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Positive solutions for a class of singular elliptic systems

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Abstract. In this paper, we mainly study the existence, boundary behavior and uniqueness of solutions for the following singular elliptic systems involving weights $-\triangle u = w(x)u^{-p}v^{-q}, -\triangle v = \lambda(x)u^{-r}v^{-s}, u > 0, v > 0,\ x \in \Omega,\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$, $p, s \geq 0$, $q, r > 0$ and the weight functions $w(x), \lambda(x) \in C^\alpha(\bar{\Omega})$ which are positive in Ω and may be blow-up on the boundary.

Keywords: singular elliptic systems, Dirichlet problems, existence, boundary behavior, uniqueness.

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1 Introduction

In this paper, we mainly consider the existence, boundary behavior and uniqueness of solutions for the following singular elliptic systems involving weights

$$\begin{cases} -\Delta u = w(x)u^{-p}v^{-q}, & \text{in } \Omega, \\ -\Delta v = \lambda(x)u^{-r}v^{-s}, & \text{in } \Omega, \\ u > 0,\ v > 0,\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$, $p, s \geq 0$ $q, r > 0$. Assume w, λ satisfies

($\mathbf{H_0}$) $w, \lambda \in C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$, are positive in Ω, and there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ and positive constants c_1, c_2 such that

$$\lim_{d(x)\to 0} \frac{w(x)}{d(x)^{\gamma_1}} = c_1, \qquad \lim_{d(x)\to 0} \frac{\lambda(x)}{d(x)^{\gamma_2}} = c_2.$$

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The first motivation for the study of problem (1.1) comes from the so-called Lane–Emden equation (see [4,5])

$$-\Delta u = u^p \quad \text{in } B_R(0),\ R > 0.$$

Systems of type (1.1) with $p, s \le 0$ and $q, s < 0$ have received considerably attention in the last decade (see, e.g., [1,3,15–18,20,23] and the references therein). It has been shown that for such range of exponents system (1.1) has a rich mathematical structure. Various techniques such as moving plane method, Pohozaev-type identities, rescaling arguments have been developed and suitably adapted to deal with (1.1) in this case.

Recently, there has been some interest in systems of type (1.1) where not all the exponents are negative. Ghergu [8] first established the existence, non-existence, C^1-regularity and uniqueness of classical solutions (in $C^2(\Omega) \cap C(\bar{\Omega})$) in terms of p, q, r and s.

Later, Zhang [21] also study the existence, boundary behavior and uniqueness of solutions for problem (1.1), which results are obtained in a range of p, q, r, s different from those in [8].

In [13,14], Lee et al. studied the existence of solutions for the singular systems

$$\begin{cases} -\Delta_p u = \lambda(f_1(u,v) - u^{-\gamma_1}), & \text{in } \Omega, \\ -\Delta_q u = \lambda(f_2(u,v) - u^{-\gamma_2}), & \text{in } \Omega, \\ u > 0,\ v > 0,\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\gamma_i \in (0,1)$, $f_i \in C([0,\infty) \times [0,\infty))$, f_i is non-decreasing for both u and v, $i = 1,2$, $\lambda > 0$, and $\Delta_r u := \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, $r = p(> 1)$, $q(> 1)$.

Inspired by the above works, in this paper, we wish to further deal with the existence, boundary behavior and uniqueness of solutions to problem (1.1) under appropriate conditions on weight function $w(x)$ and $\lambda(x)$, which have a precise asymptotic behavior near $\partial\Omega$.

Our main results are summarized as follows.

Theorem 1.1 (Existence)**.** *Let $-2 < \gamma_1 < p - 1, -2 < \gamma_2 < s - 1$ and p, q, r, s be such that one of the following conditions hold:*

(H$_1$)
$$(1+p)(1+s) - qr > 0, \qquad \frac{2+\gamma_1}{2+\gamma_2} > \max\left\{\frac{q}{1+s}, \frac{r}{1+p}\right\},$$
$$p + \frac{q(2+\gamma_2 - r)}{1+s} > 1 + \gamma_1, \quad \text{and} \quad s + \frac{r(2+\gamma_2 - q)}{1+p} > 1 + \gamma_1.$$

(H$_2$)
$$(1+p)(1+s) - qr < 0, \qquad \frac{2+\gamma_1}{2+\gamma_2} < \min\left\{\frac{q}{1+s}, \frac{r}{1+p}\right\},$$
$$p + \frac{q(2+\gamma_2 - r)}{1+s} < 1 + \gamma_1, \quad \text{and} \quad s + \frac{r(2+\gamma_2 - q)}{1+p} < 1 + \gamma_1.$$

Then system (1.1) *has at least one classical solution (u, v) satisfying*

$$m_0 d(x) \le u(x) \le M_0 (d(x))^\alpha, \qquad x \in \bar{\Omega}, \tag{1.3}$$

$$m_0 d(x) \le v(x) \le M_0 (d(x))^\beta, \qquad x \in \bar{\Omega}, \tag{1.4}$$

where m_0 and M_0 are positive constants, $d(x) = \operatorname{dist}(x, \partial\Omega)$ and

$$\alpha = \frac{(2+\gamma_1)(1+s) - q(2+\gamma_2)}{(1+p)(1+s) - qr}, \qquad \beta = \frac{(2+\gamma_1)(1+p) - r(2+\gamma_2)}{(1+p)(1+s) - qr}. \tag{1.5}$$

Theorem 1.2 (Exact boundary behavior)**.** *Let p, q, r, s satisfy (H_1) and the following conditions:*

(H3) $$p > 0, \qquad p + q > 1 + \gamma_1 \quad \textit{and} \quad q < 2 + \gamma_1;$$

(H4) $$s > 0, \qquad s + r > 1 + \gamma_2 \quad \textit{and} \quad r < 2 + \gamma_2.$$

Then for any classical solution (u, v) of system (1.1)

$$\lim_{d(x)\to 0} \frac{u(x)}{(d(x))^{\alpha}} = \left(c_1^{1+s} c_2^{-q} \frac{(\beta(1-\beta))^q}{(\alpha(1-\alpha))^{1+s}} \right)^{1/((1+p)(1+s)-qr)},$$
$$\lim_{d(x)\to 0} \frac{v(x)}{(d(x))^{\beta}} = \left(c_1^{-r} c_2^{1+q} \frac{(\beta(1-\beta))^r}{(\alpha(1-\alpha))^{1+p}} \right)^{1/((1+p)(1+s)-qr)},$$
$$\lim_{d(x)\to 0} \frac{\nabla u(x)\nu(x)}{(d(x))^{\alpha-1}} = -\alpha \left(c_1^{1+s} c_2^{-q} \frac{(\beta(1-\beta))^q}{(\alpha(1-\alpha))^{1+s}} \right)^{1/((1+p)(1+s)-qr)},$$
$$\lim_{d(x)\to 0} \frac{\nabla v(x)\nu(x)}{(d(x))^{\beta-1}} = -\beta \left(c_1^{-r} c_2^{1+q} \frac{(\beta(1-\beta))^r}{(\alpha(1-\alpha))^{1+p}} \right)^{1/((1+p)(1+s)-qr)},$$

where $\nu(x)$ is the outer unit normal vector to $\partial\Omega$ at x.

Theorem 1.3 (Uniqueness)**.** *Under the conditions of Theorem 1.2, system (1.1) has a unique classical solution (u, v).*

Corollary 1.4 (Existence)**.** *Let $p = q = r = s = \text{constant} =: \mathcal{C}$ and $-2 < \gamma_1, \gamma_2 < \mathcal{C} - 1$. If the following conditions holds:*

(H5) $$(\gamma_2 - \gamma_1)\mathcal{C} < 2 + \gamma_1, \quad \textit{and} \quad (2 + \gamma_2 + \gamma_1)\mathcal{C} > 1 + \gamma_1,$$

then system (1.1) has at least one classical solution (u, v) satisfying

$$m_0 d(x) \le u(x) \le M_0 (d(x))^{\alpha}, \qquad x \in \bar{\Omega}, \tag{1.6}$$

$$m_0 d(x) \le v(x) \le M_0 (d(x))^{\alpha}, \qquad x \in \bar{\Omega}, \tag{1.7}$$

where m_0 and M_0 are positive constants, $d(x) = \operatorname{dist}(x, \partial\Omega)$ and

$$\alpha = \frac{2 + \gamma_1 + \mathcal{C}(\gamma_1 - \gamma_2)}{1 + 2\mathcal{C}}. \tag{1.8}$$

Corollary 1.5 (Exact boundary behavior)**.** *Let p, q, r, s satisfy the assumption in Corollary 1.4 and the following conditions:*

(H6) $$\mathcal{C} > 0, \qquad \mathcal{C} > \max\left\{ \frac{1+\gamma_1}{2}, \frac{1+\gamma_2}{2} \right\} \quad \textit{and} \quad \mathcal{C} < \max\left\{ 2 + \gamma_1, 2 + \gamma_2 \right\}.$$

Then for any classical solution (u, v) of system (1.1)

$$\lim_{d(x)\to 0} \frac{u(x)}{(d(x))^{\alpha}} = \left(\frac{c_1^{1+\mathcal{C}} c_2^{-\mathcal{C}}}{\alpha(1-\alpha)} \right)^{1/(1+2\mathcal{C})},$$
$$\lim_{d(x)\to 0} \frac{v(x)}{(d(x))^{\alpha}} = \left(\frac{c_1^{-\mathcal{C}} c_2^{1+\mathcal{C}}}{\alpha(1-\alpha)} \right)^{1/(1+2\mathcal{C})},$$

$$\lim_{d(x)\to 0} \frac{\nabla u(x)\nu(x)}{(d(x))^{\alpha-1}} = -\alpha \left(\frac{c_1^{1+\mathcal{C}} c_2^{-\mathcal{C}}}{\alpha(1-\alpha)} \right)^{1/(1+2\mathcal{C})},$$
$$\lim_{d(x)\to 0} \frac{\nabla v(x)\nu(x)}{(d(x))^{\alpha-1}} = -\alpha \left(\frac{c_1^{-\mathcal{C}} c_2^{1+\mathcal{C}}}{\alpha(1-\alpha)} \right)^{1/(1+2\mathcal{C})},$$

where $\nu(x)$ is the outer unit normal vector to $\partial\Omega$ at x.

The outline of this paper is as follows. In Section 2, we give some preliminary results that will be used in the following sections. Theorems 1.1–1.3 are proved in next sections.

2 Some preliminary results

In this section, we collect some useful results about the following singular Dirichlet problem

$$-\triangle w = (d(x))^{-\sigma} w^{-\gamma}, \quad w > 0,\ x \in \Omega,\ w|_{\partial\Omega} = 0, \tag{2.1}$$

where $\sigma \in \mathbb{R}$ and $\gamma > 0$.

Problem (2.1) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials, and was discussed and extended in a number of works; see, for instance, [2, 6, 9, 11, 12, 19, 22] and the references therein.

Definition 2.1. A function $\bar{w}$ is called a super-solution of problem (2.1) if $\bar{w} \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$-\triangle \bar{w} \geq (d(x))^{-\sigma} \bar{w}^{-\gamma}, \quad \bar{w} > 0,\ x \in \Omega,\ \bar{w}|_{\partial\Omega} \geq 0. \tag{2.2}$$

Definition 2.2. A function $\underline{w}$ is called a sub-solution of problem (2.1) if $\underline{w} \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$-\triangle \underline{w} \leq (d(x))^{-\sigma} \underline{w}^{-\gamma}, \quad \underline{w} > 0,\ x \in \Omega,\ \underline{w}|_{\partial\Omega} \leq 0. \tag{2.3}$$

Since Ω is C^2, we see by Lemma 14.16 in [10] that d is C^2 in a neighborhood of $\partial\Omega$. Redefining $d(x)$ outside this neighborhood if necessary, we can always assume that $d \in C^2(\bar{\Omega})$.

Let (λ_1, φ_1) be the first eigenvalue/eigenfunction of

$$-\triangle \varphi = \lambda\varphi, \quad \varphi > 0,\ x \in \Omega,\ \varphi|_{\partial\Omega} = 0. \tag{2.4}$$

It is well known that $\lambda_1 > 0$ and $\varphi_1 \in C^2(\bar{\Omega})$. Furthermore, using the smoothness of Ω and normalizing φ_1 with a suitable constant, we can assume

$$c_0 d(x) \leq \varphi_1(x) \leq d(x), \qquad x \in \Omega \tag{2.5}$$

for some $0 < c_0 < 1$.

By Hopf's boundary point lemma, we have $\frac{\partial\varphi_1(x)}{\partial\nu} > 0$, $\forall x \in \Omega$. Hence,

$$|\nabla\varphi_1| > 0 \quad \text{near } \partial\Omega$$

and

$$C_\mu = \max_{x\in\bar{\Omega}}(\lambda_1\varphi_1^2(x) + (1-\mu)|\nabla\varphi_1|^2), \tag{2.6}$$
$$c_\mu = \min_{x\in\bar{\Omega}}(\lambda_1\varphi_1^2(x) + (1-\mu)|\nabla\varphi_1|^2), \tag{2.7}$$

are well defined with $c_\mu > 0$ for $\mu \in (0,1)$.

Lemma 2.3 (Lemma 3 in [2] and Proposition 2.1 in [8]). *If problem (2.1) has a super-solution $\bar{w}_{\gamma,\sigma}$ and a sub-solution $\underline{w}_{\gamma,\sigma}$, then*

(**i**) $\underline{w}_{\gamma,\sigma} \le \bar{w}_{\gamma,\sigma}$ *in* $\bar{\Omega}$;

(**ii**) *problem (2.1) have a unique solution $W_{\gamma,\sigma} \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying*

$$\underline{w}_{\gamma,\sigma} \le W_{\gamma,\sigma} \le \bar{w}_{\gamma,\sigma} \quad \text{in } \Omega.$$

Lemma 2.4 (Theorem 1.2 in [22]).

(**i**) *If $\sigma \ge 2$, then problem (2.1) has no classical solution;*

(**ii**) *If $\sigma \in (1-\gamma, 2)$, then problem (2.1) has a unique classical solution $W_{\gamma,\sigma}$ satisfying*

$$c_\tau \varphi_1^\tau(x) \le W_{\gamma,\sigma} \le C_\tau \varphi_1^\tau(x), \qquad x \in \Omega,$$

where C_τ and c_τ are as in (2.6) and (2.7),

$$\tau = \frac{2-\sigma}{1+\gamma}. \tag{2.8}$$

Lemma 2.5 (Lemma 2.3 in [21]). *Let $\lambda > 0, \sigma < 2, \gamma > 0$ and let $\bar{w}_\lambda \in C^2(\Omega)$ verify*

$$-\triangle \bar{w}_\lambda \ge \lambda (d(x))^{-\sigma} \bar{w}_\lambda^{-\gamma}, \quad \bar{w}_\lambda > 0,\ x \in \Omega,\ \bar{w}_\lambda|_{\partial\Omega} = 0,$$

then

$$\bar{w}_\lambda(x) \ge \lambda^{1/(1+\gamma)} W_{\gamma,\sigma}, \qquad x \in \Omega.$$

Similarly, if $\underline{w}_\lambda \in C^2(\Omega)$ satisfies

$$-\triangle \underline{w}_\lambda \le \lambda (d(x))^{-\sigma} \underline{w}_\lambda^{-\gamma}, \quad \underline{w}_\lambda > 0,\ x \in \Omega,\ \underline{w}_\lambda|_{\partial\Omega} = 0,$$

then

$$\underline{w}_\lambda(x) \le \lambda^{1/(1+\gamma)} W_{\gamma,\sigma}, \qquad x \in \Omega.$$

The following lemma is an extension of Lemmas 2.4 and 2.5 to the case where Ω is a half-space $D = \{x \in \mathbb{R}^N : x_1 > 0\}$ (for a point $x \in \mathbb{R}^N$ we write $x = (x_1, x')$, with $x' \in \mathbb{R}^{N-1}$). This result is useful when dealing with the boundary estimates for solutions to system (1.1).

Lemma 2.6 (Lemma 2.4 in [21]). *Let $C_0 > 0, \gamma > 0, \sigma \in (1-\gamma, 2)$ and $\bar{w}, \underline{w} \in C^2(D)$ verify*

$$-\triangle \bar{w} \ge C_0 x_0^{-\sigma} \bar{w}^{-\gamma}, \quad (\text{resp.} -\triangle \underline{w} \le C_0 x_0^{-\sigma} \underline{w}^{-\gamma}) \quad \text{in } D,$$

and

$$\bar{w}(x) \ge C x_1^\tau \qquad (\underline{w}(x) \le C x_1^\tau),$$

where C is positive constants and τ is in (2.8). Then

$$\bar{w}(x) \ge A x_1^\tau \quad (\text{resp. } \underline{w}(x) \le A x_1^\tau), \qquad x \in D, \tag{2.9}$$

where

$$A = \left(\frac{C_0}{\tau(1-\tau)} \right)^{1/(1+\gamma)}.$$

3 Existence and estimates of solutions

In this section, we quote the sub-supersolution method in [13].

Consider the more general systems

$$\begin{cases} -\Delta u = h_1(x,u,v), & \text{in } \Omega, \\ -\Delta v = h_2(x,u,v), & \text{in } \Omega, \\ u > 0,\ v > 0,\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $h_i : \Omega \times (0,\infty) \times (0,\infty) \to \mathbb{R}$ is continuous for $i = 1,2$.

Definition 3.1. A pair of function $(\bar{u},\bar{v}) : \bar{\Omega} \to \mathbb{R}^2$ is called a super-solution of system (3.2) if $\bar{u},\bar{v} \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{cases} -\Delta \bar{u} \geq h_1(x,\bar{u},\bar{v}), & \text{in } \Omega, \\ -\Delta \bar{v} \geq h_2(x,\bar{u},\bar{v}), & \text{in } \Omega, \\ \bar{u} > 0,\ \bar{v} > 0,\ \bar{u}|_{\partial\Omega} = \bar{v}|_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

Definition 3.2. A pair of function $(\underline{u},\underline{v}) : \bar{\Omega} \to \mathbb{R}^2$ is called a sub-solution of system (3.2) if $\underline{u},\underline{v} \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{cases} -\Delta \underline{u} \leq h_1(x,\underline{u},\underline{v}), & \text{in } \Omega, \\ -\Delta \underline{v} \leq h_2(x,\underline{u},\underline{v}), & \text{in } \Omega, \\ \underline{u} > 0,\ \underline{v} > 0,\ \underline{u}|_{\partial\Omega} = \underline{v}|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

Lemma 3.3 (The extension of Lemma 1.8 in [13]). *If $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ in $\bar{\Omega}$, then the system (3.2) has at least one solution (u,v) satisfying $u,v \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$ on $\bar{\Omega}$.*

Proof of Theorem 1.1. By (H_0), we deduce that there exist positive constants w_i, Λ_i $(i = 1,2)$ such that $w_1 d(x)^{\gamma_1} \leq w(x) \leq w_2 d(x)^{\gamma_1}$ and $\Lambda_1 d(x)^{\gamma_2} \leq \Lambda(x) \leq \Lambda_2 d(x)^{\gamma_2}$ in Ω.

Let $\underline{u} = \underline{v} = m_0\varphi_1$, where

$$m_0 = \min\left\{ (\lambda_1^{-1}w_1)^{\frac{1}{1+p+q}} \left(\max_{x\in\bar{\Omega}} \varphi_1(x) \right)^{-\frac{1+p+q-\gamma_1}{1+p+q}}, (\lambda_1^{-1}\Lambda_1)^{\frac{1}{1+p+q}} \left(\max_{x\in\bar{\Omega}} \varphi_1(x) \right)^{-\frac{1+p+q-\gamma_2}{1+p+q}} \right\}.$$

By a direct calculation, one can see that (u,v) is a sub-solution of system (1.1).

By (H_1) or (H_2) and the definitions of α,β, we see that $\alpha,\beta \in (0,1)$.

Let

$$\underline{u} = M_0\varphi_1^{\alpha}, \qquad \underline{v} = M_0\varphi_1^{\beta},$$

where

$$M_0 = \max\left\{ (w_2^{-1}\alpha c_\alpha)^{-1/(1+p+q)}, (\Lambda_2^{-1}\beta c_\beta)^{-1/(1+r+s)}, m_0\left(\max_{x\in\bar{\Omega}} \varphi_1(x) \right)^{1-\alpha}, m_0\left(\max_{x\in\bar{\Omega}} \varphi_1(x) \right)^{1-\beta} \right\}$$

and c_α and c_β are as in (2.7).

By a direct calculation, one can see that

$$\begin{aligned} -\Delta\bar{u} &= M_0\alpha\varphi_1^{\alpha-2}\left(\lambda_1\varphi_1^2 + (1-\alpha)|\nabla\varphi_1|^2\right) \\ &\geq w(x)M_0^{-(p+q)}\varphi_1^{-(p\alpha+q\beta)} = w(x)\underline{u}^{-p}\underline{v}^{-q} \quad \text{in } \Omega \end{aligned}$$

$$-\Delta\bar{v} = M_0\beta\varphi_1^{\beta-2}\left(\lambda_1\varphi_1^2 + (1-\beta)\,|\nabla\varphi_1|^2\right)$$
$$\geq \lambda(x)M_0^{-(r+s)}\varphi_1^{-(r\alpha+s\beta)} = \lambda(x)\underline{u}^{-r}\underline{v}^{-s} \quad \text{in } \Omega$$

and

$$\bar{u} \geq \underline{u} \quad \text{and} \quad \bar{v} \geq \underline{v} \quad \text{in } \Omega$$

Thus the result follows by Lemma 3.3. □

In the following, by using an iteration method, we consider the global estimates of solutions.

Lemma 3.4. *Let (u, v) be any classical solution of system (1.1), $-2 < \gamma_1 < p - 1$ and $-2 < \gamma_2 < s - 1$. Then there exists a constant $\tilde{c}_0 > 0$ such that*

$$u(x) > \tilde{c}_0 d(x) \quad \textit{and} \quad v(x) > \tilde{c}_0 d(x) \quad \textit{in } \Omega.$$

Proof. Since $-\triangle u \geq C(d(x))^{\gamma_1}u^{-p}$ for some constant $C > 0$, combined with Lemma 2.5, we can find a suitable constant $\tilde{c}_0 > 0$ such that $u(x) > \tilde{c}_0 d(x)$ and similarly $v(x) \geq \tilde{c}_0 d(x)$ in Ω, where $\tilde{c}_0$ is a positive constant. □

Lemma 3.5. *Under the conditions of Theorem 1.2, for any classical solution (u, v)*

$$A(d(x))^{\alpha} \leq u(x) \leq B(d(x))^{\alpha} \quad \textit{and} \quad A(d(x))^{\beta} \leq u(x) \leq B(d(x))^{\beta}, \qquad x \in \Omega, \tag{3.4}$$

where A and B are positive constants, α and β are in Theorem 1.1.

Proof. Let (H_3) hold. By (2.5) and Lemma 3.4, $v(x) \geq C_0 d(x), x \in \Omega$, where $C_0 = \min\{c_0, c_1\}$. Then

$$-\triangle u \leq w_2(d(x))^{\gamma_1}C_0^{-q}(d(x))^{-q}u^{-p}, \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0.$$

By (H_3), Lemmas 2.4 and 2.5, we see that

$$u \leq a_0 C_{\alpha_0}(d(x))^{\alpha_0}, \qquad x \in \Omega,$$

where C_{α_0} is in (2.7) and

$$a_0 = \left(w_2 C_0^{-q}\right)^{1/(1+p)}, \qquad \alpha_0 = \frac{(2+\gamma_1) - q}{1+p} \in (0, 1).$$

Inserting this into the second equation in system (1.1), we have

$$-\triangle v \geq \Lambda_1(d(x))^{\gamma_2}\left(a_0 C_{\alpha_0}\right)^{-r}\left(d(x)\right)^{-r\alpha_0}v^{-s}, \quad v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0.$$

By (H_1), (H_3) and $\alpha_0 \in (0, 1)$, we have

$$r\alpha_0 < 2 + \gamma_2, \qquad s + r\alpha_0 = s + r\frac{(2+\gamma_1) - q}{1+p} > 1 + \gamma_2.$$

Then Lemmas 2.4 and 2.5 give that

$$v \geq C_0^{\beta_0} c_{\beta_0} b_0 (d(x))^{\beta_0}, \qquad x \in \Omega,$$

where C_{β_0} is in (2.7) and

$$b_0 = \left(\Lambda_1\left(a_0 C_{\beta_0}\right)^{-r}\right)^{1/(1+s)}, \qquad \beta_0 = \frac{(2+\gamma_2) - r\alpha_0}{1+s} \in (0, 1).$$

Proceeding inductively, we obtain

$$u \le a_n C_{\alpha_n} (d(x))^{\alpha_n}, \qquad v \ge C_0^{\beta_n} c_{\beta_n} b_n (d(x))^{\beta_n}, \qquad x \in \Omega, \tag{3.5}$$

where $n = 0, 1, \ldots,$

$$\begin{aligned} \alpha_n &= \frac{(2+\gamma_1) - q\beta_{n-1}}{1+p} \\ &= \frac{(2+\gamma_1)(1+s) - q(2+\gamma_2)}{(1+p)(1+s)} + \frac{qr}{(1+p)(1+s)} \alpha_{n-1} \in (0,1), \end{aligned} \tag{3.6}$$

$$\begin{aligned} \beta_n &= \frac{(2+\gamma_2) - r\alpha_n}{1+s} \\ &= \frac{(2+\gamma_1)(1+p) - r(2+\gamma_2)}{(1+p)(1+s)} + \frac{qr}{(1+p)(1+s)} \beta_{n-1} \in (0,1), \end{aligned} \tag{3.7}$$

$$\begin{aligned} a_n &= w_2^{1/(1+p)} \left(C_0^{\beta_{n-1}} C_{\beta_{n-1}} b_{n-1} \right)^{-q/(1+p)} \\ &= w_2^{1/(1+p)} \Lambda_1^{-q/(1+p)(1+s)} \left(C_0^{\beta_{n-1}} C_{\beta_{n-1}} C_{\alpha_{n-1}}^{-r/(1+s)} \right)^{-q/(1+p)} a_{n-1}^{qr/(1+s)(1+p)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} b_n &= \Lambda_1^{1/(1+s)} \left(C_{\alpha_n} a_n \right)^{-r/(1+s)} \\ &= \Lambda_1^{1/(1+s)} w_2^{-r/(1+s)(1+p)} \left(C_{\alpha_n} (C_0^{\beta_{n-1}} C_{\beta_{n-1}})^{-q/(1+p)} \right)^{-r/(1+s)} b_{n-1}^{qr/(1+s)(1+p)}. \end{aligned} \tag{3.9}$$

Since

$$\frac{qr}{(1+s)(1+p)} \in (0,1),$$

we deduce that

$$\lim_{n\to\infty} \beta_n = \frac{(2+\gamma_1)(1+p) - r(2+\gamma_2)}{(1+p)(1+s) - qr} \tag{3.10}$$

and

$$\lim_{n\to\infty} \alpha_n = \frac{(2+\gamma_1)(1+s) - q(2+\gamma_2)}{(1+p)(1+s) - qr}. \tag{3.11}$$

Then, we have

$$\lim_{n\to\infty} a_n = a = (w_2^{1+s} \Lambda_1^{-q})^{\frac{1}{(1+p)(1+s)-rs}} \left(C_0^{\beta} C_{\beta} C_{\alpha}^{-r/(1+s)} \right)^{-\frac{q(1+s)}{(1+p)(1+s)-qr}}, \tag{3.12}$$

$$\lim_{n\to\infty} b_n = b = (w_2^{-r} \Lambda_1^{1+p})^{\frac{1}{(1+p)(1+s)-qr}} \left(C_{\alpha} (C_0^{\beta} C_{\beta})^{-q/(1+p)} \right)^{-\frac{r(1+p)}{(1+p)(1+s)-qr}} \tag{3.13}$$

and

$$u \le a C_{\alpha} (d(x))^{\alpha}, \qquad v \ge b c_{\beta} C_0^{\beta} (d(x))^{\beta}.$$

The symmetric argument and (H_4) prove the reversed inequalities and thus the results are established □

4 Boundary behavior

In this section, we prove Theorems 1.2. The proof is an adaptation of the arguments used in [7].

Proof of Theorem 1.2. Let (u,v) be a classical solution of system (1.1). Taking $x_0 \in \partial\Omega$ and $x_n \in \Omega$ such that $x_n \to x_0$ as $n \to \infty$. Choose an open neighborhood U of x_0 so that $\partial\Omega$ admits $C^{2,\mu}$ local coordinates $\xi : U \to \mathbb{R}^N$, and $x \in U \cap \Omega$ if and only if $\xi_1(x) > 0$ ($\xi = (\xi_1, \xi_2, \dots, \xi_N)$). We can moreover assume $\xi(x_0) = 0$. If $u(x) = \bar{u}(\xi(x)), v(x) = \bar{v}(\xi(x))$ then we have the systems

$$\begin{cases} \sum\limits_{i,j=1}^{N} a_{i,j}(\xi)\frac{\partial^2 \bar{u}}{\partial\xi_i\partial\xi_i} + \sum\limits_{i=1}^{N} b_i(\xi)\frac{\partial\bar{u}}{\partial\xi_i} = -w(x)\bar{u}^{-p}\bar{v}^{-q}, \\ \sum\limits_{i,j=1}^{N} a_{i,j}(\xi)\frac{\partial^2 \bar{v}}{\partial\xi_i\partial\xi_i} + \sum\limits_{i=1}^{N} b_i(\xi)\frac{\partial\bar{v}}{\partial\xi_i} = -\lambda(x)\bar{u}^{-r}\bar{v}^{-s}, \end{cases}$$

in $\xi(U \cap \Omega)$, where a_{ij}, b_i are C^μ, and $a_{ij}(0) = \delta_{ij}$.

Denote by t_n the projections onto $\xi(U \cap \Omega)$ of $\xi(x_n)$, and introduce the functions

$$u_n(y) = d^\alpha \bar{u}(t_n + d_n y), \qquad v_n(y) = d^\beta \bar{u}(t_n + d_n y),$$

where $d_n = d(\xi(x_n))$, and α, β are given in (1.5). Then the functions (u_n, v_n) verify

$$\begin{cases} \sum\limits_{i,j=1}^{N} a_{i,j}(t_n + d_n y)\frac{\partial^2 \bar{u}}{\partial\xi_i\partial\xi_i} + d_n \sum\limits_{i=1}^{N} b_i(t_n + d_n y)\frac{\partial\bar{u}}{\partial\xi_i} = -c_1(d_n(x))^{\gamma_1}\bar{u}^{-p}\bar{v}^{-q}, \\ \sum\limits_{i,j=1}^{N} a_{i,j}(t_n + d_n y)\frac{\partial^2 \bar{v}}{\partial\xi_i\partial\xi_i} + d_n \sum\limits_{i=1}^{N} b_i(t_n + d_n y)\frac{\partial\bar{v}}{\partial\xi_i} = -c_2(d_n(x))^{\gamma_2}\bar{u}^{-r}\bar{v}^{-s}. \end{cases}$$

On the other hand, estimates (3.4) imply that

$$Ay_1^\alpha \le u_n(y) \le By_1^\alpha \quad \text{and} \quad Ay_1^\beta \le v_n(y) \le By_1^\beta,$$

for y in compact subsets K of $D := \{y \in \mathbb{R}^N : y_1 > 0\}$. These estimates, together with the system, a bootstrap argument and a diagonal procedure, allow us to obtain a subsequence (still labeled by u_n) such that $u_n \to u_0, v_n \to v_0$ in $C^2_{loc}(D)$. In particular, we obtain that

$$\begin{cases} -\Delta u_0 = c_1 y_1^{\gamma_1} u_0^{-p} v_0^{-q} & \text{in } D, \\ -\Delta v_0 = c_2 y_1^{\gamma_2} u_0^{-r} v_0^{-s} & \text{in } D, \end{cases}$$

which verifies

$$Ay_1^\alpha \le u_0(y) \le By_1^\alpha \quad \text{and} \quad Ay_1^\beta \le v_0(y) \le By_1^\beta, \qquad y \in D.$$

We claim

$$u_0(y) = C_1 y_1^\alpha \quad \text{and} \quad v_0(y) = C_2 y_1^\beta, \qquad y \in D,$$

where

$$C_1 = \left(c_1^{1+s} c_2^{-q} \frac{(\beta(1-\beta))^q}{(\alpha(1-\alpha))^{1+s}} \right)^{1/((1+p)(1+s)-qr)} \tag{4.1}$$

and

$$C_2 = \left(c_1^{-r} c_2^{1+q} \frac{(\beta(1-\beta))^r}{(\alpha(1-\alpha))^{1+p}} \right)^{1/((1+p)(1+s)-qr)}. \tag{4.2}$$

Let us prove the claim by an iteration method.

Notice that

$$-\triangle u_0(y) \ge c_1 y_1^{\gamma_1} B^{-q} y_1^{-q\beta} u_0^{-p}(y), \qquad y \in D.$$

Lemma 2.6 implies

$$u_0(y) \ge A_1 y_1^{\alpha}, \qquad y \in D,$$

where

$$A_1 = \left(\frac{c_1}{B^q \alpha(1-\alpha)} \right)^{1/(1+p)}.$$

Similarly, since

$$-\triangle v_0(y) \le c_2 y_1^{\gamma_2} A_1^{-r} y_1^{-r\alpha} v_0^{-s}(y), \qquad y \in D,$$

Lemma 2.6 again gives

$$v_0(y) \le B_1 y_1^{\beta}, \qquad y \in D,$$

where

$$B_1 = \left(\frac{c_2}{A_1^r \beta(1-\beta)} \right)^{1/(1+s)}.$$

Iterating this procedure, we obtain that

$$u_0(y) \ge A_n y_1^{\alpha}, \qquad v_0(y) \le B_n y_1^{\beta}, \qquad y \in D,$$

where

$$\begin{aligned} A_{n+1} &= \left(\frac{c_1}{B_n^q \alpha(1-\alpha)} \right)^{1/(1+p)} \\ &= \left(c_1 c_2^{-q/(1+s)} \right)^{\frac{1}{1+p}} \left(\frac{(\beta(1-\beta))^{q/(1+s)}}{\alpha(1-\alpha)} \right)^{\frac{1}{1+p}} A_n^{\frac{qr}{(1+s)(1+p)}} \end{aligned}$$

and

$$\begin{aligned} B_{n+1} &= \left(\frac{c_2}{A_{n+1}^r \beta(1-\beta)} \right)^{1/(1+s)} \\ &= \left(c_2 c_1^{-r/(1+p)} \right)^{\frac{1}{1+s}} \left(\frac{(\alpha(1-\alpha))^{r/(1+p)}}{\beta(1-\beta)} \right)^{\frac{1}{1+s}} B_n^{\frac{qr}{(1+s)(1+p)}}. \end{aligned}$$

Consequently,

$$\ln A_{n+1} = \ln C_3 + \theta \ln A_n$$

and

$$\ln B_{n+1} = \ln C_4 + \theta \ln B_n$$

where

$$\theta = \frac{qr}{(1+s)(1+p)} \in (0,1),$$

$$C_3 = \left(c_1 c_2^{-q/(1+s)} \right)^{\frac{1}{1+p}} \left(\frac{(\beta(1-\beta))^{q/(1+s)}}{\alpha(1-\alpha)} \right)^{\frac{1}{1+p}}$$

and

$$C_4 = \left(c_2 c_1^{-r/(1+p)}\right)^{\frac{1}{1+s}} \left(\frac{(\alpha(1-\alpha))^{r/(1+p)}}{\beta(1-\beta)}\right)^{\frac{1}{1+s}}.$$

By the iteration, we have

$$\lim_{n\to\infty} \ln A_n = \frac{\ln C_3}{1-\theta} \quad \text{and} \quad \lim_{n\to\infty} \ln B_n = \frac{\ln C_4}{1-\theta},$$

i.e.,

$$\lim_{n\to\infty} A_n = C_3^{1/(1-\theta)} = C_1 \quad \text{and} \quad \lim_{n\to\infty} B_n = C_4^{1/(1-\theta)} = C_2,$$

where C_1 and C_2 are given in (4.1) and (4.2).

Thus

$$u_0(y) \geq C_1 y_1^{\alpha} \quad \text{and} \quad v_0(y) \leq C_2 y_1^{\beta}, \qquad y \in D.$$

The symmetric argument provides with the reversed inequality, and the claim is proved.

To summarize, we have shown that $u_n \to C_1 y_1^{\alpha}$ and $v_n \to C_2 y_1^{\beta}$ in $C^2_{loc}(D)$. Thus, taking $y = e_1 = (1, 0, \ldots, 0)$ and recalling that $\xi(x_n) = t_n + d_n e_1$, we arrive at

$$\begin{aligned}
\frac{u(x_n)}{(d_n(x))^{\alpha}} &\to \left(c_1^{1+s} c_2^{-q} \frac{(\beta(1-\beta))^q}{(\alpha(1-\alpha))^{1+s}}\right)^{1/((1+p)(1+s)-qr)}, \\
\frac{v(x_n)}{(d_n(x))^{\beta}} &\to \left(c_1^{-r} c_2^{1+q} \frac{(\beta(1-\beta))^r}{(\alpha(1-\alpha))^{1+p}}\right)^{1/((1+p)(1+s)-qr)}, \\
\frac{\frac{\partial u}{\partial \xi_1}(x_n)}{(d_n(x))^{\alpha-1}} &\to -\alpha \left(c_1^{1+s} c_2^{-q} \frac{(\beta(1-\beta))^q}{(\alpha(1-\alpha))^{1+s}}\right)^{1/((1+p)(1+s)-qr)}, \\
\frac{\frac{\partial v}{\partial \xi_1} v(x_n)}{(d_n(x))^{\beta-1}} &\to -\beta \left(c_1^{-r} c_2^{1+q} \frac{(\beta(1-\beta))^r}{(\alpha(1-\alpha))^{1+p}}\right)^{1/((1+p)(1+s)-qr)}.
\end{aligned}$$

Then Theorem 1.2 follows by the arbitrariness of the sequence x_n. □

5 Uniqueness of solutions

In this section, we prove the uniqueness of solutions.

Proof of Theorem 1.3. Let (u_1, v_1) and (u_2, v_2) be positive solutions to system (1.1).

Let

$$\omega = \frac{u_1}{u_2},$$

and assume $k = \sup_{x\in\Omega} \omega(x) > 1$.

It follows by Theorem 1.2 that

$$\lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1.$$

Then, there exists x_0 such that $\omega(x_0) = k$, and hence

$$\omega(x_0) = 0, \qquad \nabla\omega(x_0) = 0.$$

In particular,

$$u_2 \triangle u_1 - u_1 \triangle u_2 \leq 0$$

at x_0. This leads to

$$v_2(x_0) \geq k^{(p+1)/q} v_1(x_0).$$

We now claim that $v_2 \leq k^{r/(s+1)} v_1$ in Ω. Assume on the contrary that $\Omega_0 := \{v_2 \geq k^{r/(s+1)} v_1\}$ is nonempty. Notice that $\partial\Omega_0 \subset \Omega$, since $k > 1$ and $v_1/v_2 = 1$ on $\partial\Omega$, thus $v_2 = k^{r/(s+1)} v_1$ on Ω_0. Then

$$-\triangle v_2 = \lambda(x) u_2^{-r} v_2^{-s} < \lambda(x) k^{r/(s+1)} u_1^{-r} v_1^{-s} = -\triangle(k^{r/(s+1)} v_1)$$

on in Ω_0 and the maximum principle implies $v_2 \leq k^{r/(s+1)} v_1$ in Ω_0 which is impossible. Hence $v_2 \leq k^{r/(s+1)} v_1$ in Ω and by the strong maximum principle it follows that $v_2 \leq k^{r/(s+1)} v_1$ in Ω. Combining the two assertions we have

$$k^{(1+p)/q} v_1(x_0) < k^{r/(s+1)} v_1(x_0),$$

i.e.

$$k^{\frac{(1+p)(s+1)-qr}{q(1+s)}} < 1.$$

By $(1+s)(1+p) > qr$, we obtain $k < 1$, which is also a contradiction. Thus we conclude $k \leq 1$, i.e., $u_1 \leq u_2$. The symmetric argument proves $u_1 \geq u_2$, and using the equation for u_1 and u_2, we deduce $v_1 = v_2$. The result is proved. □

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