A characterization of positive normal functionals on the full operator algebra

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Abstract. Using the recent theory of Krein–von Neumann extensions for positive functionals we present several simple criteria to decide whether a given positive functional on the full operator algebra $B(H)$ is normal. We also characterize those functionals defined on the left ideal of finite rank operators that have a normal extension.

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The aim of this short note is to present a theoretical application of the generalized Krein–von Neumann extension, namely to offer a characterization of positive normal functionals on the full operator algebra. To begin with, let us fix our notations. Given a complex Hilbert space H, denote by $B(H)$ the full operator algebra, i.e., the C^*-algebra of continuous linear operators on H. The symbols $B_F(H), B_1(H), B_2(H)$ are referring to the ideals of continuous finite rank operators, trace class operators, and Hilbert–Schmidt operators, respectively. Recall that $B_2(H)$ is a complete Hilbert algebra with respect to the inner product

$$(X \,|\, Y)_2 = \mathrm{Tr}(Y^*X) = \sum_{e\in\mathcal{E}} (Xe \,|\, Ye), \qquad X, Y \in B_2(H).$$

Here Tr refers to the the trace functional and $\mathcal{E}$ is an arbitrary orthonormal basis in H. Recall also that $B_1(H)$ is a Banach *-algebra under the norm $\|X\|_1 := \mathrm{Tr}(|X|)$, and that $B_F(H)$ is dense in both $B_1(H)$ and $B_2(H)$, with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. It is also known that $X \in B_1(H)$ holds if and only if X is the product of two elements of $B_2(H)$. For the proofs and further basic properties of Hilbert-Schmidt and trace class operators we refer the reader to [1].

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Let $\mathscr{A}$ be a von Neumann algebra, that is a strongly closed *-subalgebra of $B(H)$ containing the identity. A bounded linear functional $f:\mathscr{A}\to\mathbb{C}$ is called normal if it is continuous in the ultraweak topology, that is f belongs to the predual of $\mathscr{A}$. It is well known that the predual of $B(H)$ is $B_1(H)$, hence every normal functional can be represented by a trace class operator. We will use this property as the definition.

Definition. *A linear functional $f:B(H)\to\mathbb{C}$ is called a normal functional if there exists a trace class operator F such that*

$$f(X) := \operatorname{Tr}(XF) = \operatorname{Tr}(FX), \qquad X\in B(H).$$

Remark that such a functional is always continuous due to the inequality

$$|\operatorname{Tr}(XF)| \le \|F\|_1\cdot\|X\|.$$

Our main tool is a canonical extension theorem for linear functionals which is analogous with the well-known operator extension theorem named after the pioneers of the 20th century operator theory M.G. Krein [2] and J. von Neumann [3]. For the details see Section 5 in [5], especially Theorem 5.6 and the subsequent comments. Let us recall the cited theorem:

A Krein–von Neumann type extension. *Let $\mathscr{I}$ be a left ideal of the complex Banach *-algebra $\mathscr{A}$, and consider a linear functional $\varphi:\mathscr{I}\to\mathbb{C}$. The following statements are equivalent:*

(a) *There is a representable positive functional $\varphi^\bullet:\mathscr{A}\to\mathbb{C}$ extending φ, which is minimal in the sense that*

$$\varphi^\bullet(x^*x)\le\widetilde{\varphi}(x^*x), \qquad \text{holds for all } x\in\mathscr{A},$$

whenever $\widetilde{\varphi}:\mathscr{A}\to\mathbb{C}$ is a representable extension of φ.

(b) *There is a constant $C\ge 0$ such that $|\varphi(a)|^2\le C\cdot\varphi(a^*a)$ for all $a\in\mathscr{I}$.*

We remark that the construction used in the proof of the above theorem is closely related to the one developed in [4] for Hilbert space operators. The main advantage of that construction is that we can compute the values of the smallest extension $\varphi^\bullet$ on positive elements, namely

$$\varphi^\bullet(x^*x) = \sup\left\{|\varphi(x^*a)|^2 \,\middle|\, a\in\mathscr{I},\ \varphi(a^*a)\le 1\right\} \qquad \text{for all } x\in\mathscr{A}. \qquad (*)$$

The minimal extension $\varphi^\bullet$ is called the *Krein–von Neumann extension of φ*.

The characterization we are going to prove is stated as follows.

Main Theorem. *For a given positive functional $f:B(H)\to\mathbb{C}$ the following statements are equivalent:*

(i) *f is normal.*

(ii) *There exists a normal positive functional g such that $f\le g$.*

(iii) *$f\le g$ holds for any positive functional g that agrees with f on $B_F(H)$.*

(iv) *For any $X\in B(H)$ we have*

$$f(X^*X) = \sup\{|f(X^*A)|^2 \mid A\in B_F(H), f(A^*A)\le 1\}. \qquad (**)$$

(v) $f(I)\le\sup\{|f(A)|^2 \mid A\in B_F(H), f(A^*A)\le 1\}$.

Proof. The proof is divided into three claims, which might be interesting on their own right. Before doing that we make some observations. For a given trace class operator S let us denote by f_S the normal functional defined by

$$f_S(X) := \operatorname{Tr}(XS), \qquad X \in B(H).$$

The map $S \mapsto f_S$ is order preserving between positive trace class operators and normal positive functionals. Indeed, if $S \geq 0$ then

$$f_S(A^*A) = \operatorname{Tr}(A^*AS) = \|AS^{1/2}\|_2^2 \geq 0.$$

Conversely, if f_S is a positive functional and $P_{\langle h\rangle}$ denotes the orthogonal projection onto the subspace spanned by $h \in H$, we obtain $S \geq 0$ by

$$(Sh \,|\, h) = \operatorname{Tr}(P_{\langle h\rangle}S) = f_S(P_{\langle h\rangle}^* P_{\langle h\rangle}) \geq 0, \qquad \text{for all } h \in H.$$

Our first two claims will prove that (i) and (iv) are equivalent.

Claim 1. *Let f be a normal positive functional and set $\varphi := f|_{B_F(H)}$. Then f is the smallest positive extension of φ, i.e $\varphi^\bullet = f$.*

Proof of Claim 1. Since $f \geq 0$ is normal, there is a positive $S \in B_1(H)$ such that $f = f_S$. By assumption φ has a positive extension (namely f itself is one), thus there exists also the Krein–von Neumann extension denoted by $\varphi^\bullet$. As $f_S - \varphi^\bullet$ is a positive functional due to the minimality of $\varphi^\bullet$, its norm is attained at identity I. Therefore it is enough to show that

$$\varphi^\bullet(I) \geq f_S(I) = \operatorname{Tr}(S).$$

We know from $(*)$ that

$$\varphi^\bullet(X^*X) = \sup\{|\varphi(X^*A)|^2 \,|\, A \in B_F(H), \varphi(A^*A) \leq 1\}$$

for any $X \in B(H)$. Choosing $A = \operatorname{Tr}(S)^{-1/2}P$ for any projection P with finite rank, we see that $\varphi(A^*A) = \operatorname{Tr}(S)^{-1}\operatorname{Tr}(PS) \leq 1$, whence

$$\varphi^\bullet(I) \geq |\varphi(A)|^2 = \frac{\operatorname{Tr}(PS)^2}{\operatorname{Tr}(S)}.$$

Taking supremum in P on the right hand side we obtain $\varphi^\bullet(I) \geq \operatorname{Tr}(S)$, which proves the claim.

Claim 2. *The smallest positive extension of φ, i.e. $(f|_{B_F(H)})^\bullet$ is normal.*

Proof of Claim 2. First observe that the restriction of f to $B_2(H)$ defines a continuous linear functional on $B_2(H)$ with respect to the norm $\|\cdot\|_2$. Due to the Riesz representation theorem, there exists a unique representing operator $S \in B_2(H)$ such that

$$f(A) = (A \,|\, S)_2 = \operatorname{Tr}(S^*A), \qquad \text{for all } A \in B_2(H). \qquad (***)$$

We are going to show that $S \in B_1(H)$. Indeed, let $\mathcal{E}$ be an orthonormal basis in H and let $\mathcal{F}$ be any non-empty finite subset of $\mathcal{E}$. Denoting by $P_{\mathcal{F}}$ the orthogonal projection onto the subspace spanned by $\mathcal{F}$ we get

$$\sum_{e\in\mathcal{F}} (Se \,|\, e) = (P_{\mathcal{F}} \,|\, S)_2 = f(P_{\mathcal{F}}) \leq f(I).$$

Taking supremum in $\mathcal{F}$ we obtain that S is in trace class. By Claim 1, the smallest positive extension $\varphi^\bullet$ of φ equals f_S which is normal. This proves Claim 2.

Now, we are going to prove (ii)⇒(i).

Claim 3. *If there exists a normal positive functional g such that $f \leq g$ holds, then f is normal as well.*

Proof of Claim 3. Let g be a normal positive functional dominating f, and let T be a trace class operator such that $g = f_T$. According to Claim 2 it is enough to prove that $f = \varphi^\bullet$. Since $h := f - \varphi^\bullet$ is positive, this will follow by showing that $h(I) = 0$. We see from $(***)$ that $h(A) = 0$ for any finite rank operator A. Consequently, as $h \leq f \leq f_T$, it follows that

$$h(I) = h(I - P) \leq f_T(I - P) = \mathrm{Tr}(T) - \mathrm{Tr}(TP),$$

for any finite rank projection P. Taking infimum in P we obtain $h(I) = 0$ and therefore Claim 3 is established.

Completing the proof we mention all the missing trivial implications. Taking $g := f$, (i) implies (ii). As $(**)$ means that $\varphi^\bullet = f$, equivalence of (iii) and (iv) follows from the minimality of the Krein-von Neumann extension. Replacing X with I in $(**)$ we obtain (v). Conversely, (v) implies (iv) as $\varphi^\bullet \leq f$ and $f - \varphi^\bullet$ attains its norm at I. □

Finally, we remark that the above proof contains a characterization of having normal extension for a functional defined on $B_F(H)$.

Corollary. *Let $\varphi : B_F(H) \to \mathbb{C}$ be a linear functional. The following statements are equivalent to the existence of a normal extension.*

(a) *There is a $C \geq 0$ such that $|\varphi(A)|^2 \leq C \cdot \varphi(A^*A)$ for all $A \in B_F(H)$.*
(b) *There is a positive functional f such that $f|_{B_F(H)} = \varphi$.*
(c) *There is an $F \in B_1(H)$ such that $\varphi(A) = \mathrm{Tr}(FA)$ for all $A \in B_F(H)$.*

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