ON THE POINTWISE ENTANGLED ERGODIC THEOREM

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Dedicated to our advisor Rainer Nagel on the occasion of his 75^{th} birthday

Abstract. We present some twisted compactness conditions for almost everywhere convergence of one-parameter entangled ergodic averages of Dunford-Schwartz operators $T_0, \dots, T_a$ on a Borel probability space of the form

$$\frac{1}{N}\sum_{n=1}^{N} T_a^n A_{a-1} T_{a-1}^n A_{a-1} \cdot \dots \cdot A_0 T_0^n f$$

for $f \in L^p(X,\mu)$, $p \geq 1$. We also discuss examples and present a continuous version of the result.

1. Introduction

For the proof of a central limit theorem for certain models in quantum probability, Accardi, Hashimoto, Obata [1] introduced the study of entangled ergodic averages. These were studied further by Liebscher [22], Fidaleo [13, 14, 15], and the authors [11]. We refer to [11] for more information and the connection to noncommutative multiple ergodic theorems.

The setting of the entangled ergodic theorems is the following. Let $k \leq m$ be positive integers and $\alpha : \{1, \dots, m\} \to \{1, \dots, k\}$ be a surjective map. Let further E be a Banach space, $T_1, \dots, T_m$ and $A_1, \dots, A_{m-1}$ be bounded operators on E. As shown in [11], the entangled ergodic averages

$$\frac{1}{N^k}\sum_{n_1,\dots,n_k=1}^{N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_1 T_1^{n_{\alpha(1)}}$$

converge in norm under quite weak compactness conditions on the operators T_j and the pairs (A_j, T_j).

In our knowledge, pointwise convergence of the entangled ergodic averages for $E := L^p(X,\mu)$, where (X,μ) is a probability space and $p \geq 1$, and for Koopman or Dunford-Schwartz operators $T_1, \dots, T_m$ has not yet been investigated. The aim of this paper is to close this gap partially and to present sufficient conditions in the spirit of those in [11] for the case $k = 1$. The general case remains open. In what follows, we shall denote by $\mathbb{N}$ the set of positive integers.

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Our main result is the following. (Recall that a Borel probability space is a compact metrizable space with a Borel probability measure, see e.g. Einsiedler, Ward [7, Def. 5.13]. For the Jacobs-deLeeuw-Glicksberg decomposition and basics on Dunford-Schwartz operators see Section 2.)

Theorem 1. *For $a \in \mathbb{N}$, let $T_0, T_1, \dots T_a$ be Dunford-Schwartz operators on a Borel probability space (X, μ) with $\mathrm{Fix}\,|T_1| = \dots = \mathrm{Fix}\,|T_a| = \langle \mathbf{1} \rangle$. For $p \in [1, \infty)$ and $E := L^p(X, \mu)$, let $E = E_{0,r} \oplus E_{0,s}$ be the Jacobs-deLeeuw-Glicksberg decomposition corresponding to T_0, and let further $A_j \in \mathcal{L}(E)$ $(0 \le j < a)$ be bounded operators. For a function $f \in E$ and an index $0 \le j < a$, write $\mathscr{A}_{j,f} := \left\{ A_j T_j^n f \mid n \in \mathbb{N} \right\}$. Suppose that the following conditions hold:*

(A1) (Twisted compactness) *For every $f \in E$, $0 \le j < a$ and $\varepsilon > 0$, there exists a decomposition (depending on f, j and ε) $E = \mathcal{U} \oplus \mathcal{R}$ with $\dim \mathcal{U} < \infty$ such that*

$$P_{\mathcal{R}} \mathscr{A}_{j,f} \subset B_\varepsilon(0, L^\infty(X, \mu)),$$

where $P_{\mathcal{R}}$ denotes the projection onto $\mathcal{R}$ along $\mathcal{U}$.

(A2) (Joint L^∞-boundedness) *There exists a constant $C > 0$ such that*

$$\{A_j T_j^n | n \in \mathbb{N}, 1 \le j < a\} \subset B_C(0, \mathcal{L}(L^\infty(X, \mu)).$$

Then we have the following:

(1) *for each $f \in E_{0,s}$, $\frac{1}{N} \sum_{n=1}^N |T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n A_0 T_0^n f| \to 0$ pointwise a.e.;*

(2) *if $p = 2$, then for each $f \in E_{0,r}$, $\frac{1}{N} \sum_{n=1}^N T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n A_0 T_0^n f$ converges pointwise a.e. to*

$$\sum_{\substack{\lambda_j \in \sigma_j\ (1 \le j \le a) \\ \lambda_1 \cdot \ldots \cdot \lambda_a = 1}} P_{\lambda_a}^{(a)} A_{a-1} P_{\lambda_{a-1}}^{(a-1)} A_{a-2} \dots A_1 P_{\lambda_1}^{(1)} f, \tag{1}$$

where $\sigma_j = P_\sigma(T_j) \cap \mathbb{T}$ and $P_{\lambda_j}^{(j)}$ is the projection onto the eigenspace of T_j corresponding to λ_j, i.e., the mean ergodic projection of the operator $\overline{\lambda_j} T_j$.

Note that the above conditions are stronger than the conditions in [11] for norm convergence. (In particular, the total mean ergodicity assumption on T_a follows from the discussion in Section 2). Since the pointwise limit coincides with the norm limit, the above representation of the limit in Theorem 1 is the same as in [11, Theorem 3].

Note further that a sufficient condition for (A2) is that every A_j is bounded as an operator on $L^\infty(X, \mu)$.

An interesting question not studied in this paper is to find analogues of the above result for non-commutative multiple ergodic averages. While norm convergence results can just be translated into corresponding results for convergence of non-commutative multiple ergodic averages in the strong sense, see, e.g., [11, Section 4], the situation with pointwise convergence is more delicate. Several different analogues of pointwise convergence in the non-commutative case are provided by Egorov's theorem (see e.g. Junge, Xu [16], Lance [21], Yeadon [26] for non-commutative Birkhoff's theorem), but the use of the uniform topology combined with projections makes a direct connection to our setting difficult.

The paper is organized as follows. After showing the main ideas in a simpler case in Section 3 and presenting the proof of Theorem 1 in Section 4, we discuss some examples and the continuous case in Section 5.

2. Notations and tools

We denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$. We further denote by $\mathcal{N}$ the set of all bounded sequences $\{a_n\} \subset \mathbb{C}$ with the property

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |a_n| = 0.$$

By the Koopman-von Neumann lemma, see e.g. Petersen [24, p. 65], $(a_n) \in \mathcal{N}$ if and only if it is bounded and converges to 0 along a sequence of density 1.

Let E be a Banach space and let $T \in \mathcal{L}(E)$ be *weakly almost periodic*, i.e., such that for every $f \in E$ the set $\{T^n f, n \in \mathbb{N}\}$ is relatively weakly compact in E. We will use the following version of the Jacobs-deLeeuw-Glicksberg decomposition, see [8, Theorem II.4.8] or [10, Section 16.3]:

$$E = E_r \oplus E_s,$$

where

$$\begin{aligned} E_r &:= \overline{\mathrm{lin}}\{f \in E : \; Tf = \lambda f \text{ for some } \lambda \in \mathbb{T}\}, \\ E_s &:= \{f \in E : \; (\varphi(T^n f)) \in \mathcal{N} \text{ for every } \varphi \in E'\}. \end{aligned}$$

Here, E_r is called the *reversible* subspace and E_s the *stable* subspace. Note that Jacobs, deLeeuw, Glicksberg and some other authors use(d) the terminology "flight vectors" for elements of E_s. Our preference of the name "(almost weakly) stable vectors" is justified by the fact that the orbit of such a vector converges to 0 weakly along a subsequence of density 1, see, e.g., [10, Section 16.4].

Note that every power bounded operator on a reflexive Banach space has relatively weakly compact orbits and hence the above decomposition is valid for e.g. every contraction on $L^p(X,\mu)$ for $p \in (1,\infty)$. Moreover, if T is a *Dunford-Schwartz operator* on $L^1(X,\mu)$, i.e., a contraction in L^1 which is also a contraction in L^∞, then T has relatively weakly compact orbits as well, see Lin, Olsen, Tempelman [23, Prop. 2.6] and Kornfeld, Lin [18, pp. 226–227]. Note that every Dunford-Schwartz operator is also a contraction on $L^p(X,\mu)$ for every $p \in (1,\infty)$, see, e.g., [10, Theorem 8.23]. Thus, the Jacobs-deLeeuw-Glicksberg decomposition is valid for Dunford-Schwartz operators on $L^p(X,\mu)$ for every $p \in [1,\infty)$.

Let T be a Dunford-Schwartz operator on (X,μ) (we will write so since T is a contraction on every $L^p(X,\mu)$, $p \geq 1$). The *(linear) modulus* $|T|$ of T is the unique positive operator on $L^1(X,\mu)$ having the same L^1- and L^∞-norm as T such that $|T^n f| \leq |T|^n |f|$ holds a.e. for every $f \in L^1(X,\mu)$ and every $n \in \mathbb{N}$. It is again a Dunford-Schwartz operator. For details, see Dunford, Schwartz [6, p. 672] and Krengel [19, pp. 159–160]. Note that for T Dunford-Schwartz, the operators λT for $\lambda \in \mathbb{T}$ are again Dunford-Schwartz and have the same modulus.

For example, every Koopman operator (i.e., the operator induced by a μ-preserving transformation on X) is a positive Dunford-Schwartz operator, hence coincides with its modulus, and thus ergodic Koopman operators satisfy the condition $\mathrm{Fix}\,|T| = \langle \mathbf{1} \rangle$ appearing in Theorem 1. See e.g. [10] and [24] for more information on Koopman operators and an introduction to ergodic theory.

An important property of Dunford-Schwartz operators which we will need is the validity of the pointwise ergodic theorem, i.e., for every $f \in L^1(X,\mu)$ the ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} T^n f \tag{2}$$

converge a.e. as $N \to \infty$, see Dunford, Schwartz [6, p. 675].

Remark 1. *Let T be a mean ergodic contraction on $L^1(X,\mu)$ with $\mathrm{Fix}\,T = \langle \mathbf{1} \rangle$, and let $f \in L^1(X,\mu)$. Then the L^1-limit of (2) equals $c \cdot \mathbf{1}$, where c is a constant satisfying $|c| \le \|f\|_1$. Indeed,*

$$c \;=\; \|c\mathbf{1}\|_1 = \lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^n f \right\|_1 \le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \|T^n f\|_1 \le \|f\|_1.$$

In particular, if T is a Dunford-Schwartz operator with $\mathrm{Fix}\,T = \langle \mathbf{1} \rangle$, then the pointwise limit of (2) equals $c \cdot \mathbf{1}$ with $|c| \le \|f\|_1$.

We finally denote by $\mathscr{P} \subset \ell^\infty$ the set of Bohr almost periodic sequences, i.e., uniform limits of finite linear combinations of sequences of the form (λ^n), $\lambda \in \mathbb{T}$. The set $\mathscr{P}$ has the following properties: It is closed in ℓ^∞, closed under multiplication, and is a subclass of (Weyl) almost periodic sequences $AP(\mathbb{N})$, i.e., sequences whose orbit under the left shift is relatively compact in l^∞. In fact, $AP(\mathbb{N}) = \mathscr{P} \oplus c_0$ holds, see Bellow, Losert [3, p. 316], corresponding to the Jacobs-deLeeuw-Glicksberg decomposition of $AP(\mathbb{N})$ induced by the left shift, see, e.g., [8, Theorem I.1.20].

Every element $(a_n)_{n=1}^\infty$ of $AP(\mathbb{N})$, and hence of $\mathscr{P}$, is a good weight for the pointwise ergodic theorem for Dunford-Schwartz operators, i.e., for every Dunford-Schwartz operator T on a probability space and every $f \in L^1(X,\mu)$, the weighted ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} a_n T^n f$$

converge almost everywhere as $N \to \infty$, see Çömez, Lin, Olsen [5, Theorem 2.5]. (Note that also every element of $\mathscr{N}$ is such a good weight, which is clear for bounded functions and follows from the Banach principle for L^1-functions. We will however not use it in this paper.)

For more information and the first part of the following example see, e.g., Lin, Olsen, Tempelman [23] and Eisner [9].

Example 1. (1) *If T has relatively weakly compact orbits on a Banach space E, $f \in E_r$ and $\varphi \in E'$, then $(\varphi(T^n f)) \in \mathscr{P}$.*

(2) *Let* $(q_k)_{k\in\mathbb{N}} \in \ell^1$ *and* $(\gamma_k) \subset \mathbb{T}$. *Define* $(a_n)_{n\in\mathbb{N}} \subset \ell^\infty$ *by*

$$a_n = \sum_{k=1}^{\infty} \gamma_k^n \cdot q_k \quad \forall n \in \mathbb{N}.$$

Then $(a_n) \in \mathscr{P}$.

3. A model case

Before presenting the proof of the general case, we first explain its ideas on a simpler model where $a = 1$, $p = 2$ and the decompositions in (A1) are orthogonal.

Theorem 2. *Let* (X, μ) *be a Borel probability space,* T_0 *be a Dunford-Schwartz operator on* (X, μ), $H := L^2(X, \mu)$ *and let* $H = H_r \oplus H_s$ *be the corresponding Jacobs-deLeeuw-Glicksberg decomposition induced by* T_0. *Let further* $A_0 \in \mathcal{L}(H)$ *be a bounded operator. For a function* $f \in H$, *write* $\mathscr{A}_f := \{A_0 T_0^n f \mid n \in \mathbb{N}\}$. *Suppose that the following holds true:*

For any function $f \in H$ *and* $\varepsilon > 0$, *there exists a finite dimensional subspace* $\mathcal{U} = \mathcal{U}(f, \varepsilon) \subset H$ *such that* $P_{\mathcal{U}^\perp} \mathscr{A}_f \subset B_\epsilon(0, L^\infty(X, \mu))$.

Then for any further T_1 *on* (X, μ) *with* $\operatorname{Fix} |T_1| = \langle \mathbf{1} \rangle$ *we have the following:*

(1) *for each* $f \in H_s$, $\frac{1}{N} \sum_{n=1}^{N} |T_1^n A_0 T_0^n f| \to 0$ *pointwise a.e.;*
(2) *for each* $f \in H_r$, $\frac{1}{N} \sum_{n=1}^{N} T_1^n A_0 T_0^n f$ *converges pointwise a.e..*

Proof. Let $f \in H$ and $\epsilon > 0$ be given. By assumption we have a finite-dimensional subspace $\mathcal{U} = \mathcal{U}(f, \varepsilon) \subset H$ such that $P_{\mathcal{U}^\perp} \mathscr{A}_f \subset B_\varepsilon(0, L^\infty(X, \mu))$. Let $g_1, \ldots, g_k$ be an orthonormal basis in $\mathcal{U}$. Then we may for each $n \in \mathbb{N}$ write

$$A_0 T_0^n f = \lambda_{1,n} g_1 + \ldots + \lambda_{k,n} g_k + r_n \tag{3}$$

for appropriate $\lambda_{j,n} \in \mathbb{C}$ and $r_n \in \mathcal{U}^\perp$ with $\|r_n\|_\infty < \varepsilon$. Note that

$$\lambda_{j,n} = \langle A_0 T_0^n f, g_j \rangle = \langle T_0^n f, A_0^* g_j \rangle,$$

and so $|\lambda_{j,n}| \le \|f\|_2 \cdot \|A_0^*\| =: c$.

For part (1), assume that $f \in H_s$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\lambda_{j,n}| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle T_0^n f, A_0^* g_j \rangle| = 0 \tag{4}$$

by the definition of H_s. For $\delta := \varepsilon / ck$ and for each $1 \le j \le k$ choose a function $\widetilde{g}_j \in L^\infty(X, \mu)$ such that $\|g_j - \widetilde{g}_j\|_1 < \delta$. By Birkhoff's theorem applied to the functions $g_j - \widetilde{g}_j$ and the operator $|T_1|$, see Section 2 and in particular Remark 1, there exists a set $S_\varepsilon \subset X$ with $\mu(S_\varepsilon) = 1$ such that for every $x \in S_\varepsilon$ and every $j \in \{1, \ldots, k\}$ the following conditions hold:

- $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (|T_1|^n |g_j - \widetilde{g}_j|)(x) \le \|g_j - \widetilde{g}_j\|_1$,
- $|T_1^n r_n(x)| \le \|r_n\|_\infty$ and $|T_1^n \widetilde{g}_j(x)| \le \|\widetilde{g}_j\|_\infty$ for every $n \in \mathbb{N}$.

In particular, we have the following inequalities for each $1 \le j \le k$ and $x \in S_\varepsilon$

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} |\lambda_{j,n} T_1^n (g_j - \widetilde{g}_j)| (x) \le \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} c \left(|T_1|^n |g_j - \widetilde{g}_j|\right)(x) \le c\delta. \tag{5}$$

Consequently, using that T_1 is a Koopman operator and hence preserves the $\|\cdot\|_\infty$-norm, we have for each $x \in S_\varepsilon$ using (4)

$$\begin{aligned}
&\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|(T_1^n A_0 T_0^n f)(x)\right| = \overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|\left(T_1^n r_n + \sum_{j=1}^{k}\lambda_{j,n}T_1^n g_j\right)(x)\right| \\
\leq\ & \overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|(T_1^n r_n)\right|(x) + \sum_{j=1}^{k}\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|\lambda_{j,n}T_1^n g_j\right|(x) \\
\leq\ & \overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\|T_1^n r_n\|_\infty + \sum_{j=1}^{k}\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|\lambda_{j,n}T_1^n g_j\right|(x) \\
\leq\ & \varepsilon + \sum_{j=1}^{k}\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|\lambda_{j,n}T_1^n(g_j-\widetilde{g}_j)\right|(x) + \sum_{j=1}^{k}\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|\lambda_{j,n}T_1^n\widetilde{g}_j\right|(x) \\
\leq\ & \varepsilon + kc\delta + \sum_{j=1}^{k}\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\left|\lambda_{j,n}T_1^n\widetilde{g}_j\right|(x) \\
\leq\ & 2\varepsilon + \sum_{j=1}^{k}\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}|\lambda_{j,n}|\,\|T_1^n\widetilde{g}_j\|_\infty \leq 2\varepsilon + \sum_{j=1}^{k}\|\widetilde{g}_j\|_\infty\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}|\lambda_{j,n}| = 2\varepsilon.
\end{aligned}$$

Thus for each $x \in \bigcap_{m\in\mathbb{N}} S_{1/m} =: S$ we have that

$$\left(\frac{1}{N}\sum_{n=1}^{N}\left|T_1^n A_0 T_0^n f\right|\right)(x) \to 0.$$

Since $\mu(S) = 1$, we are done.

For part (2), note that eigenfunctions in H_r pertaining to different unimodular eigenvalues are always orthogonal. Take $f \in H_r$ and let $\{h_j\}_{j=1}^\infty$ be an orthonormal basis in H_r of eigenvectors pertaining to unimodular eigenvalues $\{\alpha_j\}_{j=1}^\infty$. (Note that the space H and hence H_r is separable, and we write here an infinite sequence for notational convenience whereas the finite dimensional case can be treated analogously.) Then we can write $f = \sum_{m=1}^\infty d_m h_m$ for some ℓ^2-sequence $(d_m)_m$ and obtain by the definition of $\lambda_{j,n}$'s in equality (3)

$$\lambda_{j,n} = \langle T_0^n f, A_0^* g_j\rangle = \Big\langle \sum_{m=1}^{\infty}\alpha_m^n d_m h_m, A_0^* g_j\Big\rangle = \sum_{m=1}^{\infty}\alpha_m^n\left(d_m\langle h_m, A_0^* g_j\rangle\right).$$

By the Cauchy-Schwarz and Bessel inequalities, $(d_m\langle h_m, A_0^* g_j\rangle)_{m=1}^\infty \in l^1$ with the l^1-norm bounded by $\|f\|_2\|A_0^* g_j\|_2$. So for each $1 \leq j \leq k$, we have $(\lambda_{j,n})_n \in \mathscr{P}$, so this sequence is a good weight for the pointwise ergodic theorem for Dunford-Schwartz operators, see Example 1(2). In other words, there exists a set $S_\varepsilon \subset X$ with $\mu(S_\varepsilon) = 1$ such that for each $1 \leq j \leq k$ and all $x \in S_\varepsilon$, the Cesàro means

$$\frac{1}{N}\sum_{n=1}^{N}\lambda_{j,n}\left(T_1^n g_j\right)(x)$$

converge. But $\frac{1}{N}\sum_{n=1}^{N}\|T_1^n r_n\|_\infty \le \varepsilon$, and so for each $x \in S_\varepsilon$ we have by (3) that

$$\begin{aligned}
&\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_1^n A_0 T_0^n f\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_1^n A_0 T_0^n f\right)(x)\right| \\
&\le \sum_{j=1}^{k}\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_1^n \lambda_{j,n} g_j\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_1^n \lambda_{j,n} g_j\right)(x)\right| \\
&\quad + \left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_1^n r_n\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_1^n r_n\right)(x)\right| \\
&\le \sum_{j=1}^{k}\left|\overline{\lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda_{j,n}\left(T_1^n g_j\right)(x) - \underline{\lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda_{j,n}\left(T_1^n g_j\right)(x)\right| \\
&\quad + \left|\overline{\lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\|T_1^n r_n\|_\infty\right| + \left|\underline{\lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\|T_1^n r_n\|_\infty\right| \\
&\le 0 + \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Thus for each $x \in \bigcap_{m\in\mathbb{N}} S_{1/m} =: S$ the limit

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_1^n A_0 T_0^n f\right)(x),$$

exists. Since $\mu(S) = 1$, the proof is complete. □

4. Proof of Theorem 1

Theorem 2, up to the orthogonality assumption, provides the proof for the simplest case $a = 1$ (and $p = 2$).

We shall proceed by iterated splitting. To avoid cumbersome notations, however, we shall only provide all details for the case $a = 2$, and sketch how the ideas carry over to the general case. We start with facts concerning the general case; the assumption $a = 2$ will be introduced later on.

Take $f \in E$ and $\epsilon > 0$. Then by assumption (A1) we have a finite-dimensional subspace $\mathcal{U} = \mathcal{U}(f, \varepsilon/C^{a-1}) \subset E$ and a decomposition $E = \mathcal{U} \oplus \mathcal{R}$ such that

$$P_{\mathcal{R}}\mathscr{A}_{0,f} \subset B_{\varepsilon/C^{a-1}}(0, L^\infty(X,\mu)).$$

Let $g_1, \dots, g_k$ be a maximal linearly independent set in $\mathcal{U}$. Then we may for each $n \in \mathbb{N}$ write

$$A_0 T_0^n f = \lambda_{1,n} g_1 + \ldots + \lambda_{k,n} g_k + r_n$$

for appropriate $\lambda_{j,n} \in \mathbb{C}$ and $r_n \in \mathcal{R}$ with $\|r_n\|_\infty < \varepsilon/C^{a-1}$. By the Hahn-Banach theorem we may consider linear forms $\varphi_1, \dots \varphi_k \in E'$ such that

$$\varphi_j(g_i) = \delta_{i,j} \quad \text{and} \quad \varphi_j|_{\mathcal{R}} = 0 \quad \text{for every } i, j \in \{1, \dots, k\}.$$

We then have

$$\lambda_{j,n} = \varphi_j(A_0 T_0^n f) = (A_0^* \varphi_j)(T_0^n f), \tag{6}$$

therefore

$$|\lambda_{j,n}| \le \|f\|_p \cdot \|A_0^*\|_q \max_{j\in\{1,\dots,k\}} \|\varphi_j\|_q =: c \tag{7}$$

for the dual index q. Note that c depends on ε.

Now we have that

$$\begin{aligned} & T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n A_0 T_0^n f \\ = \quad & T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n r_n + \sum_{j=1}^k T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n \lambda_{j,n} g_j, \end{aligned}$$

and we shall investigate the Cesàro convergence of each term separately.

The first term satisfies, by (A2), the inequality

$$\frac{1}{N}\sum_{n=1}^N |T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n r_n|(x) \le C^{a-1}\|r_n\|_\infty < \varepsilon$$

for almost every $x \in X$.

For part (1), assume that $f \in E_{0,s}$. Then (6) and (7) imply $(\lambda_{j,n})_{n\in\mathbb{N}} \in \mathscr{N}$ for each $1 \le j \le k$. Fix $1 \le j \le k$ and consider the term

$$\frac{1}{N}\sum_{n=1}^N |T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n \lambda_{j,n} g_j|.$$

As in the proof of Theorem 2, we may choose a function $\widetilde{g}_j \in L^\infty$ such that $\|g_j - \widetilde{g}_j\|_1 \le \|g_j - \widetilde{g}_j\|_p < \varepsilon/ck$. Then

$$\begin{aligned} & \frac{1}{N}\sum_{n=1}^N |T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n \lambda_{j,n} g_j| \\ \le \quad & \frac{1}{N}\sum_{n=1}^N |T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n \lambda_{j,n} (g_j - \widetilde{g}_j)| + \frac{1}{N}\sum_{n=1}^N |T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n \lambda_{j,n} \widetilde{g}_j|. \end{aligned}$$

Since $(\lambda_{j,n})_{n\in\mathbb{N}} \in \mathscr{N}$, the second term satisfies by (A2)

$$\frac{1}{N}\sum_{n=1}^N |T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n \lambda_{j,n} \widetilde{g}_j|(x) \le C^{a-1}\|\widetilde{g}_j\|_\infty \cdot \frac{1}{N}\sum_{n=1}^N |\lambda_{j,n}| \to 0$$

for almost every $x \in X$.

It now remains to treat the first term. Again using our assumption (A1), there exists a finite dimensional subspace $\mathcal{U}_j = \mathcal{U}(g_j - \widetilde{g}_j, \varepsilon/kC^{a-2}) \subset E$ and a decomposition $E = \mathcal{U}_j \oplus \mathcal{R}_j$ such that $P_{\mathcal{R}_j}\mathscr{A}_{1,g_j-\widetilde{g}_j} \subset B_{\varepsilon/kC^{a-2}}(0, L^\infty(X,\mu))$. Let $g_{1,j}, g_{2,j}, \dots, g_{k_j,j}$ be a maximal linearly independent set in $\mathcal{U}_j$ and choose $\varphi_{1,j}, \dots, \varphi_{k_j,j} \in E'$ to have the property

$$\varphi_{i,j}(g_{l,j}) = \delta_{i,l} \quad \text{and} \quad \varphi_{i,j}|_{\mathcal{R}_j} = 0 \quad \text{for every } i, l \in \{1, \dots, k_j\}$$

which is possible by the Hahn-Banach theorem. Then, for each $n \in \mathbb{N}$, we write

$$A_1 T_1^n (g_j - \widetilde{g}_j) = \lambda_{1,j,n} g_{1,j} + \dots + \lambda_{k_j,j,n} g_{k_j,j} + r_{j,n}$$

for appropriate $\lambda_{i,j,n} \in \mathbb{C}$ $(1 \leq i \leq k_j)$ and $r_{j,n} \in \mathcal{R}_j$ with $\|r_{j,n}\|_\infty < \varepsilon/kC^{a-2}$ and obtain

$$\lambda_{i,j,n} = \varphi_{i,j}(A_1 T_1^n(g_j - \widetilde{g}_j)) = (A_1^* \varphi_{i,j})(T_1^n(g_j - \widetilde{g}_j)).$$

It follows that

$$|\lambda_{i,j,n}| \leq \|g_j - \widetilde{g}_j\|_p \cdot \|A_1^*\|_q \cdot \max_{i \in \{1,\ldots,k_j\}} \|\varphi_{i,j}\|_q =: c_j$$

for the dual index q. Now for each $1 \leq i \leq k_j$ choose a function $\widetilde{g}_{i,j} \in L^\infty$ such that $\|g_{i,j} - \widetilde{g}_{i,j}\|_1 \leq \|g_{i,j} - \widetilde{g}_{i,j}\|_p < \varepsilon/(cc_j k_j k)$.

Thus we write

$$\begin{aligned} & T_a^n A_{a-1} T_{a-1}^n \ldots A_1 T_1^n \lambda_{j,n}(g_j - \widetilde{g}_j) \\ = & \sum_{i=1}^{k_j} T_a^n \ldots A_2 T_2^n \lambda_{j,n} \lambda_{i,j,n}(g_{i,j} - \widetilde{g}_{i,j}) + \sum_{i=1}^{k_j} T_a^n \ldots A_2 T_2^n \lambda_{j,n} \lambda_{i,j,n} \widetilde{g}_{i,j} \\ + & T_a^n \ldots A_2 T_2^n \lambda_{j,n} r_{j,n}. \end{aligned}$$

When taking the Cesàro averages over n of the absolute values, the contribution of last term tends to 0 for almost every $x \in X$, since $(\lambda_{j,n})_{n \in \mathbb{N}} \in \mathscr{N}$. The contribution of the second sum also tends to zero almost everywhere, due to $(\lambda_{j,n}\lambda_{i,j,n})_{n \in \mathbb{N}} \in \mathscr{N}$ (as $\mathscr{N}$ is closed under multiplication by bounded sequences) and by $\widetilde{g}_{i,j} \in L^\infty(X, \mu)$ and (A2).

Now, when $a = 2$, using the fact that T_2 is a Dunford-Schwartz operator, the contribution of the first sum is bounded by

$$\frac{1}{N} \sum_{n=1}^{N} \sum_{i=1}^{k_j} |T_2|^n |\lambda_{j,n} \lambda_{i,j,n}(g_{i,j} - \widetilde{g}_{i,j})|.$$

In this case we can use the boundedness of the λ_* sequences and the pointwise ergodic theorem for Dunford-Schwartz operators (cf. Remark 1 and equation (5) from the proof of Theorem 2) to see that there is a set $S_{j,\varepsilon}$ with $\mu(S_{j,\varepsilon}) = 1$ such that for all $x \in S_{j,\varepsilon}$ this contribution has a limes superior not exceeding $c \cdot c_j \cdot (\varepsilon/cc_j k_j k) = \varepsilon/k_j k$. All other contributions discussed above have limit zero.

Summing over all $1 \leq i \leq k_j$ and then $1 \leq j \leq k$, we have for each $x \in \cap_{j=1}^k S_{j,\varepsilon} =: S_\varepsilon$ that

$$
\begin{aligned}
& \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \left| \left(T_2^n A_1 T_1^n A_0 T_0^n f\right)(x) \right| \\
= \ & \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \Bigg| \Bigg(T_2^n A_1 T_1^n r_n + \sum_{j=1}^{k} T_2^n \lambda_{j,n} r_{j,n} + \sum_{j=1}^{k} \sum_{i=1}^{k_j} T_2^n \lambda_{j,n} \lambda_{i,j,n} \widetilde{g}_{i,j} \\
& + \sum_{j=1}^{k} \sum_{i=1}^{k_j} T_2^n \lambda_{j,n} \lambda_{i,j,n} (g_{i,j} - \widetilde{g}_{i,j}) \Bigg)(x) \Bigg| \\
\leq \ & \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \left| \left(T_2^n A_1 T_1^n r_n\right)(x) \right| + \sum_{j=1}^{k} \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \left| \left(T_2^n \lambda_{j,n} r_{j,n}\right)(x) \right| \\
& + \sum_{j=1}^{k} \sum_{i=1}^{k_j} \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \left| \left(T_2^n \lambda_{j,n} \lambda_{i,j,n} \widetilde{g}_{i,j}\right)(x) \right| \\
& + \sum_{j=1}^{k} \sum_{i=1}^{k_j} \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \left| \left(\sum_{j=1}^{k} \sum_{i=1}^{k_j} T_2^n \lambda_{j,n} \lambda_{i,j,n} (g_{i,j} - \widetilde{g}_{i,j}) \right)(x) \right| \\
\leq \ & \varepsilon + \sum_{j=1}^{k} 0 + \sum_{j=1}^{k} \sum_{i=1}^{k_j} 0 + \sum_{j=1}^{k} \sum_{i=1}^{k_j} \varepsilon / k_j k = 2\varepsilon.
\end{aligned}
$$

Thus for each $x \in \bigcap_{m\in\mathbb{N}} S_{1/m} =: S$ we have that

$$
\left(\frac{1}{N} \sum_{n=1}^{N} \left| T_2^n A_1 T_1^n A_0 T_0^n f \right| \right)(x) \to 0,
$$

and, since $\mu(S) = 1$, we are done.

If $a > 2$, then we from here iterate the following for each operator pair $A_z T_z^n$ $(2 \leq z \leq a-1)$.

We consider the last untreated sum from the previous step, the one containing the contribution arising from the functions $A_z T_z^n (g_* - \widetilde{g}_*)$. Using assumption $(A1)$, we split each such function further into a linear combination of finitely many functions $g_{\ell,*} \in E$ and a remainder term $r_{*,n} \in L^\infty$. The new coefficient sequences $\lambda_{\ell,*,n}$ will also lie in $\mathscr{N}$, hence the contribution of remainder terms to the Cesàro means will be zero. Then, as seen for g_j, we split each of the $g_{\ell,*}$ into an essentially bounded part $\widetilde{g}_{\ell,*} \in L^\infty$ and a remainder small in L^1. In the Cesàro means, using that all coefficient sequences lie in $\mathscr{N}$ and by assumption $(A2)$, the terms with $\widetilde{g}_{\ell,*}$ all have zero contribution, and so we are left with the functions $g_{\ell,*} - \widetilde{g}_{\ell,*}$, from where we continue the iteration.

At the end, we reach T_a^n, applied to functions $g_* - \widetilde{g}_*$ (sufficiently small in L^1) with coefficients being products of λ-s. At this point, as detailed for T_2^n when we assumed $a = 2$, we use the boundedness of the coefficient sequences, and apply Birkhoff's pointwise ergodic theorem for Dunford-Schwartz operators to $|g_* - \widetilde{g}_*|$

to obtain a contribution to the limsup of the Cesàro means that adds up to 2ε over all – finitely many – multiindices $*$.

For part (2), assume $p = 2$, write $H := L^2(X, \mu)$ and note that eigenfunctions in $H_{0,r}$ pertaining to different unimodular eigenvalues are orthogonal. For notational convenience we again assume that $H_{0,r}$ is infinite-dimensional, whereas the finite dimensional case can be treated analogously. Take $f \in H_{0,r}$ and let $\{h_j\}_{j=1}^{\infty}$ be an orthonormal basis in $H_{0,r}$ of eigenvectors of T_0 pertaining to unimodular eigenvalues $\{\alpha_j\}_{j=1}^{\infty}$. Then we can write $f = \sum_{m=1}^{\infty} d_m h_m$ for some ℓ^2-sequence $(d_m)_m$ and obtain

$$\lambda_{j,n} = \langle T_0^n f, A_0^* \varphi_j \rangle = \Big\langle \sum_{m=1}^{\infty} \alpha_m^n d_m h_m, A_0^* \varphi_j \Big\rangle = \sum_{m=1}^{\infty} \alpha_m^n \left(d_m \langle h_m, A_0^* \varphi_j \rangle \right).$$

So for each $1 \le j \le k$ we have that $(\lambda_{j,n})_n \in \mathscr{P}$ since $(d_m \langle h_m, A_0^* \varphi_j \rangle) \in l^1$ by the Cauchy-Schwarz inequality.

For each $1 \le j \le k$, we may split g_j into the (almost weakly) stable and the reversible part with respect to T_1, i.e. $g_j = g_j^s + g_j^r$ with $g_j^s \in H_{1,s}$ and $g_j^r \in H_{1,r}$. Then we have

$$\sum_{j=1}^{k} T_a^n A_{a-1} T_{a-1}^n \ldots A_1 T_1^n \lambda_{j,n} g_j = \sum_{j=1}^{k_j} T_a^n \ldots A_1 T_1^n \lambda_{j,n} g_j^r + \sum_{j=1}^{k} T_a^n \ldots A_1 T_1^n \lambda_{j,n} g_j^s.$$

We first look at the contribution of the second sum to the Cesàro averages. Observe that

$$\left| \frac{1}{N} \sum_{n=1}^{N} \sum_{j=1}^{k} T_a^n \ldots A_1 T_1^n \lambda_{j,n} g_j^s \right| (x) \le \frac{1}{N} \sum_{n=1}^{N} \sum_{j=1}^{k} \left| T_a^n \ldots A_1 T_1^n \lambda_{j,n} g_j^s \right| (x)$$

$$\le \frac{1}{N} \sum_{n=1}^{N} \sum_{j=1}^{k} c \left| T_a^n \ldots A_1 T_1^n g_j^s \right| (x) = \sum_{j=1}^{k} \left(\frac{1}{N} \sum_{n=1}^{N} c \left| T_a^n \ldots A_1 T_1^n g_j^s \right| (x) \right) \to 0$$

for almost all $x \in X$, using part (1) applied to $(a-1)$ pairs $A_i T_i^n$.

We now turn our attention to the first sum, involving the reversible parts g_j^r. For each $1 \le j \le k$ there exists a finite dimensional subspace

$$\mathcal{U}_j = \mathcal{U}(g_j^r, \varepsilon / C^{a-2}) \subset H$$

and a decomposition $E = \mathcal{U}_j \oplus \mathcal{R}_j$ such that

$$P_{\mathcal{R}_j} \mathscr{A}_{1, g_j^r} \subset B_{\varepsilon / C^{a-2}}(0, L^{\infty}(X, \mu)).$$

Let $g_{1,j}, g_{2,j}, \ldots, g_{k_j,j}$ be an orthonormal basis in $\mathcal{U}_j$. Then we write for each $n \in \mathbb{N}$

$$A_1 T_1^n (g_j^r) = \lambda_{1,j,n} g_{1,j} + \ldots + \lambda_{k_j,j,n} g_{k_j,j} + r_{j,n}$$

for appropriate $\lambda_{i,j,n} \in \mathbb{C}$ $(1 \le i \le k_j)$ and $r_{j,n} \in \mathcal{R}_j$ with $\|r_{j,n}\|_{\infty} < \varepsilon / C^{a-2}$ and observe

$$\lambda_{i,j,n} = \langle A_1 T_1^n g_j^r, \varphi_{i,j} \rangle = \langle T_1^n g_j^r, A_1^* \varphi_{i,j} \rangle,$$

where as above each $\varphi_{i,j}$ is orthogonal to $\mathcal{R}_j$ and $\langle g_{l,j}, \varphi_{i,j} \rangle = \delta_{l,i}$. (Note that if $\mathcal{R}_j \perp \mathcal{U}_j$, then we can choose $\varphi_{i,j} := g_{i,j}$.) So

$$|\lambda_{i,j,n}| \le \|g_j^r\|_2 \cdot \|A_1^*\| \max\{\|\varphi_{i,j}\|_2,\, i = 1, \ldots, k_j\} =: c_j$$

and $(\lambda_{i,j,n})_{n\in\mathbb{N}} \in \mathscr{P}$ by Example 1.

Thus for each $1 \le j \le k$ we have for almost every $x \in X$

$$\begin{aligned}
&\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n \dots A_2T_2^n\lambda_{j,n}g_j^r\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n \dots A_2T_2^n\lambda_{j,n}g_j^r\right)(x)\right| \\
\le\ &\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n \dots A_2T_2^n\lambda_{j,n}r_{j,n}\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n \dots A_2T_2^n\lambda_{j,n}r_{j,n}\right)(x)\right| \\
&+\sum_{i=1}^{k_j}\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n \dots A_2T_2^n\lambda_{j,n}\lambda_{i,j,n}g_{i,j}\right)(x)\right. \\
&\left.-\underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n \dots A_2T_2^n\lambda_{j,n}\lambda_{i,j,n}g_{i,j}\right)(x)\right|.
\end{aligned}$$

The first difference on the right hand side is bounded by $2C^{a-2}\|r_{j,n}\|_\infty \le 2\varepsilon$.

If now $a = 2$, then the sum at the end consists of terms of the form

$$\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^n\lambda_{j,n}\lambda_{i,j,n}g_{i,j}\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^n\lambda_{j,n}\lambda_{i,j,n}g_{i,j}\right)(x)\right|.$$

Note that $(\lambda_{j,n})_n \in \mathscr{P}$ and $(\lambda_{i,j,n})_n \in \mathscr{P}$ implies $(\lambda_{j,n}\lambda_{i,j,n})_n \in \mathscr{P}$, and since elements in $\mathscr{P}$ are good weights for the pointwise ergodic theorem for Dunford-Schwartz operators, this absolute value is zero for almost all x.

Summing up, we obtain

$$\begin{aligned}
&\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^nA_0T_0^nf\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^nA_0T_0^nf\right)(x)\right| \\
\le\ &\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^nr_n\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^nr_n\right)(x)\right| \\
&+\sum_{j=1}^{k}\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j\right)(x)\right| \\
\le\ &2\varepsilon+\sum_{j=1}^{k}\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j^s\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j^s\right)(x)\right| \\
&+\sum_{j=1}^{k}\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j^r\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j^r\right)(x)\right| \\
=\ &2\varepsilon+\sum_{j=1}^{k}\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j^r\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^nA_1T_1^n\lambda_{j,n}g_j^r\right)(x)\right| \\
\le\ &2\varepsilon+2\varepsilon = 4\varepsilon
\end{aligned}$$

for all $x \in S_\varepsilon$ for some appropriate $S_\varepsilon \subset X$ with $\mu(S_\varepsilon) = 1$. Thus for each $x \in \bigcap_{m\in\mathbb{N}} S_{1/m} =: S$ we have that

$$\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^n A_1 T_1^n A_0 T_0^n f\right)(x) - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_2^n A_1 T_1^n A_0 T_0^n f\right)(x)\right| \to 0.$$

Since $\mu(S) = 1$, this completes the case $a = 2$.

For the case when $a > 2$, for each pair (i, j), we split the function $g_{i,j}$ into its stable and reversible part with respect to T_2, and apply the above arguments until we reach the last operator T_a. In each split, the stable parts g_*^s will contribute with a pointwise almost everywhere zero Cesàro average limit each, and the remainder parts r_* have a total spread between the limes superior and the limes inferior bounded by 2ε. The last reversible parts $T_a g_*^r$ converge pointwise almost everywhere since the sequence of weights is a product of elements of $\mathscr{P}$, and hence an element of $\mathscr{P}$ itself, being a good sequence of weights.

In total, we obtain that

$$\begin{aligned} &\left|\overline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n A_0 T_0^n f\right)(x)\right. \\ &\left. - \underline{\lim}_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} T_a^n A_{a-1} T_{a-1}^n \dots A_1 T_1^n A_0 T_0^n f\right)(x)\right| \\ \leq\ & a \cdot 2\varepsilon \end{aligned}$$

for all x outside of a nullset, completing the proof. (Recall that the form (1) of the limit is the same as in the norm case and follows from [11, Theorem 3].)

Remark 2. *For eigenfunctions $f \in L^p(X, \mu)$ of T_0, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^n A_0 T_0^n f$$

converge a.e. for every operator A_0 on $E := L^p(X, \mu)$, $p \in [1, \infty)$. Indeed, if $T_0 f = \lambda f$ for some $\lambda \in \mathbb{T}$, then the above averages take the form

$$\frac{1}{N}\sum_{n=1}^{N} (\lambda T_1)^n A_0 f.$$

Since λT_1 is again a Dunford-Schwartz operator, a.e. convergence of the above averages follows from the pointwise ergodic theorem. However, due to the lack of a Banach principle, it is not clear how to conclude convergence for arbitrary $f \in E_r$, E_r being the reversible part of E corresponding to T_0, for $p \neq 2$.

5. Examples and a continuous analogue

5.1. Examples: powers of the Volterra operator.

Consider on $H := L^2([0, 1])$ the Volterra operator V given by

$$(Vf)(x) := \int_0^x f(t)\mathrm{d}t.$$

We first check that V can be written as a sum of three operators which satisfy conditions (A1) and (A2) of Theorem 1 for any Dunford-Schwartz operators.

With the orthonormal base $e_m(x) := e^{2\pi i m x}$, we have for $0 \neq m \in \mathbb{Z}$

$$(Ve_m)(x) = \int_0^x e^{2\pi i m t} \mathrm{dt} = \frac{1}{2\pi i m} \left(e_m(x) - 1 \right),$$

and thus for an $f \in H$ with the base decomposition $f = \sum_{m\in\mathbb{Z}} c_m e_m$ (where $(c_m)_m$ is an ℓ^2-sequence) we may write

$$(Vf)(x) = \left(\frac{1}{2\pi i} \sum_{0\neq m\in\mathbb{Z}} c_m \frac{e_m(x) - 1}{m} \right) + c_0 x.$$

Consider now the decomposition of the Volterra operator into the sum $V = V_1 + V_2 + V_3$ with

$$V_1 f := c_0 \cdot J, \quad V_2 f := -\frac{1}{2\pi i} \sum_{0\neq m\in\mathbb{Z}} \frac{c_m}{m} e_0, \quad V_3 f := \frac{1}{2\pi i} \sum_{0\neq m\in\mathbb{Z}} \frac{c_m}{m} e_m,$$

where $f = \sum_{m=-\infty}^{\infty} c_m e_m$ and $J(x) = x$.

The operators V_1 and V_2 both have one-dimensional range and are bounded with respect to the L^∞-norm. Indeed, the last assertion for V_2 follows from

$$\|V_2 f\|_\infty \leq \frac{1}{2\pi} \sum_{0\neq m\in\mathbb{Z}} \frac{|c_m|}{|m|} \leq \frac{1}{2\pi} \left(\sum_{0\neq m\in\mathbb{Z}} |c_m|^2 \sum_{0\neq m\in\mathbb{Z}} \frac{1}{m^2} \right)^{1/2} \leq \frac{\|f\|_2}{2\sqrt{3}} \leq \frac{\|f\|_\infty}{2\sqrt{3}}.$$

Thus, assumptions (A1) and (A2) are satisfied for both V_1 and V_2 as well as any choice of Dunford-Schwartz operators T_j. It remains to show that the same holds for V_3, too.

Assumption (A2) is satisfied for V_3 and any Dunford-Schwartz operator by the same calculation as for V_2. To show (A1), let $\varepsilon > 0$ and $f \in H$ with $\|f\|_2 \leq 1$ be fixed. We may choose $M \in \mathbb{N}$ such that $\sum_{|m|\geq M} \frac{1}{m^2} < 4\pi^2\varepsilon^2$. Then with the decomposition $V_3 f = g_1 + g_2$, where

$$g_1 := \frac{1}{2\pi i} \sum_{0<|m|<M} \frac{c_m}{m} e_m,$$

we have by the Cauchy-Schwarz and Bessel's inequalities

$$\begin{aligned} \|g_2\|_\infty \leq \frac{1}{2\pi} \sum_{|m|\geq M} \frac{|c_m|}{|m|} \quad &\leq \quad \frac{1}{2\pi} \left(\sum_{|m|\geq M} |c_m|^2 \right)^{1/2} \left(\sum_{|m|\geq M} \frac{1}{m^2} \right)^{1/2} \\ &\leq \quad \frac{1}{2\pi} \|f\|_2 \left(\sum_{|m|\geq M} \frac{1}{m^2} \right)^{1/2} < \varepsilon. \end{aligned}$$

Thus taking $\mathcal{U} := \operatorname{span}\{e_m : |m| < M\}$ and $\mathcal{R} := \operatorname{span}\{e_m : |m| \geq M\}$ we have the desired (orthogonal) decomposition in condition (A1) for the operator V_3 and any Dunford-Schwartz operators.

Analogously, for every $k \in \mathbb{N}$ the operator $V^k = (V_1+V_2+V_3)^k$ decomposes into a finite sum of one-dimensional operators (each term containing at least one V_1 or V_2) which are bounded with respect to the L^∞-norm and the operator V_3^k of the form $V_3^k f = (2\pi i)^{-k} \sum_{0\neq m\in\mathbb{Z}} \frac{c_m}{m^k} e_m$. The properties (A1) and (A2) for V_3^k follow analogously to the above calculations for V_3. Hence, (A1) and (A2) are satisfied for V^k and any choice of Dunford-Schwartz operators T_j.

Putting everything together, we obtain for any choice of Dunford-Schwartz operators $T_0, \dots, T_a$ with $\operatorname{Fix}|T_1| = \dots = \operatorname{Fix}|T_a| = \langle \mathbf{1} \rangle$ and for every $k_0, \dots, k_{a-1} \in \mathbb{N}$ that

(1) for each $f \in H_{0,s}$, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N |T_a^n V^{k_{a-1}} T_{a-1}^n \dots V^{k_1} T_1^n V^{k_0} T_0^n f| = 0$ pointwise a.e.;

(2) for each $f \in H_{0,r}$, $\frac{1}{N} \sum_{n=1}^N T_a^n V^{k_{a-1}} T_{a-1}^n \dots V^{k_1} T_1^n V^{k_0} T_0^n f$ converges pointwise a.e..

5.2. **Continuous version.** In this section we consider strongly continuous (shortly: C_0-) semigroups $(T_j(t))_{t\in[0,\infty)}$ instead of discrete semigroups $(T_j^n)_{n=0}^\infty$, $j \in \{0, \dots, a\}$.

Let $T(\cdot) := (T(t))_{t\in[0,\infty)}$ be a C_0-semigroup of Dunford-Schwartz operators on $L^1(X,\mu)$. Then, since the unit ball in $L^\infty(X,\mu)$ is invariant under the semigroup, $T(\cdot)$ is by the standard approximation argument automatically a C_0-semigroup (of contractions) on $L^p(X,\mu)$ for every $\infty > p \geq 1$ (note that the reverse implication also holds). Moreover, for every $f \in L^1(X,\mu)$ the function $(T(\cdot)f)(x)$ is Lebesgue integrable over finite intervals in $[0,\infty)$ for almost every $x \in X$ by Fubini's theorem, see, e.g., Sato [25, p. 3]. Analogously, for C_0-semigroups $T_0(\cdot), \dots, T_a(\cdot)$ on $E := L^p(X,\mu)$, operators $A_0, \dots, A_{a-1} \in \mathcal{L}(E)$ and $f \in E$, the function

$$(T_a(\cdot) A_{a-1} T_{a-1}(\cdot) \dots A_1 T_1(\cdot) A_0 T_0(\cdot) f)(x)$$

is Lebesgue integrable over finite intervals in $[0,\infty)$ for almost every $x \in X$.

The pointwise ergodic theorem extends to every strongly measurable semigroup $T(\cdot)$ of Dunford-Schwartz operators, see Dunford, Schwartz [6, pp. 694, 708]. Moreover, as in Remark 1, $\cap_{t>0} \operatorname{Fix} T(t) = \langle \mathbf{1} \rangle$ implies that

$$\lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} T(t) f \, dt = c \cdot \mathbf{1}$$

with $|c| \leq \|f\|_1$. Furhermore, a natural modification of Lin, Olsen, Tempelman [23, Proof of Prop. 2.6] shows that every C_0-semigroup of Dunford-Schwartz operators has relatively weakly compact orbits in $L^1(X,\mu)$. Thus, the continuous version of the Jacobs-deLeeuw-Glicksberg decomposition (see e.g. [8, Theorem III.5.7]) is valid for such semigroups.

We also need a continuous analogue of the concept of the modulus. By e.g. Kipnis [17] or Kubokawa [20], for a C_0-semigroup $T(\cdot)$ of contractions there exists a minimal C_0-semigroup dominating $T(\cdot)$ which is also contractive. We denote this positive semigroup by $|T|(\cdot)$ and refer to Becker, Greiner [2] for related results. (Note that $|T|(t) \neq |T(t)|$ in general.) Of course, $|T|(\cdot) = T(\cdot)$ for positive semigroups. Moreover, the construction in [17, pp. 372-3] implies that if $T(\cdot)$ consists Dunford-Schwartz operators then so does $|T|(\cdot)$.

Analogously to the proof of Theorem 1 we obtain the following continuous version of Theorem 1. (Cf. Bergelson, Leibman, Moreira [4] for an abstract method of transferring discrete results into continuous ones.)

Theorem 3. *For $a \in \mathbb{N}$, let $(T_0(t))_{t\in[0,\infty)}$,$(T_1(t))_{t\in[0,\infty)}$, …, $(T_a(t))_{t\in[0,\infty)}$ be C_0-semigroups of Dunford-Schwartz operators on $L^1(X,\mu)$ of a Borel probability space (X,μ), with*

$$\cap_{t>0} \operatorname{Fix} |T_1|(t) = \ldots = \cap_{t>0} \operatorname{Fix} |T_a|(t) = \langle \mathbf{1} \rangle.$$

For $p \in [1,\infty)$ and $E := L^p(X,\mu)$, let $E = E_{0,r} \oplus E_{0,s}$ be the Jacobs-deLeeuw-Glicksberg decomposition corresponding to $T_0(\cdot)$. Let further $A_j \in \mathcal{L}(E)$ $(0 \le j < a)$ be bounded operators. For a function $f \in E$ and an index $0 \le j < a$, write $\mathscr{A}_{j,f} := \{A_j T_j(t) f \mid t \in (0,\infty)\}$. Suppose that the following conditions hold:

- (A1) (Twisted compactness) *For any function $f \in E$, index $0 \le j < a$ and $\varepsilon > 0$, there exists a decomposition $E = \mathcal{U} \oplus \mathcal{R}$ with $\dim \mathcal{U} < \infty$ such that $P_{\mathcal{R}} \mathscr{A}_{j,f} \subset B_\varepsilon(0, L^\infty(X,\mu))$.*
- (A2) (Joint L^∞-boundedness) *There exists a constant $C > 0$ such that we have*

$$\{A_j T_j(t) \mid t \in (0,\infty), 1 \le j < a\} \subset B_C(0, \mathcal{L}(L^\infty(X,\mu)).$$

Then

(1) *for each $f \in E_{0,s}$,*

$$\lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} |T_a(t) A_{a-1} T_{a-1}(t) \ldots A_1 T_1(t) A_0 T_0(t) f| \, dt = 0 \quad \textit{pointwise a.e.};$$

(2) *if $p = 2$, then for each $f \in E_{0,r}$,*

$$\frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} T_a(t) A_{a-1} T_{a-1}(t) \ldots A_1 T_1(t) A_0 T_0(t) f \, dt$$

converges pointwise a.e..

Remark 3. *If for some $j \in \{1,\ldots,a\}$ the semigroup $T_j(\cdot)$ consists of positive operators, then one can replace the condition $\cap_{t>0} \operatorname{Fix} |T_j|(t) = \langle \mathbf{1} \rangle$ by $\ker(G_j) = \langle \mathbf{1} \rangle$ for the generator G_j of $T_j(\cdot)$, see, e.g., Engel, Nagel* [12, Cor. IV.3.8]. *Moreover, this condition for the semigroup induced by a measure preserving flow is equivalent to the ergodicity of the flow.*

Note that the examples of powers of the Volterra operator discussed above are valid in the continuous setting as well.

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