Electronic Journal of Qualitative Theory of Differential Equations
2017, No. 92, 1–12; https://doi.org/10.14232/ejqtde.2017.1.92 www.math.u-szeged.hu/ejqtde/

Bifurcation in nonlinearizable eigenvalue problems for ordinary differential equations of fourth order with indefinite weight

Ziyatkhan S. Aliyev[✉ 1, 2] and **Rada A. Huseynova**[2]

[1]Department of Mathematical Analysis, Baku State University, Z. Khalilov Str. 23, Baku, AZ-1148, Azerbaijan
[2]Institute of Mathematics and Mechanics NAS of Azerbaijan, B. Vahabzadeh Str. 9, Baku, AZ-1141, Azerbaijan

Received 30 May 2017, appeared 19 December 2017
Communicated by Gennaro Infante

Abstract. We consider a nonlinearizable eigenvalue problem for the beam equation with an indefinite weight function. We investigate the structure of bifurcation set and study the behavior of connected components of the solution set bifurcating from the line of trivial solutions and contained in the classes of positive and negative functions.

Keywords: nonlinear eigenvalue problem, bifurcation point, principal eigenvalues, global continua, indefinite weight.

2010 Mathematics Subject Classification: 34C10, 34C23, 47J10, 47J15.

1 Introduction

We consider the following fourth order boundary value problem

$$(\ell u) \equiv (p(t)u'')'' - (q(t)u')' = \lambda r(t)u + h(t,u,u',u'',u''',\lambda), \qquad t \in (0,1), \tag{1.1}$$

$$\begin{aligned} u'(0)\cos\alpha - (pu'')(0)\sin\alpha &= 0,\\ u(0)\cos\beta + Tu(0)\sin\beta &= 0,\\ u'(1)\cos\gamma + (pu'')(1)\sin\gamma &= 0,\\ u(1)\cos\delta - Tu(1)\sin\delta &= 0, \end{aligned} \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (pu'')' - qu'$, the function $p(t)$ is strictly positive and continuous on $[0,1]$, $p(t)$ has an absolutely continuous derivative on $[0,1]$, $q(t)$ is nonnegative and absolutely continuous on $[0,1]$, the weight function $r(t)$ is sign-changing continuous on $[0,1]$ (i.e. $\text{meas}\{t \in (0,1) : \sigma r(t) > 0\} > 0$ for each $\sigma \in \{+,-\}$) and α, β, γ, δ are real constants such that $0 \le \alpha$, β, γ, $\delta \le \pi/2$ except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$

✉Corresponding author. Email: z_aliyev@mail.ru

and $\alpha = \beta = \gamma = \delta = \pi/2$. The nonlinear term has the representation $h = f + g$, where $f, g \in C([0,1] \times \mathbb{R}^5)$ are real-valued functions satisfying the following conditions:

$$uf(t, u, s, v, w, \lambda) \leq 0, \qquad (t, u, s, v, w, \lambda) \in [0,1] \times \mathbb{R}^5, \tag{1.3}$$

there exists constants $M > 0$ such that

$$\left| \frac{f(t, u, s, v, w, \lambda)}{u} \right| \leq M, \qquad (t, u, s, v, w, \lambda) \in [0,1] \times \mathbb{R}^5, \tag{1.4}$$

and

$$g(t, u, s, v, w, \lambda) = o(|u| + |s| + |v| + |w|) \tag{1.5}$$

in a neighborhood of $(u, s, v, w) = (0, 0, 0, 0)$ uniformly in $t \in [0,1]$ and in $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

It is well known that fourth-order problems arise in many applications (see [7,21]) and the references therein); problem (1.1)–(1.2) in particular, is often used to describe the deformation of an elastic beam, which is subject to axial forces (see [7]). Problems with sign-changing weight arise from population modeling. In this model, weight function g changes sign corresponding to the fact that the intrinsic population growth rate is positive at same points and is negative at others, for details, see [9,14].

The purpose of this work is to study the global bifurcation of solutions of problem (1.1)–(1.2) in the classes of positive and negative functions, bifurcating from the intervals of the line of trivial solutions.

The problem (1.1)–(1.2) for the case of $f \equiv 0$ is studied in [16]. In the case of $f \equiv 0$ the linearization of (1.1)–(1.2) at $u = 0$ is the linear eigenvalue problem

$$\begin{gathered} (p(t)u''(t))'' - (q(t)u'(t))' = \lambda r(t)u(t), \qquad t \in (0,1), \\ u \in B.C., \end{gathered} \tag{1.6}$$

where by $B.C.$ we denote the set of boundary conditions (1.2). In [16] it was shown that there exist two positive and negative principal eigenvalues (i.e., eigenvalues corresponding to eigenfunctions which have no zeros in $(0,1)$), λ_1^+ and λ_1^-, of problem (1.6). Moreover, in [16] it was also proved that for each $\sigma \in \{+, -\}$ and each $\nu \in \{+, -\}$ there exists a continuum (connected closed set) $C_1^{\sigma,\nu}$ of solutions of problem (1.1)–(1.2) with $f \equiv 0$ bifurcating from the point $(\lambda_1^\sigma, 0)$, which is unbounded in $\mathbb{R} \times C^3[0,1]$, and $\nu \operatorname{sgn} u(t) = 1$, $t \in (0,1)$, for each $(\lambda, u) \in C_1^{\sigma,\nu}$.

Because of the presence of the term f, problem (1.1)–(1.2) does not in general have a linearization about zero. For this reason, the set of bifurcation points for (1.1)–(1.2) with respect to the line of trivial solutions need not be discrete (cf. the example in [6, p. 381]). Therefore, to investigate bifurcation for (1.1)–(1.2), one has to consider bifurcation from intervals rather than from bifurcation points. We say that bifurcation occurs from an interval if this interval contains at least one bifurcation point [6].

The problem (1.1)–(1.2) with $r > 0$ was considered in a recent paper [3] where, in particular, it was shown that for each $k \in \mathbb{N}$ and $\nu = +$ or $-$, there exists a connected component (maximal connected subset) D_k^ν of the set of solutions that emanating from the bifurcation interval $\left[\lambda_k - \frac{K}{r_0}, \lambda_k - \frac{K}{r_0}\right] \times \{0\}$ $(r_0 = \min_{t \in [0,1]} r(t))$ of the line of trivial solutions, has the standard oscillation properties (the number of zeros of a function is equal to the index of the eigenvalue of the corresponding linear problem minus one), is unbounded in $\mathbb{R} \times C^3$, and $\lim_{t \to 0} \nu \operatorname{sgn} u(t) = 1$ for each $(\lambda, u) \in D_k^\nu$. Similar results on global bifurcation of solutions of nonlinear Sturm–Liouville problems obtained before by Rabinowitz [22], Berestycki [6], Schmitt and Smith [24], Chiappinelli [10], Aliyev and Mamedova [4], Rynne [23] and Dai [12].

It should be noted that to study the global bifurcation of solutions of problem (1.1)–(1.2) in the classes of positive and negative functions the method of [3] cannot be applied. This is due to the fact that the weight function $r(x)$ changes sign in the interval $(0, 1)$ and the eigenfunctions of linear problem (1.6) corresponding to the principal eigenvalues have no zeros in the interval $(0, 1)$. Therefore, in investigating global bifurcation in the nonlinear problem (1.1)–(1.2) the following questions must be addressed: using new approaches to finding bifurcation intervals of solutions to (1.1)–(1.2) and to the study of the behavior of the connected components of the set of solutions emanating from these intervals.

The structure of this paper is as follows.

In Section 2, a family of sets to exploit oscillatory properties of eigenfunctions of problem (1.6) and their derivatives is introduced. Although problem (1.1)–(1.2) is not linearizable in a neighborhood of the origin (when $f \not\equiv 0$), it is nevertheless related to a linear problem which is perturbation of problem (1.6). In Section 3, we estimate the distance between the principal eigenvalues of the perturbed and unperturbed problem. Using this estimation in Section 4 we find the bifurcation intervals. We show the existence of two pair of unbounded continua of solutions emanating from the bifurcation intervals and contained in the classes of positive and negative functions.

2 Preliminary

Let $E = C^3[0, 1] \cap B.C.$ be a Banach space with the norm $\|u\|_3 = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|u'''\|_\infty$, where $\|\cdot\|_\infty$ is the standard sup-norm in $C[0, 1]$.

Let

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(t) \neq 0, Tu(t) \neq 0, \ t \in [0, 1], \ i = 0, 1, 2\}$$

and

$$\begin{aligned} S_2 = \big\{ u \in E : \ &\text{there exists } i_0 \in \{0, 1, 2\} \text{ and } t_0 \in (0, 1) \text{ such that } u^{(i_0)}(t_0) = 0, \\ &\text{or } Tu(t_0) = 0 \text{ and if } u(t_0)u''(t_0) = 0, \text{ then } u'(t)Tu(t) < 0 \text{ in a neighborhood of } t_0, \\ &\text{and if } u'(t_0)Tu(t_0) = 0, \text{ then } u(t)u''(t) < 0 \text{ in a neighborhood of } t_0 \big\}. \end{aligned}$$

Note that if $u \in S$ then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ (see [2, 3, 5]) of the Prüfer-type transformation

$$\begin{cases} u(t) = \rho(t) \sin\psi(t) \cos\theta(t), \\ u'(t) = \rho(t) \cos\psi(t) \sin\varphi(t), \\ (pu'')(t) = \rho(t) \cos\psi(t) \cos\varphi(t), \\ Tu(t) = \rho(t) \sin\psi(t) \sin\theta(t), \end{cases} \tag{2.1}$$

does not vanish on $(0, 1)$.

For each $u \in S$ we define $\rho(u, t)$, $\theta(u, t)$, $\varphi(u, t)$ and $w(u, t)$ to be the continuous functions on $[0,1]$ satisfying

$$\rho(u,t) = u^2(t) + u'^2(t) + (p(t)u''(t))^2 + (Tu(t))^2,$$
$$\theta(u,t) = \arctan\frac{Tu(t)}{u(t)}, \qquad \theta(u,0) = \beta - \pi/2,$$
$$\varphi(u,t) = \arctan\frac{u'(t)}{(pu'')(t)}, \qquad \varphi(u,0) = \alpha,$$
$$w(u,t) = \cot\psi(u,t) = \frac{u'(t)\cos\theta(u,t)}{u(t)\sin\varphi(u,t)}, \qquad w(u,0) = \frac{u'(0)\sin\beta}{u(0)\sin\alpha},$$

and $\psi(u,t) \in (0,\pi/2)$, $t \in (0,1)$, in the cases of $u(0)u'(0) > 0$; $u(0) = 0$; $u'(0) = 0$ and $u(0)u''(0) > 0$, $\psi(u,t) \in (\pi/2,\pi)$, $t \in (0,1)$, in the cases $u(0)u'(0) < 0$; $u'(0) = 0$ and $u(0)u''(0) < 0$; $u'(0) = u''(0) = 0$, $\beta = \pi/2$ in the case $\psi(u,0) = 0$ and $\alpha = 0$ in the case $\psi(u,0) = \pi/2$.

It is obvious that ρ, θ, $\varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

Remark 2.1. By (2.1) for each $u \in S$ the function $w(u,t)$ can be determined from one of the following relations:

(a) $$w(u,t) = \cot\psi(u,t) = \frac{(pu'')(t)\cos\theta(u,t)}{u(t)\cos\varphi(u,t)}, \qquad w(u,0) = \frac{(pu'')(0)\sin\beta}{u(0)\cos\alpha},$$

(b) $$w(u,t) = \cot\psi(u,t) = \frac{(pu'')(t)\sin\theta(u,t)}{Tu(t)\cos\varphi(u,t)}, \qquad w(u,0) = -\frac{(pu'')(0)\cos\beta}{Tu(0)\cos\alpha},$$

(c) $$w(u,t) = \cot\psi(u,t) = \frac{u'(t)\sin\theta(u,t)}{Tu(t)\sin\varphi(u,t)}, \qquad w(u,0) = -\frac{u'(0)\cos\beta}{Tu(0)\sin\alpha}.$$

For each $\nu \in \{+,-\}$ let S_1^ν denote the subset of such $u \in S$ that:

1) $\theta(u,1) = \pi/2 - \delta$, where $\delta = \pi/2$ in the case $\psi(u,1) = 0$;

2) $\varphi(u,1) = 2\pi - \gamma$ or $\varphi(u,1) = \pi - \gamma$ in the case $\psi(u,0) \in [0,\pi/2)$; $\varphi(u,1) = \pi - \gamma$ in the case $\psi(y,0) \in [\pi/2,\pi)$, where $\gamma = 0$ in the case $\psi(y,l) = \pi/2$;

3) for fixed u, as t increases from 0 to 1, the function $\theta(u,t)$ ($\varphi(u,t)$) strictly increasingly takes values of $m\pi/2$, $m \in \{-1, 0, 1\}$ ($s\pi$, $s \in \{0, 1, 2\}$); as t decreases from 1 to 0, the function $\theta(u,t)$ ($\varphi(u,t)$), strictly decreasing takes values of $m\pi/2$, $m \in \{-1, 0, 1\}$ ($s\pi$, $s \in \{0, 1, 2\}$);

4) the function $\nu u(t)$ is positive in a neighborhood of $t = 0$.

By the results [2, 3, 5] it follows that the sets S_1^+ and S_1^- are nonempty. It immediately follows from the definition of these sets that they are disjoint and open in E. Moreover, by [2, Lemma 2.2], if $u(t) \in \partial S_1^\nu \cap C^4[0,1]$, $\nu \in \{+,-\}$, then $u(t)$ has at least one zero of multiplicity 4 in $(0,1)$.

Lemma 2.2. *If $(\lambda,u) \in \mathbb{R} \times E$ is a solution of* (1.1)–(1.2) *and $u \in \partial S_1^\nu$, $\nu \in \{+,-\}$, then $u \equiv 0$.*

The proof of this lemma is similar to that of [3, Lemma 1.1] (see also [2]).

3 Principal eigenvalues of perturbation linear problem

For the linear eigenvalue problem (1.6) we have the following result.

Theorem 3.1 ([16, Theorem 2.1])**.** *The spectral problem* (1.6) *has two sequences of real eigenvalues*

$$0 < \lambda_1^+ < \lambda_2^+ \leq \ldots \leq \lambda_k^+ \mapsto +\infty,$$

and

$$0 > \lambda_1^- > \lambda_2^- \geq \ldots \geq \lambda_k^- \mapsto -\infty$$

and no other eigenvalues. Moreover, λ_1^+ and λ_1^- are simple principal eigenvalues, i.e. the corresponding eigenfunctions $u_1^+(t)$ and $u_1^-(t)$ have no zeros in the interval $(0,1)$.

Similar problems have been considered in [1,8,13,15,18].

Remark 3.2. The problem (1.6) with $r > 0$ is a completely regular Sturmian system as defined by S. A. Janczewsky (see [17, p. 523]) provided that the excluded the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$. Then the eigenvalues of this problem are positive, simple and form an infinitely increasing sequence $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ The eigenfunction $u_k(t)$, corresponding to λ_k, has exactly $k-1$ simple zeros in $(0,1)$ (more precisely, $u_1(t) \in S_1$) (see [3,5]). Therefore, leaving these exceptional cases out of our consideration is essential.

Note that the proof of Theorem 3.1 is based on a method used by Brown and Lin [8]. Now we analyze the existence of principal eigenvalues using the method of Hess and Kato [15] (see also [1]). This is due to the fact that we will need further reasoning in order to find the bifurcation intervals of problem (1.1)–(1.2) corresponding to the principal eigenvalues of (1.6).

Define the linear differential operator $L : D(L) \to L_2(0,1)$ by

$$(Lu)(t) = (\ell u)(t)$$

and

$$D(L) = \{u \in L_2(0,1) \,:\, u \in W_2^4(0,1),\ \ell u \in L_2(0,1),\ u \in B.C.\}.$$

It is known that the differential operator L is a densely defined self-adjoint operator on H whose spectrum contains only positive eigenvalues [5] (see also Remark 3.2).

For fixed $\lambda \in \mathbb{R}$ we consider the following eigenvalue problem

$$\begin{gathered}(\ell u)(t) - \lambda r(t)u(t) = \mu u(t), \qquad t \in (0,1),\\ u \in B.C.\end{gathered} \tag{3.1}$$

By [3, Theorem 1.2] the problem has a sequence of real and simple eigenvalues

$$\mu_1(\lambda) < \mu_2(\lambda) < \ldots < \mu_k(\lambda) \mapsto +\infty.$$

Moreover, for each $k \in \mathbb{N}$ the eigenfunction $u_k(t,\lambda)$ corresponding to the eigenvalue $\mu_k(\lambda)$ has $k-1$ simple zeros in the interval $(0,1)$ (it should be noted that $u_1(t,\lambda) \in S_1$). Let

$$T_\lambda = \left\{\int_0^1 \{p(t)|u''(t)|^2 + q(t)|u'(t)|^2\}dt + N(u) - \lambda\int_0^1 r(t)|u(t)|^2 dt : u \in D(L),\ \int_0^1 |u(t)|^2 dt = 1\right\},$$

where $N(u) = [u'(0)]^2 \cot\alpha + [u(0)]^2 \cot\beta + [u'(1)]^2 \cot\gamma + [u(1)]^2 \cot\delta$. It is clear that T_λ is bounded below. It is shown in Courant and Hilbert [11] by variational arguments that

$\mu_1(\lambda) = \min T_\lambda$. Moreover, it follows by the above argument that the eigenfunction $u_1(t, \lambda)$ corresponding to $\mu_1(\lambda)$ does not vanish on $(0,1)$. Thus, clearly, λ is a principal eigenvalue of (1.6) if and only if $\mu_1(\lambda) = 0$. For fixed $u \in D(L)$ the mapping

$$\lambda \to \int_0^1 \{p(t)|u''(t)|^2 + q(t)|u'(t)|^2\}\, dt + N(u) - \lambda \int_0^1 r(t)|u\,(t)|^2 dt$$

is an affine and therefore a concave function. Since the minimum of any collection of concave functions is concave, it follows that $\lambda \to \mu_1(\lambda)$ is a concave function. Besides, by considering test functions u_1, u_2 such that $\int_0^1 r(t)|u_1(t)|^2 dt > 0$ and $\int_0^1 r(t)|u_2(t)|^2 dt < 0$, it is easy to see that $\mu_1(\lambda) \to -\infty$ as $\lambda \to \pm\infty$. Thus $\mu_1(\lambda)$ is an increasing function until it attains its maximum, and is a decreasing function thereafter.

Then, as can be seen from the variational characterization of $\mu_1(\lambda)$ or the fact that L has a positive principal eigenvalue, $\mu_1(0) > 0$ and thus $\mu_1(\lambda)$ must has a graph which intersects the real axis in two points first of which is to the left, and second to the right from origin of coordinates. Hence, problem (1.6) has exactly two simple principal eigenvalues, one positive and one negative, which coincide with the λ_1^+ and λ_1^-, respectively. Moreover, we have $u_1(t, \lambda_1^+) = u_1^+(t)$ and $u_1(t, \lambda_1^-) = u_1^-(t)$, $t \in [0, 1]$.

Lemma 3.3. *For each $\sigma \in \{+, -\}$ the following relation is true:*

$$\frac{d\mu_1(\lambda_1^\sigma)}{d\lambda} = -\frac{\int_0^1 r(t)\, \left(u_1^\sigma(t)\right)^2 dt}{\int_0^1 \left(u_1^\sigma(t)\right)^2 dt}. \tag{3.2}$$

Proof. By (3.1) we have

$$\begin{aligned} \ell u_1(t,\lambda) - \lambda r(t) u_1(t,\lambda) = \mu_1(\lambda) u_1(t,\lambda), \qquad & t \in (0,1), \\ u_1(t,\lambda) \in B.C. \end{aligned} \tag{3.3}$$

Let $v_1(t,\lambda) = \frac{du_1(t,\lambda)}{d\lambda}$. Then, by virtue of (3.3), $v_1(t,\lambda)$ satisfies

$$\begin{aligned} \ell v_1(t,\lambda) - \lambda r(t) v_1(t,\lambda) - \mu_1(\lambda) v_1(t,\lambda) = r(t) u_1(t,\lambda) + \frac{d\mu_1(\lambda)}{d\lambda} u_1(t,\lambda), \qquad & t \in (0,1), \\ v_1(t,\lambda) \in B.C. \end{aligned} \tag{3.4}$$

Multiplying (3.4) by $u_1(t,\lambda)$ and integrating this relation from 0 to 1 while taking into account the self-adjointness of the operator L we obtain

$$-\mu_1(\lambda) \int_0^1 v_1(t,\lambda)\, u_1(t,\lambda) dt = \int_0^1 r(t)\, u_1^2(t,\lambda) dt + \frac{d\mu_1(\lambda)}{d\lambda} \int_0^1 u_1^2(t,\lambda) dt.$$

Since $\mu_1(\lambda_1^\sigma) = 0$, $\sigma \in \{+, -\}$, it follows that

$$0 = \int_0^1 r(t)\, u_1^2(t,\lambda_1^\sigma) dt + \frac{d\mu_1(\lambda_1^\sigma)}{d\lambda} \int_0^1 u_1^2(t,\lambda_1^\sigma) dt,$$

which implies (3.2). The proof of this lemma is complete. □

Together with problems (1.6) and (3.1) we consider the following spectral problems

$$\begin{aligned} \ell u(t) + \varphi(t)u(t) = \lambda r(t)u(t), \qquad t \in (0,1), \\ u \in B.C., \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\ell u)(t) - \lambda r(t)u(t) + \varphi(t)u(t) = \mu u(t), \qquad t \in (0,1), \\ u \in B.C., \end{aligned} \tag{3.6}$$

where $\varphi(t) \in C[0,1]$ and $\varphi(t) \geq 0$, $t \in [0,1]$.

By $\varphi(t) \geq 0$, $t \in [0,1]$, it follows from the proof of [3, Lemma 4.2] that

$$0 \leq \tilde{\mu}_1(\lambda) - \mu_1(\lambda) \leq \tilde{K}, \tag{3.7}$$

where $\tilde{\mu}_1(\lambda)$ is the smallest eigenvalue of problem (3.6) and $\tilde{K} = \max_{t\in[0,1]} \varphi(t)$.

Remark 3.4. Since $\lambda \to \tilde{\mu}_1(\lambda)$ is also a concave function on $\mathbb{R}$ and $\tilde{\mu}_1(\lambda) \geq \mu_1(\lambda)$ for any $\lambda \in \mathbb{R}$ it follows that $\tilde{\lambda}_1^+ > \lambda_1^+$ and $\tilde{\lambda}_1^- < \lambda_1^-$, where $\tilde{\lambda}_1^+$ and $\tilde{\lambda}_1^-$ are the positive and negative principal eigenvalues of problem (3.5), respectively.

We need the following result which is basic in the sequel.

Lemma 3.5. *For each $\sigma \in \{+,-\}$ the following relation is true:*

$$|\tilde{\lambda}_1^\sigma - \lambda_1^\sigma| \leq \frac{\sigma\tilde{K}\int_0^1 (u_1^\sigma(t))^2\,dt}{\int_0^1 r(t)\,(u_1^\sigma(t))^2\,dt}. \tag{3.8}$$

Proof. Let

$$l^\sigma(\lambda) = a_1^\sigma(\lambda - \lambda_1^\sigma), \qquad a_1^\sigma = \frac{d\mu_1(\lambda_1^\sigma)}{d\lambda}, \qquad \sigma \in \{+,-\},$$

i.e. l^σ is the line which tangent to the graph of the function $\mu_1(\lambda)$ at point λ_1^σ. We introduce the following notation:

$$A = (\lambda_1^\sigma, 0), \quad B = (\tilde{\lambda}_1^\sigma, 0), \quad C = (\tilde{\lambda}_1^\sigma, l^\sigma(\tilde{\lambda}_1^\sigma)), \quad \text{and} \quad D = (\tilde{\lambda}_1^\sigma, \mu_1(\tilde{\lambda}_1^\sigma)), \quad \sigma \in \{+,-\}.$$

Note that

$$|AB| = |\tilde{\lambda}_1^\sigma - \lambda_1^\sigma|,$$

where $|AB|$ is the distance between the points A and B.

Since $\lambda \to \mu_1(\lambda)$ is a concave function it follows that the graph of the function $\mu_1(\lambda)$ lies under the tangent l^σ for each $\sigma \in \{+,-\}$. Hence, by Remark 3.4, we have

$$|BC| \leq |BD|. \tag{3.9}$$

Moreover, from a right-angled triangle we find that

$$|AB| = |BC| \tan \angle BAC = -\sigma|BC|\frac{d\mu_1(\lambda_1^\sigma)}{d\lambda}. \tag{3.10}$$

Combining (3.10), (3.9), (3.7) and (3.2) we obtain (3.8) which completes the proof. □

Remark 3.6. Since the class of continuous functions $C[0,1]$ is dense in $L_1[0,1]$ Lemma 3.5 also holds for $\varphi(t) \in L_1[0,1]$.

4 Global bifurcation from intervals of the set of solutions of problem (1.1)–(1.2)

For the problem (1.1)–(1.2) with $f \equiv 0$ we have the following global result.

Theorem 4.1 ([16, Theorem 3.1]). *For each $\sigma \in \{+,-\}$ and each $\nu \in \{+,-\}$ there exists a continuum $C_1^{\sigma,\nu}$ of solutions of problem (1.1)–(1.2) with $f \equiv 0$ in $S_1^\nu \cup \{(\lambda_1^\sigma, 0)\}$ which meets $(\lambda_1^\sigma, 0)$ and ∞ in $\mathbb{R} \times E$.*

Now we consider problem (1.1)–(1.2) with $f \not\equiv 0$.

We say that $(\lambda, 0)$ is a bifurcation point of (1.1)–(1.2) with respect to the set S_1^ν if in every small neighborhood of this point there is a solution to this problem which is contained in $\mathbb{R} \times S_1^\nu$.

Lemma 4.2. *For each $\nu \in \{+,-\}$ and for each sufficiently small $\tau > 0$ problem (1.1)–(1.2) has a solution (λ_τ, v_τ) such that $v_\tau \in S_1^\nu$ and $\|v_\tau\|_3 = \tau$.*

Proof. We consider the following approximation problem

$$\begin{cases} \ell u = \lambda r(t) u + f(t, |u|^\varepsilon u, u', u'', u''', \lambda) + g(t, u, u', u'', u''', \lambda), & t \in (0,1), \\ u \in B.C., \end{cases} \tag{4.1}$$

where $\varepsilon \in (0,1]$.

By virtue of (1.4) the function $f(t, |u|^\varepsilon u, u', u'', u''', \lambda)$ satisfies the condition (1.5), i.e.

$$f(t,\, |u|^\varepsilon u, s, v, w, \lambda) = o\,(|u| + |s| + |v| + |w|) \tag{4.2}$$

in a neighborhood of $(u,s,v,w) = (0,0,0,0)$ uniformly in $t \in [0,1]$ and in $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$. Then by Theorem 4.1, for each $\sigma \in \{+,-\}$ and each $\nu \in \{+,-\}$ there exists an unbounded continuum $C_{1,\varepsilon}^{\sigma,\nu}$ of solutions of (4.1) such that

$$(\lambda_1^\sigma, 0) \in C_{1,\varepsilon}^{\sigma,\nu} \subset S_1^\nu \cup \{(\lambda_1^\sigma, 0)\}.$$

Then for any $\varepsilon \in (0,1]$ there exists a solution $(\lambda_{\tau,\varepsilon}, v_{\tau,\varepsilon}) \in \mathbb{R} \times E$ of (4.1) such that $v_{\tau,\varepsilon} \in \partial B_\tau \cap S_1^\nu$, where ∂B_τ is the boundary of the open ball $B_\tau \subset E$ of radius τ centered at 0. Clearly, $(\lambda_{\tau,\varepsilon}, v_{\tau,\varepsilon})$ solves the nonlinear problem

$$\begin{cases} \ell u + \varphi_\varepsilon(t) u = \lambda r(t) u + g(t, u, u', u'', u''', \lambda), & t \in (0,1), \\ u \in B.C., \end{cases} \tag{4.3}$$

where

$$\varphi_\varepsilon(t) = \begin{cases} -\dfrac{f(t, |v_{\tau,\varepsilon}(t)|^\varepsilon v_{\tau,\varepsilon}(t), v'_{\tau,\varepsilon}(t), v''_{\tau,\varepsilon}(t), v'''_{\tau,\varepsilon}(t), \lambda)}{v_{\tau,\varepsilon}(t)}, & \text{if } v_{\tau,\varepsilon}(t) \neq 0, \\ 0, & \text{if } v_{\tau,\varepsilon}(t) = 0. \end{cases} \tag{4.4}$$

By (1.3) and (1.4), from (4.4) we obtain

$$\varphi_\varepsilon(t) \geq 0 \quad \text{and} \quad |\varphi_\varepsilon(t)| \leq K |\, v_{\tau,\varepsilon}(t)|^\varepsilon \leq K \quad \text{for all } t \in [0,1]. \tag{4.5}$$

Since $v_{\tau,\varepsilon}$ does not vanish in $(0,1)$ and is bounded on the closed interval $[0,1]$, Remark 3.6 shows that the result of Lemma 3.5 also holds for the following linear problem

$$\begin{cases} \ell u + \varphi_\varepsilon(t) u = \lambda r(t) u, & t \in (0,1), \\ u \in B.C. \end{cases} \tag{4.6}$$

Then, taking (4.5) into account it follows from (3.8) that the principal eigenvalue $\lambda_{1,\varepsilon}^{\sigma}$, $\sigma \in \{+,-\}$, of the linear problem (4.6) lies in J_1^{σ}, where

$$J_1^+ = [\lambda_1^+, \lambda_1^+ + d_1^+], \qquad J_1^- = [\lambda_1^- - d_1^-, \lambda_1^-], \qquad d_1^{\sigma} = \frac{\sigma K \int_0^1 (u_1^{\sigma}(t))^2\, dt}{\int_0^1 r(t)\, (u_1^{\sigma}(t))^2\, dt}.$$

By [19, Ch. 4, § 2, Theorem 2.1] and Theorems 3.1 and 4.1, for each $\sigma \in \{+,-\}$, $(\lambda_{1,\varepsilon}^{\sigma}, 0)$ is the bifurcation point of (4.3) with respect to the set of S_1^{ν} and a continuous branch of nontrivial solutions corresponds to this point. Hence to each small $\tau > 0$ we can assign a small $\rho_{\tau,\varepsilon}^{\sigma} > 0$, $\sigma \in \{+,-\}$ such that

$$\lambda_{\tau,\varepsilon} \in (\lambda_{1,\varepsilon}^{+} - \rho_{\tau,\varepsilon}^{+}, \lambda_{1,\varepsilon}^{+} + \rho_{\tau,\varepsilon}^{+}) \subset [\lambda_1^+ - \rho_0^+, \lambda_1^+ + d_0^+ + \rho_0^+],$$

or

$$\lambda_{\tau,\varepsilon} \in (\lambda_{1,\varepsilon}^{-} - \rho_{\tau,\varepsilon}^{-}, \lambda_{1,\varepsilon}^{-} + \rho_{\tau,\varepsilon}^{-}) \subset [\lambda_1^- - d_0^- - \rho_0^-, \lambda_1^- + \rho_0^-],$$

where $\rho_0^{\sigma} = \sup\limits_{\tau,\varepsilon} \rho_{\tau,\varepsilon}^{\sigma} > 0$.

Since $\{v_{\tau,\varepsilon} \in E \ : \ 0 < \varepsilon \leq 1\}$ is a bounded subset of $C^3[0,1]$, the functions f and g are continuous in $[0,1] \times \mathbb{R}^5$, and the set $\{\lambda_{\tau,\varepsilon} \in \mathbb{R} \ : \ 0 < \varepsilon \leq 1\}$ is bounded in $\mathbb{R}$, it follows from (4.1) that $\{v_{\tau,\varepsilon} \in E \ : \ 0 < \varepsilon \leq 1\}$ is also bounded in $C^4[0,1]$. Hence it is precompact in E by the Arzelà–Ascoli theorem.

Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,\ 1)$ be a sequence such that $\varepsilon_n \to 0$ and $(\lambda_{\tau,\varepsilon_n} v_{\tau,\varepsilon_n}) \to (\lambda_\tau, v_\tau)$ as $n \to \infty$. Taking the limit in (4.1) we see that (λ_τ, v_τ) is a solution of (1.1)–(1.2). Since $\|v_\tau\|_3 = \tau > 0$, it follows from Lemma 2.2 that $v_\tau \in S_1^{\nu}$. The proof of Lemma 4.2 is complete. □

Corollary 4.3. *The set of bifurcation points for problem* (1.1)–(1.2) *with respect to the set* S_1^{ν} *is nonempty.*

Lemma 4.4. *Let* $\{\varepsilon_n\}_{n=1}^{\infty} \subset [0,1]$ *and* $\varepsilon_n \to 0$. *If* $(\zeta_n, w_n) \in \mathbb{R} \times S_1^{\nu}$ *is a solution of* (4.1) *for* $\varepsilon = \varepsilon_n$ *and* $\{(\zeta_n, w_n)\}_{n=1}^{\infty}$ *converges to* $(\zeta, 0)$ *in* $\mathbb{R} \times E$, *then* $\zeta \in J_1^+$ *or* $\zeta \in J_1^-$.

The proof of this lemma is similar to that of [3, Lemma 5.4] with considering of Lemma 4.2 and Corollary 4.3.

Corollary 4.5. *If* $(\lambda, 0)$ *is a bifurcation point for* (1.1)–(1.2) *with respect to the set* S_1^{ν}, *then* $\lambda \in J_1^+ \cup J_1^-$.

Let $\mathfrak{L}$ denote the closure of the set of nontrivial solutions of (1.1)–(1.2).

For each $\sigma \in \{+,-\}$ and $\nu \in \{+,-\}$, let $\tilde{D}_1^{\sigma,\nu}$ denote the union of the connected components $D_{1,\lambda}^{\sigma,\nu}$ of the set of solutions of (1.1)–(1.2) emanating from bifurcation points $(\lambda, 0) \in J_1^{\sigma}$ with respect to S_1^{ν}. It is clear that $\tilde{D}_1^{\sigma,\nu} \neq \emptyset$. Note that $D_1^{\sigma,\nu} = \tilde{D}_1^{\sigma,\nu} \cup (J_1^{\sigma} \times \{0\})$ is a connected subset of $\mathbb{R} \times E$, but $\tilde{D}_1^{\sigma,\nu}$ is not necessarily connected in $\mathbb{R} \times E$.

Let

$$I_1 = [\lambda_1^- - d_0^-, \lambda_1^+ + d_0^+].$$

Remark 4.6. Since $J_1^+ \subset I_1$ and $J_1^- \subset I_1$ it follows from Corollary 4.5 that all bifurcation points of (1.1)–(1.2) with respect to the set S_1^{ν} lie in $I_1 \times \{0\}$.

Let D^{ν}, $\nu \in \{+,-\}$, denote the union of the sets $D_1^{+,\nu}$, $D_1^{-,\nu}$ and $I_1 \times \{0\}$, i.e.

$$D_1^{\nu} = D_1^{+,\nu} \cup D_1^{-,\nu} \cup (I_1 \times \{0\}).$$

Theorem 4.7. *For each $\nu \in \{+,-\}$ the connected component D_1^ν of $\mathfrak{L}$, containing $I_1 \times \{0\}$, lies in $(\mathbb{R} \times S_1^\nu) \cup (I_1 \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

The proof of this theorem is similar to that of [3, Theorem 1.3] with considering of Lemma 2.2, Theorem 3.1, Theorem 4.1, Lemma 4.2, Corollary 4.3 and Remark 4.6.

The main result of this paper is the following theorem.

Theorem 4.8. *For each $\nu \in \{+,-\}$ and each $\nu \in \{+,-\}$ the connected component $D_1^{\sigma,\nu}$ of $\mathfrak{L}$, containing $J_1^\sigma \times \{0\}$, lies in $(\mathbb{R} \times S_1^\nu) \cup (J_1^\sigma \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

Proof. It follows from Corollary 4.5 that

$$D_1^{\sigma,\nu} \cap \left(\mathbb{R} \backslash (J_1^+ \cup J_1^-)\right) = \emptyset, \qquad \sigma \in \{+,-\}.$$

Then, by [20, Theorem 3.1], for each $\sigma \in \{+,-\}$ either $D_1^{\sigma,+} \cup D_1^{\sigma,-}$ is unbounded in $\mathbb{R} \times E$, or $D_1^{\sigma,+} \cup D_1^{\sigma,-}$ meets $J_1^{-\sigma} \times \{0\}$. Since

$$(D_1^{\sigma,+} \backslash (\mathbb{R} \times \{0\}) \cap (D_1^{\sigma,-} \backslash (\mathbb{R} \times \{0\}) = \emptyset \quad \text{for each } \sigma \in \{+,-\},$$

it follows that if $D_1^{\sigma,+} \cup D_1^{\sigma,-}$ meets $J_1^{-\sigma} \times \{0\}$ (where $-\sigma$ is interpreted in the natural way), then

$$D_1^{+,\nu} = D_1^{-,\nu} \quad \text{for each } \nu \in \{+,-\}.$$

Hence it follows that for each $\nu \in \{+,-\}$ the set D_1^ν is bounded in $\mathbb{R} \times E$ which contradicts Theorem 4.7. The proof of this theorem is complete. □

Corollary 4.9. *Let $g \equiv 0$. Then for each $\nu \in \{+,-\}$ and each $\nu \in \{+,-\}$ the connected component $D_1^{\sigma,\nu}$ of $\mathfrak{L}$, containing $(J_1^\sigma \times \{0\})$, lies in $(J_1^\sigma \times S_1^\nu) \cup (J_1^\sigma \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

The proof of this corollary follows from Theorem 4.8 with considering the following lemma.

Lemma 4.10. *Let $g \equiv 0$ and (λ, u) is a solution of problem (1.1)–(1.2) such that $u \in S_1$. Then $\lambda \in J_1^+$ or $\lambda \in J_1^-$.*

Proof. Let $(\lambda, u) \in \mathbb{R} \times S_1$. Then (λ, u) solves the linear problem

$$\begin{cases} \ell u + \varphi(t) u = \lambda r(t) u, & t \in (0,1), \\ u \in B.C., \end{cases} \tag{4.7}$$

where

$$\varphi(t) = \begin{cases} -\frac{f(t,u(t),u'(t),u''(t),u'''(t),\lambda)}{u(t)}, & \text{if } u(t) \neq 0, \\ 0, & \text{if } u(t) = 0. \end{cases} \tag{4.8}$$

Taking (1.3) and (1.4) into account, (4.8) yields

$$\varphi(t) \geq 0 \quad \text{and} \quad |\varphi(t)| \leq K, \qquad t \in [0,1].$$

Hence λ is a principal eigenvalue of problem (4.7). By Remark 3.6 it follows from Lemma 3.5 that $\lambda \in J_1^+$ or $\lambda \in J_1^-$. The proof of Lemma 4.10 is complete. □

References

[1] G. A. Afrouzi, K. J. Brown, On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions, *Proc. Amer. Math. Soc.* **127**(1999), 125–130. MR1469392; https://doi.org/10.1090/S0002-9939-99-04561-X

[2] Z. S. Aliyev, Some global results for nonlinear fourth order eigenvalue problems, *Cent. Eur. J. Math.* **12**(2014), No. 12, 1811–1828. MR3232641; https://doi.org/10.2478/s11533-014-0416-z

[3] Z. S. Aliyev, Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.* **207**(2016), No. 12, 1625–1649. MR3588983; https://doi.org/10.1070/SM8369

[4] Z. S. Aliyev, G. M. Mamedova, Some global results for nonlinear Sturm–Liouville problems with spectral parameter in the boundary condition, *Ann. Polon. Math.* **115**(2015), No. 1, 75–87. MR3392699; https://doi.org/10.4064/ap115-1-6

[5] D. O. Banks, G. J. Kurowski, A Prüfer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations* **24**(1977), No. 1, 57–74. MR0457834; https://doi.org/10.1016/0022-0396(77)90170-X

[6] H. Berestycki, On some nonlinear Sturm–Liouville problems, *J. Differential Equations* **26**(1977), No. 3, 375–390. MR0481230; https://doi.org/10.1016/0022-0396(77)90086-9

[7] B. B. Bolotin, *Vibrations in technique. Handbook in 6 volumes. The vibrations of linear systems. I.* (in Russian), Engineering Industry, Moscow, 1978.

[8] K. J. Brown, S. S. Lin, On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function, *J. Math. Anal. Appl.* **75**(1980), No. 1, 112–120. MR0576277; https://doi.org/10.1016/0022-247X(80)90309-1

[9] R. S. Cantrell, C. Cosner, *Spatial ecology via reaction–diffusion equations*, Wiley, Chichester, 2003. https://doi.org/10.1002/0470871296

[10] R. Chiappinelli, On eigenvalues and bifurcation for nonlinear Sturm–Liouville operators, *Boll. Un. Mat. Ital. A(6)* **4**(1985), No. 1, 77–83. MR0781798

[11] R. Courant, D. Hilbert, *Methods of mathematical physics. Vol. I.*, Interscience, New York, 1953. MR0065391

[12] G. Dai, Global bifurcation from intervals for Sturm–Liouville problems which are not linearizable, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 65, 1–7. MR3141789; https://doi.org/10.14232/ejqtde.2013.1.65

[13] J. Fleckinger, M. L. Lapidus, Eigenvalues of elliptic boundary value problems with an indefinite weight function, *Trans. Amer. Math. Soc.* **295**(1986), No. 1, 305–324. MR0831201; https://doi.org/10.1090/S0002-9947-1986-0831201-3

[14] W. H. Fleming, A selection-migration model in population genetics, *J. Math. Biol.* **2**(1975), No. 3, 219–233. MR0403720; https://doi.org/10.1007/BF00277151

[15] P. Hess, T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, *Comm. Partial Differential Equations* **5**(1980), No. 10, 999–1030. MR0588690; https://doi.org/10.1080/03605308008820162

[16] R. A. Huseynova, Global bifurcation from principal eigenvalues for nonlinear fourth order eigenvalue problem with indefinite weight, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **42**(2016), No. 2, 202–211. MR3581769

[17] S. N. Janczewsky, Oscillation theorems for the differential boundary value problems of the fourth order, *Ann. Math.* **29**(1927/28), No. 1–4, 521–542. MR1502859; https://doi.org/10.2307/1968021

[18] A. El Khalil, S. Kellati, A. Touzani, On the spectrum of the p-biharmonic operator, in: *Proceedings of the 2002 Fez Conference on Partial Differential Equations*, *Electron. J. Differ. Equ. Conf.* **9**, Southwest Texas State Univ., San Marcos, TX, 2002, pp. 161–170. MR1976693

[19] M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, Macmillan, New York, 1964. MR0159197

[20] A. P. Makhmudov, Z. S. Aliev, Global bifurcation of solutions of certain nonlinearizable eigenvalue problems, *Differ. Equ.* **25**(1989), No. 1, 71–76. MR986400

[21] T. G. Myers, Thin films with high surface tension, *SIAM Rev.* **40**(1998), No. 3, 441–462. https://doi.org/10.1137/S003614459529284X

[22] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7**(1971), No. 3, 487–513. MR0301587 ; https://doi.org/10.1016/0022-1236(71)90030-9

[23] B. P. Rynne, Bifurcation from zero or infinity in Sturm–Liouville problems which are not linearizable, *J. Math. Anal. Appl.* **228**(1998), No. 1, 141–156. MR1659893; https://doi.org/10.1006/jmaa.1998.6122

[24] K. Schmitt, H. L. Smith, On eigenvalue problems for nondifferentiable mappings, *J. Differential Equations* **33**(1979), No. 3, 294–319. MR0543701; https://doi.org/10.1016/0022-0396(79)90067-6