof. Using (1.5) and (1.9), we get

$$\begin{aligned} \omega(x) L_{n, q}^{(\beta)} \left(\frac{1}{\omega}; x \right) &= \frac{1}{1 + x^2} \left[L_{n, q}^{(\beta)}(1; x) + L_{n, q}^{(\beta)}(t^2; x) \right] \\ &\leq \frac{1}{1 + x^2} \left(1 + x^2 + \frac{b_n}{n} (1 + 2[\beta - 1])x + \left(\frac{b_n}{n} \right)^2 [\beta - 1] \right) \end{aligned}$$

$$\begin{aligned} &\leq 1 + \frac{b_n}{n} (1 + 2[\beta - 1]) + \left(\frac{b_n}{n}\right)^2 [\beta - 1] \\ &= \mu(\beta). \end{aligned}$$

Also we have

$$\begin{aligned} \omega(x) \left| L_{n,q}^{(\beta)}(f; x) \right| &= \omega(x) \left| L_{n,q}^{(\beta)}\left(\omega \frac{f}{\omega}; x\right) \right| \\ &\leq \|f\|_* \omega(x) L_{n,q}^{(\beta)}\left(\frac{1}{\omega}; x\right) \\ &\leq \mu(\beta) \|f\|_*. \end{aligned}$$

If we take supremum both sides of above inequality over $x \in [0, \infty)$, then the proof is done. \square

Now, we acquire that the rate of convergence of the operators $L_{n,q}^{(\beta)} f$ to f with the help of usual modulus of convergence.

Firstly, we remember that the modulus of continuity of f on the closed interval $[0, D]$ is given by

$$\omega_D(f, \delta) = \sup_{\substack{x, t \in [0, D] \\ |t-x| \leq \delta}} |f(t) - f(x)|.$$

We always know that, for a function f belongs to E , there is a limit equation as $\lim_{\delta \rightarrow \infty} \omega_D(f, \delta) = 0$.

Theorem 2. Assume that $\left(\frac{b_n}{n}\right)$ is a bounded sequence of positive numbers with $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$ and $\beta > 1$. For each $f \in E$, $q \in (0, 1)$ and $x \in [0, D] \subset [0, \infty)$; ($D > 0$),

$$\left\| L_{n,q}^{(\beta)}(f; x) - f(x) \right\|_{C[0, D]} \leq M_f(\beta, D) \epsilon_n^2(\beta, D) + 2\omega_{D+1}(f, \epsilon_n(\beta, D))$$

where

$$\epsilon_n(\beta, D) = \frac{b_n}{n} (1 + 4[\beta - 1]) D + \left(\frac{b_n}{n}\right)^2 [\beta - 1]$$

and $M_f(\beta, D)$ is an absolute constant.

Proof. From the hypothesis, $x \in [0, D]$ and $t \leq D + 1$, we get

$$|f(t) - f(x)| \leq \omega_{D+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{D+1}(f, \delta)$$

where $\delta > 0$. Also we can write

$$\begin{aligned} |f(t) - f(x)| &\leq K_f(1 + x^2 + t^2) \leq K_f(2 + 3x^2 + 2(t-x)^2) \\ &\leq 6K_f(1 + D^2)(t-x)^2 \end{aligned}$$

for $x \in [0, D]$ and $t > D + 1$, getting $t - x > 1$ and K_f is a constant depending on f . From the above, we have

$$|f(t) - f(x)| \leq 6K_f(1 + D^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{D+1}(f, \delta)$$

for $x \in [0, D]$ and $t \geq 0$. By the well-known Cauchy-Schwarz inequality and from (1.8), we have

$$\begin{aligned} |L_{n,q}^{(\beta)}(f; x) - f(x)| &\leq 6K_f(1 + D^2)L_{n,q}^{(\beta)}((t - x)^2; x) \\ &\quad + \omega_{D+1}(f, \delta) \left(1 + \frac{1}{\delta} \left(L_{n,q}^{(\beta)}((t - x)^2; x)\right)^{1/2}\right) \\ &\leq 6K_f(1 + D^2) \left[\frac{b_n}{n} (1 + 4[\beta - 1])x + \left(\frac{b_n}{n}\right)^2 [\beta - 1]\right] \\ &\quad + \omega_{D+1}(f, \delta) \left[1 + \frac{1}{\delta} \left(\frac{b_n}{n} (1 + 4[\beta - 1])x + \left(\frac{b_n}{n}\right)^2 [\beta - 1]\right)^{\frac{1}{2}}\right] \\ &\leq M_f(\beta, D)\epsilon_n^2(\beta, D) + 2\omega_{D+1}(f, \epsilon_n(\beta, D)), \end{aligned}$$

where

$$\epsilon_n(\beta, D) = \left(\frac{b_n}{n} (1 + 4[\beta - 1])D + \left(\frac{b_n}{n}\right)^2 [\beta - 1]\right)^{1/2}$$

and $M_f(\beta, D) = 6(1 + D^2)K_f$. So, the proof is done. \square

4. A -STATISTICAL CONVERGENCE

Recently, some authors use A -statistical convergence by [2, 3, 12, 13]. Here for a non-negative regular summability matrix $A = (a_{jk})$, the definition of A -density of a subset K of \mathbb{N} is as

$$\delta_A(K) = \lim_j \sum_{k \in K} a_{j,k},$$

where limit exists (see [5]). If a sequence $x = (x_n)$ convergent to l meaning A -statistically converges symbolized by $st_A - \lim x = l$ where every $\varepsilon > 0$, $\delta_A \{n \in \mathbb{N} : |x - l| \geq \varepsilon\} = 0$ (see [4, 14]).

Substituting $A = C_1$ which C_1 is the Cesaro matrix of order one, then we obtain the statistical convergence instead of A -statistically convergence [6, 10]. Choosing $A = I$, which is the identity matrix, then we get the ordinary convergence instead of A -statistically convergence. In the case of $\lim_j \max_n |a_{j,n}| = 0$, Kolk [9] showed that A -statistical convergence is stronger than ordinary convergence.

Assuming that

$$st_A - \lim_n \frac{b_n}{n} = 0,$$

then one can easily see that

$$st_A - \lim_n \frac{b_n^2}{n^2} = 0.$$

Taking $A = C_1$ and defining

$$b_n := \begin{cases} n; & n = m^2 \ (m \in \mathbb{N}) \\ 1; & \text{otherwise} \end{cases},$$

so we can easily obtain that $st_A - \lim_n \frac{b_n}{n} = st_A - \lim_n \frac{b_n^2}{n^2} = 0$.

Theorem 3. Assume that $A = (a_{jk})$ is a non-negative regular summability matrix, $q \in (0, 1)$ and $\beta > 1$. We have

$$st_A - \left\| L_{n,q}^{(\beta)}(f; x) - f(x) \right\|_{C[0,D]} = 0,$$

if $st_A - \lim_n \frac{b_n}{n} = 0$ and each $f \in E$.

Proof. We choose $\varepsilon > 0$ satisfying $\varepsilon < r$ where $r > 0$. Defining the following sets as

$$\begin{aligned} T &:= \{n : \epsilon_n(\beta, D) \geq r\}, \\ T_1 &:= \left\{ n : \frac{b_n}{n} (1 + 4[\beta - 1]) D \geq \frac{r - \varepsilon}{2} \right\}, \\ T_2 &:= \left\{ n : \left(\frac{b_n}{n} \right)^2 [\beta - 1] \geq \frac{r - \varepsilon}{2} \right\}. \end{aligned}$$

Then one can see $T \subset T_1 \cup T_2$. Thus we can write

$$\sum_{k \in T} a_{jk} \leq \sum_{k \in T_1} a_{jk} + \sum_{k \in T_2} a_{jk}.$$

We get $\lim_j \sum_{k \in T} a_{jk} = 0$ for $j \rightarrow \infty$ in the above and $st_A - \lim_n \frac{b_n}{n} = 0$. Hence we have that $st_A - \lim_n \epsilon_n(\beta, D) = 0$ and this showed that

$$st_A - \lim_n \omega_{D+1}(f, \epsilon_n(\beta, D)) = 0.$$

From the Theorem 2, the proof is done. \square

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Authors’ addresses

Gürhan İçöz

Gazi University, Faculty of Science, Department of Mathematics, Ankara, Turkey

E-mail address: gurhanicoz@gazi.edu.tr

Bayram Çekim

Gazi University, Faculty of Science, Department of Mathematics, Ankara, Turkey

E-mail address: bayramcekim@gazi.edu.tr