Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **35**, 1–8; doi: 10.14232/ejqtde.2017.1.35 http://www.math.u-szeged.hu/ejqtde/

Global stability of a predator–prey model with Beddington–DeAngelis and Tanner functional response

Nai-wei Liu[✉][1,2] and **Na Li**[2]

[1]School of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China
[2]School of Mathematics and Information Science, Yantai University
Yantai, Shandong 264005, People's Republic of China

Received 30 November 2016, appeared 9 May 2017
Communicated by Leonid Berezansky

Abstract. In this paper, we study the global stability of a predator–prey system with Beddington–DeAngelis and Tanner functional response. By using the iteration method and comparison principle, we prove the global asymptotic stability of the unique positive equilibrium solution.

Keywords: global asymptotic stability, comparison principle, positive equilibrium solution, Beddington–DeAngelis and Tanner functional response.

2010 Mathematics Subject Classification: 35K57, 34C37, 92D25.

1 Introduction

The purpose of this paper is to consider the following predator–prey system with Beddington–DeAngelis and Tanner functional response

$$\begin{cases} u_t = d_1 \Delta u + u - u^2 - \frac{uv}{a+u+v}, & (x,t) \in \Omega \times (0,\infty), \\ v_t = d_2 \Delta v + v(\delta - \beta \frac{v}{u}), & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) > 0, \ \ v(x,0) = v_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are the densities of prey and predator, respectively, Ω is a bounded domain with smooth boundary $\partial\Omega$, a, δ and β are positive constants. In this paper we assume that the two diffusion coefficients d_1 and d_2 are the diffusion coefficients corresponding to u and v, respectively, and are positive and equal, but not necessary constants. We use d to represent the common value. The admissible initial data $u_0(x)$ and $v_0(x)$ are continuous functions on $\bar{\Omega}$.

✉Corresponding author. Email: liunaiwei@aliyun.com

The functional response $\frac{uv}{a+u+v}$ was introduced by Beddington [1] and DeAngelis [3]. They proposed the following predator–prey model with Beddington–DeAngelis functional response

$$\begin{cases} x' = x(r-\theta x) - \frac{Exy}{a+bx+cy}, \\ y' = -dy + \frac{\beta xy}{a+bx+cy}. \end{cases} \tag{1.2}$$

Huang et al. [9, 10] proposed a class of virus dynamics model with Beddington–DeAngelis functional response. Liu and Kong [11] studied the dynamics of a predator–prey system with Beddington–DeAngelis functional response and delays.

Besides the Beddington–DeAngelis functional responses mentioned above, there are several other well-known functional responses, such as Holling type (I, II, III, IV), Monod–Haldane type and Hassel–Verley type functional responses etc. Some authors studied and raised some open questions for structured predator–prey models with different types of functional responses. Especially, in [15], Peng and Wang considered the steady states of a diffusive Holling–Tanner prey–predador model

$$\begin{cases} u_t = d_1\Delta u + au - u^2 - \frac{uv}{m+u}, & (x,t) \in \Omega \times (0,\infty), \\ v_t = d_2\Delta v + bv - \frac{v^2}{\gamma u}, & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(0) > 0, \;\; v(x,0) = v_0(0) \geq 0, & x \in \bar{\Omega}. \end{cases} \tag{1.3}$$

They discussed the existence and non-existence of positive non-constant steady solutions for (1.3), and proved that (1.3) has no positive non-constant steady solution under a certain condition. In the another paper [16], by the construction of a Lyapunov function and a standard linearization procedure, they studied the stability of diffusive predator–prey system of Holling–Tanner type (1.3). Chen and Shi [2] concentrated on the steady states of (1.3). They used the comparison principle and defined iteration sequences to prove the global stability for the constant positive equilibrium. Their result improves the earlier one given in [16] which was established by Lyapunov method. We also note here that the (non-spatial) kinetic equation of system (1.3) was first proposed by Tanner [20] and May [14], see also [12, 13].

Recently, Qi and Zhu [17] studied the global stability of diffusive predator–prey system (1.3). Indeed, in [17], they established improved global asymptotic stability of the unique positive equilibrium solution. For more detailed biological implications of the model, besides the references mentioned above, one can see [4–8, 18, 19].

Motivated by the previous works [17], in this paper by incorporating the diffusion and ratio-dependent Beddington–DeAngelis functional response into system (1.3), we study the stability of the positive equilibrium solution of (1.1).

A direct computation gives that (1.1) has a unique positive equilibrium (u^*, v^*), where

$$u^* = \frac{\beta\left(1-a+\sqrt{(1-a)^2+4a(1+\frac{\delta}{\beta})}\right)}{2(\beta+\delta)}, \qquad v^* = \frac{\delta}{\beta}u^*.$$

2 Proof of the main result

Let $w = \frac{v}{u}$, then we have

$$w_t = \frac{v_t}{u} - \frac{u_t v}{u^2},$$
$$\nabla w = \frac{\nabla v}{u} - \frac{\nabla u}{u^2} v,$$
$$\Delta w = \frac{\Delta v}{u} - \frac{v \Delta u}{u^2} - \frac{2\nabla u \cdot \nabla v}{u^2} + \frac{2|\nabla u|^2}{u^3} v.$$

Therefore the equation satisfied by w is

$$w_t - d\Delta w = w\left(\delta - 1 + u + w\left(-\beta + \frac{u}{a+u+v}\right)\right) + 2d\frac{\nabla u}{u} \cdot \nabla w. \tag{2.1}$$

Theorem 2.1. *Suppose $d = d(x,t)$ is strictly positive, bounded and continuous in $\Omega \times [0, +\infty)$, a, δ and β are positive constants, $\delta < 1$, then the positive equilibrium solution (u^*, v^*) is globally asymptotically stable in the sense that every solution $u(x,t)$ of (1.1) satisfies*

$$\lim_{t\to\infty} (u(x,t), v(x,t)) = (u^*, v^*)$$

uniformly in $x \in \Omega$.

Proposition 2.2. *Suppose $\delta < 1$ and $\varepsilon_1 > 0$ small. There exists a sufficiently large constant $T > 0$ such that the solution u of (1.1) satisfies*

$$u \le \overline{u}_2(\varepsilon_1) \equiv \frac{1 - a - \frac{\delta}{\beta}\underline{u}_1 + \sqrt{\left(1 - a - \frac{\delta}{\beta}\underline{u}_1\right)^2 + 4a}}{2} + O(\varepsilon_1),$$

for $x \in \Omega$ and $t \ge T$, where

$$\underline{u}_1 = \frac{1 - a + \sqrt{(1-a)^2 + 4a(1 + \underline{w}_1(\varepsilon_1))}}{2(1 + \overline{w}_1(\varepsilon_1))},$$
$$\overline{w}_1 = \frac{\delta\overline{u}_1 + (\overline{u}_1)^2 - \beta\overline{u}_1 - a\beta}{2\beta\overline{u}_1} + \frac{\sqrt{(\beta\overline{u}_1 + a\beta - \delta\overline{u}_1 - (\overline{u}_1)^2)^2 + 4\beta\overline{u}_1(a(\delta - 1) + \overline{u}_1(\delta - 1 + a + \overline{u}_1))}}{2\beta\overline{u}_1},$$

and $\overline{u}_1 \equiv 1$.

Proof. Since $v > 0$, a direct computation gives

$$u_t - d\Delta u \le u(1-u), \quad \text{in } \Omega \times (0, \infty).$$

By a simple comparison argument and the well established fact that any positive solution of

$$\begin{cases} u_t - d\Delta u = u(1-u), & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty), \end{cases}$$

converges to the asymptotic stable equilibrium 1 as $t \to \infty$, we get that for all $\varepsilon_1 > 0$, there exists a constant $t_1 > 0$, such that

$$u(x,t) < \overline{u}_1(\varepsilon_1) \equiv 1 + \frac{\varepsilon_1}{5} \tag{2.2}$$

if $x \in \Omega$ and $t \geq t_1$. Thus

$$w_t - d\Delta w \leq w\left(\delta - 1 + \overline{u}_1(\varepsilon_1) + w\left(-\beta + \frac{\overline{u}_1(\varepsilon_1)}{a + \overline{u}_1(\varepsilon_1)w + \overline{u}_1(\varepsilon_1)}\right) + \frac{2d}{u}\nabla u \cdot \nabla w\right)$$

for $x \in \Omega$ and $t \geq t_1$.

It is clear that the following equation about $W(t)$

$$W_t = W\left(\delta - 1 + \overline{u}_1(\varepsilon_1) + W\left(-\beta + \frac{\overline{u}_1(\varepsilon_1)}{a + \overline{u}_1(\varepsilon_1)W + \overline{u}_1(\varepsilon_1)}\right)\right) \tag{2.3}$$

has three solutions:

$$\begin{aligned} W_0 &= 0, \\ W_{1,2} &= \frac{\delta\overline{u}_1(\varepsilon_1) + (\overline{u}_1(\varepsilon_1))^2 - \beta\overline{u}_1(\varepsilon_1) - a\beta}{2\beta\overline{u}_1(\varepsilon_1)} \\ &\quad \pm \frac{\sqrt{(\beta\overline{u}_1(\varepsilon_1) + a\beta - \delta\overline{u}_1(\varepsilon_1) - (\overline{u}_1(\varepsilon_1))^2)^2 + 4\beta\overline{u}_1(\varepsilon_1)(a + \overline{u}_1(\varepsilon_1))(\delta - 1 + \overline{u}_1(\varepsilon_1))}}{2\beta\overline{u}_1(\varepsilon_1)} \end{aligned} \tag{2.4}$$

It is clear that $W_1(t)$ is the unique asymptotically stable positive equilibrium point of (2.3), and $W_0(t) = 0$ is unstable. Thus, all positive solutions $W(t)$ of (2.3) converge to the unique positive asymptotically stable equilibrium point $W_1(t)$, since the trajectories of (2.3) cannot cross the x-axis. By a simple comparison argument, we get that there exists a positive constant $t_2 \geq t_1$ such that

$$\frac{v}{u} = w(x,t) \leq \overline{w}_1(\varepsilon_1) \equiv W_1 + \frac{\varepsilon_1}{5} \tag{2.5}$$

for all $x \in \Omega$ and $t \geq t_2$. Consequently, $v \leq \overline{w}_1(\varepsilon_1)u$, and

$$u_t - d\Delta u \geq u(1-u) - \frac{\overline{w}_1(\varepsilon_1)u}{\frac{a}{u} + 1 + \overline{w}_1(\varepsilon_1)} = \frac{u\left[(1-u)(\frac{a}{u} + 1 + \overline{w}_1(\varepsilon_1)) - \overline{w}_1(\varepsilon_1)\right]}{\frac{a}{u} + 1 + \overline{w}_1(\varepsilon_1)}$$

for all $x \in \Omega$ and $t \geq t_2$. The equation

$$(1-u)\left(\frac{a}{u} + 1 + \overline{w}_1(\varepsilon_1)\right) - \overline{w}_1(\varepsilon_1) = 0$$

has only one positive root

$$\hat{u} = \frac{1 - a + \sqrt{(1-a)^2 + 4a(1 + \overline{w}_1(\varepsilon_1))}}{2(1 + \overline{w}_1(\varepsilon_1))},$$

which is a stable equilibrium point of the ODE

$$u_t = \frac{u[(1-u)(\frac{a}{u} + 1 + \overline{w}_1(\varepsilon_1)) - \overline{w}_1(\varepsilon_1)]}{\frac{a}{u} + 1 + \overline{w}_1(\varepsilon_1)}. \tag{2.6}$$

Thus, all positive solution of (2.6) converge to $\hat{u}$, which implies that there exists $t_3 > t_2$ such that

$$u \geq \underline{u}_1(\varepsilon_1) \equiv \frac{1 - a + \sqrt{(1-a)^2 + 4a(1 + \overline{w}_1(\varepsilon_1))}}{2(1 + \overline{w}_1(\varepsilon_1))} - \frac{\varepsilon_1}{5} \tag{2.7}$$

for all $x \in \Omega$ and $t \geq t_3$. On the other hand, by using the second equation of (1.1), we get

$$v_t - d\Delta v \geq v\left(\delta - \beta\frac{v}{\underline{u}_1(\varepsilon_1)}\right)$$

for all $x \in \Omega$ and $t \geq t_3$. Thus, there exists a constant $t_4 > t_3$ such that

$$v \geq \underline{v}_1(\varepsilon_1) = \frac{\delta \underline{u}_1(\varepsilon_1)}{\beta} - \frac{\varepsilon_1}{5} \tag{2.8}$$

for all $x \in \Omega$ and $t \geq t_4$. Substituting $v \geq \underline{v}_1(\varepsilon_1)$ into the first equation of (1.1), we get

$$u_t - d\Delta u \leq u - u^2 - \frac{u\underline{v}_1(\varepsilon_1)}{a + u + \underline{v}_1(\varepsilon_1)} = \frac{u[(1-u)(a + u + \underline{v}_1(\varepsilon_1)) - \underline{v}_1(\varepsilon_1)]}{a + u + \underline{v}_1(\varepsilon_1)}.$$

The quadratic equation

$$(1-u)(a + u + \underline{v}_1(\varepsilon_1)) - \underline{v}_1(\varepsilon_1) = 0$$

has only one positive root

$$\hat{u} = \frac{1 - a - u\underline{v}_1(\varepsilon_1) + \sqrt{(1 - a - u\underline{v}_1(\varepsilon_1))^2 + 4a}}{2}. \tag{2.9}$$

By comparison principle then yields there exists $t_5 > t_4$ such that if $t \geq t_5$,

$$u \leq \overline{u}_2(\varepsilon_1) \equiv \hat{u} + \frac{\varepsilon_1}{5}. \tag{2.10}$$

Simple computation using (2.2), (2.4), (2.5) and (2.7)–(2.10) shows the expression of $\overline{u}_2(\varepsilon_1)$ and that of $\underline{u}_1(\varepsilon_1)$ and $\overline{w}_1(\varepsilon_1)$ are valid. This completes the proof. □

By repeating the above procedure, for any positive integer n, there exists T sufficiently large such that when $t \geq T$,

$$u \leq \overline{u}_{n+1}(\varepsilon_1) \equiv \frac{1 - a - u\underline{v}_n(\varepsilon_1) + \sqrt{(1 - a - u\underline{v}_n(\varepsilon_1))^2 + 4a}}{2} + \frac{\varepsilon_1}{5},$$
$$u \geq \underline{u}_n(\varepsilon_1) \equiv \frac{1 - a + \sqrt{(1-a)^2 + 4a(1 + \overline{w}_n(\varepsilon_1))}}{2(1 + \overline{w}_n(\varepsilon_1))} - \frac{\varepsilon_1}{5}$$

uniformly in Ω, where

$$\underline{v}_n(\varepsilon_1) = \frac{\delta}{\beta}\underline{u}_n(\varepsilon_1) - \frac{\varepsilon_1}{5},$$

$$\begin{aligned}\overline{w}_n = {} & \frac{\delta \overline{u}_n(\varepsilon_1) + (\overline{u}_n(\varepsilon_1))^2 - \beta \overline{u}_n(\varepsilon_1) - a\beta}{2\beta \overline{u}_n(\varepsilon_1)} \\ & + \frac{\sqrt{(\beta \overline{u}_n(\varepsilon_1) + a\beta - \delta \overline{u}_n(\varepsilon_1) - (\overline{u}_n(\varepsilon_1))^2)^2 + 4\beta \overline{u}_n(\varepsilon_1)(a + \overline{u}_n(\varepsilon_1))(\delta - 1 + \overline{u}_n(\varepsilon_1))}}{2\beta \overline{u}_n(\varepsilon_1)}.\end{aligned}$$

When $\varepsilon_1 = 0$, we have

$$\begin{aligned}\overline{u}_{n+1} &= \frac{1 - a - \frac{\delta}{\beta}\underline{u}_n + \sqrt{\left(1 - a - \frac{\delta}{\beta}\underline{u}_n\right)^2 + 4a}}{2}, \\ \underline{u}_n &= \frac{1 - a + \sqrt{(1-a)^2 + 4a(1 + \overline{w}_n)}}{2(1 + \overline{w}_n)}, \\ \underline{v}_n &= \frac{\delta}{\beta}\underline{u}_n\end{aligned}$$

and $\overline{u}_1 = 1$, $\overline{u}_1 > u^*, \underline{u}_1 < u^*$. Direct calculation gives

$$\begin{aligned}\left(1-a-\frac{\delta}{\beta}\underline{u}_1\right)^2 + 4a &= (1-a)^2 + \frac{\delta^2\underline{u}_1{}^2}{\beta^2} - 2(1-a)\frac{\delta}{\beta}\underline{u}_1 + 4a \\ &< (1+a)^2 + \frac{\delta^2\underline{u}_1{}^2}{\beta^2} + 2(1+a)\frac{\delta}{\beta}\underline{u}_1 + \frac{4a\delta\underline{u}_1}{\beta} \\ &= \left(1+a+\frac{\delta}{\beta}\underline{u}_1\right)^2,\end{aligned}$$

thus,

$$\overline{u}_2 = \frac{1-a-\frac{\delta}{\beta}\underline{u}_1 + \sqrt{(1-a-\frac{\delta}{\beta}\underline{u}_1)^2+4a}}{2} < 1 = \overline{u}_1.$$

Then, we can obtain that $\{\overline{u}_n\}$ is a decreasing sequence by induction. Similarly, since

$$\begin{aligned}\overline{w}_n = {} & \frac{\delta}{2\beta} + \frac{\overline{u}_n}{2\beta} - \frac{1}{2} - \frac{a}{2\overline{u}_n} \\ & + \sqrt{\left(\frac{\delta}{2\beta} + \frac{\overline{u}_n}{2\beta} - \frac{1}{2} - \frac{a}{2\overline{u}_n}\right)^2 + \frac{1}{\beta}\left(\frac{a(\delta-1)}{\overline{u}_n} + \overline{u}_n + a + \delta - 1\right)},\end{aligned}$$

and

$$\underline{u}_n = \frac{1}{2}\left(\frac{1-a}{1+\overline{w}_n} + \sqrt{\left(\frac{1+a}{1+\overline{w}_n}\right)^2 + 4a}\right),$$

where $\delta < 1$, we obtain that $\{\overline{w}_n\}$ is a decreasing sequence and $\{\underline{u}_n\}$ is an increasing sequence. Thus, under the assumption of Theorem 2.1, we have

$$\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} \underline{u}_n = u^*.$$

Consequently, we have

$$\lim_{n\to\infty} \overline{v}_n = \lim_{n\to\infty} \underline{v}_n = v^*.$$

Now, we show $\lim_{t\to\infty}(u,v) = (u^*,v^*)$, uniformly in Ω.

Proof of Theorem 2.1. for any $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that when $n > N$,

$$|\overline{u}_n - u^*| + |\underline{u}_n - u^*| < \frac{\varepsilon}{4}. \tag{2.11}$$

Choose $\varepsilon_1 > 0$ sufficiently small such that

$$|\overline{u}_N(\varepsilon_1) - \overline{u}_N| + |\underline{u}_N(\varepsilon_1) - \underline{u}_N| < \frac{\varepsilon}{4}. \tag{2.12}$$

and the same to $\overline{v}_n(\varepsilon_1)$, $\overline{v}_n$, $\underline{v}_n(\varepsilon_1)$, $\underline{v}_n$ and v^*. Furthermore, there exists $t_M \gg 1$ such that when $t \geq t_M$,

$$\underline{u}_N(\varepsilon_1) \leq u(x,t) \leq \overline{u}_N(\varepsilon_1) \quad \text{in } \Omega.$$

Hence, by (2.11) and (2.12), when $t \geq t_M$,

$$|u(x,t) - u^*| < \varepsilon \quad \text{in } \Omega.$$

This proves $\lim_{t\to\infty} u(x,t) = u^*$ uniformly in Ω. Similarly, $\lim_{t\to\infty} v(x,t) = v^*$ uniformly in Ω. This finished the proof of Theorem 2.1. □

Acknowledgements

The second author's work was partially supported by NSF of China (11371179, 11401513) and by China Postdoctoral Science Foundation funded project (2014M560546). We would like to thank the referee for helpful comments and suggestions.

References

[1] J. R. Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, *J. Anim. Ecol.* **44**(1975), No. 1, 331–340. url

[2] S. S. Chen, J. P. Shi, Global stability in a diffusive Holling–Tanner predator–prey model, *Appl. Math. Lett.* **25**(2012), No. 3, 614–618. MR2856044; url

[3] D. L. DeAngelis, R. A. Goldstein, R. V. O'Neill, A model for tropic interaction, *Ecology* **56**(1975), No. 4, 881–892. url

[4] Y.-H. Fan, W.-T. Li, Global asymptotic stability of a ratio-dependent predator–prey system with diffusion, *J. Comput. Appl. Math.* **188**(2006), No. 2, 205–227. MR2201577; url

[5] Y.-H. Fan, W.-T. Li, Permanence in delayed ratio-dependent predator–prey models with monotonic functional responses, *Nonlinear Anal. Real World Appl.* **8**(2007), No. 2, 424–434. MR2289565; url

[6] Y.-H. Fan, L.-L. Wang, On a generalized discrete ratio-dependent predator–prey system, *Discrete Dyn. Nat. Soc.* **2009**, Art. ID 653289, 22 pp. MR2556336; url

[7] Y.-H. Fan, L.-L. Wang, Multiplicity of periodic solutions for a delayed ratio-dependent predator–prey model with Holling type III functional response and harvesting terms, *J. Math. Anal. Appl.* **365**(2010), No. 2, 525–540. MR2587056; url

[8] Y.-H. Fan, L.-L. Wang, Average conditions for the permanence of a bounded discrete predator–prey system, *Discrete Dyn. Nat. Soc.* **2013**, Art. ID 508686, 5 pp. MR3091244; url

[9] G. Huang, W. Ma, Y. Takeuchi, Global properties for virus dynamics model with Beddington–DeAngelis functional response, *Appl. Math. Lett.* **22**(2009), No. 11, 1690–1693. MR2569065; url

[10] G. Huang, W. Ma, Y. Takeuchi, Global analysis for delay virus dynamics model with Beddington-DeAngelis functional response, *Appl. Math. Lett.* **24**(2011), No. 7, 1199–1203. MR2784182; url

[11] N.-W. Liu, T.-T. Kong, Dynamics of a predator–prey system with Beddington–DeAngelis functional response and delays, *Abstr. Appl. Anal.* **2014**, Art. ID 930762, 8 pp. MR3216082; url

[12] P. H. Leslie, Some further notes on the use of matrices in population mathematics, *Biometrika* **35**(1948), No. 3–4, 213–245. MR0027991; url

[13] P. H. Leslie, J. C. Gower, The properties of a stochastic model for the predator–prey type of interaction between two species, *Biometrika* **47**(1960), No. 3–4, 219–234. MR0122603; url

[14] R. M. May, *Stability and complexity in model ecosystems*, Princeton University Press, 1973.

[15] R. Peng, M. Wang, Positive steady states of the Holling–Tanner prey–predator model with diffusion, *Proc. Roy. Soc. Edinburgh Sect. A* **135**(2005), No. 1, 149–164. MR2119846; url

[16] R. Peng, M. Wang, Global stability of the equilibrium of a diffusive Holling–Tanner prey–predador model, *Appl. Math. Lett.* **20**(2007), No. 6, 664–670. MR2314410; url

[17] Y. Qi, Y. Zhu, The study of global stability of a diffusive Holling–Tanner predator–prey model, *Appl. Math. Lett.* **57**(2016), 132–138. MR3464119; url

[18] H.- B. Shi, Y. Li, Global asymptotic stability of a diffusive predator–prey model with ratio-dependent functional response, *Appl. Math. Comput.* **25**(2015), 71–77. MR3285519; url

[19] X. Song, A. Neumann, Global stability and periodic solution of the viral dynamics, *J. Math. Anal. Appl.* **329**(2007), No. 1, 281–297. MR2306802; url

[20] J. T. Tanner, The stability and the intrinsic growth rates of prey and predator populations, *Ecology* **56**(1975), No. 56, 855–867. url