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Upper and lower absolutely continuous functions with applications to discontinuous differential equations.

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Abstract. We use upper and lower absolutely continuous functions as subsolutions and supersolutions to discontinuous ordinary differential equations. We present sufficient conditions for the existence of extremal solutions to initial value problems. Due to a new notion of sub and supersolutions we generalize previous results and present elementary and relatively simple proofs.

Keywords: generalized solution, discontinuous differential equations.

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1 Introduction

We consider discontinuous ordinary differential equations and introduce a definition of sub and supersolution for initial value problems by means of upper absolutely and lower absolutely continuous functions. Our paper extends some previous results, in particular [9,16]. By using a new notion of sub and supersolutions our proofs are relatively short and clear.

The discontinuity in the equation is of the type given in [9] and earlier in its stronger versions in [2,3,20,23]. Our main interest is in the Cauchy problem

$$u'(t) = f(t, u(t)) \qquad \text{in } [0, T], \quad u(0) = u_0, \tag{1.1}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, and $u_0 \in \mathbb{R}$ under the following assumption.

Assumption 1.1. Suppose that

1) there exists an integrable $h : [0, T] \to \mathbb{R}$ such that $|f(t, u)| \le h(t)$ for a.e. $t \in [0, T]$ and all $u \in \mathbb{R}$,

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2) for a.e. t in $[0,T]$ and all $u \in \mathbb{R}$

$$\limsup_{v\to u^-} f(t,v) \le f(t,u) \le \liminf_{v\to u^+} f(t,v), \tag{1.2}$$

3) for every $u \in \mathbb{R}$ $f(\cdot,u)$ is Lebesgue measurable.

If we assume that $f(t,\cdot)$ is continuous for a.e. t in $[0,T]$ then the condition 2) is satisfied and 1)–3) coincide with well known Carathéodory's conditions. The first existence result for (1.1) under these conditions can be found in [5]. The paper [8] is probably the first where the idea of the original Peano existence theorem proof (see[14]) was applied to Carathéodory's solutions. A maximal solution to (1.1) is obtained there as the supremum of all subsolutions. That method is very fruitful especially when we consider discontinuous differential equations where standard analytical methods do not work.

The simplest example of a discontinuous function f such that Assumption 1.1 2) is satisfied is f nondecreasing. This is the reason why f satisfying (1.2) is sometimes called "quasi-increasing" (see [3]). On the other hand the term "quasi-semicontinuous" is also used (see [4]).

The Condition 2) of Assumption 1.1 has been the subject of intensive studies by many researchers. In the present form it appeared in [9]. In earlier papers various stronger versions had been considered (together with conditions 1), 3)). For instance, in [23] the author assumes the condition

$$\lim_{v\to u^-} f(t,v) \le f(t,u) = \lim_{v\to u^+} f(t,v).$$

In [2] it is replaced by

$$\limsup_{v\to u^-} f(t,v) \le f(t,u) = \lim_{v\to u^+} f(t,v),$$

and in [20] by

$$\limsup_{v\to u^-} f(t,v) \le f(t,u) = \liminf_{v\to u^+} f(t,v).$$

In [3] the condition 2) is assumed together with an additional assumption that the function $f(\cdot,u(\cdot))$ is measurable for every absolutely continuous function u. The existence of extremal solutions to (1.1) was proved in [2,3,9,20,23], in almost each case, in a long and difficult way. This is especially true in the case of [9] where the authors admit that a little change in the condition 2) (in comparison to [20]) causes serious troubles in the proof (see [9, Theorem 3.1]).

In the following we give a new, short and relatively easy proof of the theorem that generalizes [9, Theorem 3.1]. We use upper absolutely continuous functions as subsolutions and we find a maximal solution which is also a maximal subsolution. The reason why our proof is relatively simple and short is due to the notion of subsolutions. In particular a maximal subsolution $\bar{u}$ which is not a solution can be slightly modified on a small interval in such a way that we obtain a subsolution greater than $\bar{u}$. This new subsolution is not continuous (but still upper absolutely continuous). This procedure does not work when one considers absolutely continuous subsolutions.

It must be pointed out that the results obtained in [16,17] for sub and supersolutions (in sets BV^-, BV^+) base on [9, Theorem 3.1] and concentrate only on extremal solutions (not subsolutions). It is assumed also, by the definition, that a superposition of any subsolution and supersolution with f is integrable.

In [18] some generalization of (1.2) is considered. It is assumed that (1.2) may not be fulfilled on points of a countable family of admissible curves. The proof of the existence theorem is very long and difficult. It follows that given in [9].

We refer the reader to [7, 10, 11] for other results concerning discontinuous differential equations.

Our paper is divided into three main parts. In Section 2 we present a definition and properties of semiabsolutely continuous functions. In Section 3 we prove a theorem on the existence of maximal subsolution to (1.1) in the class of upper absolutely continuous functions. In Section 4 we prove a theorem on the existence of maximal solution to (1.1).

2 Semiabsolutely continuous functions

The origin of the notion of upper and lower absolutely continuous functions goes back to Ridder [19] (see also Lee [12], Ponomarev [15]). Roughly speaking, a scalar function is upper or lower absolutely continuous if in the definition of absolutely continuous function we replace a two-sided estimation of increments by one-sided (resp. right or left-sided). In our investigations we use an equivalent version of this definition (see also [22]).

Definition 2.1. Let $a, b \in \mathbb{R}$, $a < b$, $u : [a, b] \to R$. We say that u is an upper absolutely continuous (resp. lower absolutely continuous) if there exists a Lebesgue integrable function $l : [a, b] \to \mathbb{R}$ such that

$$u(t) - u(s) \leq \int_s^t l(\tau)d\tau \qquad (\text{resp. } \geq) \tag{2.1}$$

for $a \leq s \leq t \leq b$. We write $u \in UAC[a, b]$ (resp. $u \in LAC[a, b]$).

The expression '*semiabsolutely continuous*' means that a function is *upper absolutely continuous* or *lower absolutely continuous*. In [15] the term 'absolute upper (resp. lower) semicontinuous' is used.

Proposition 2.1. If $u \in UAC[a, b]$ (resp. $LAC[a, b]$) then u is bounded, it has at most countably many points of discontinuity and one-sided limits in every point of $[a, b]$. Moreover, the derivative u' exists a.e. in $[a, b]$ and it is integrable.

Proof. Set $l^+(t) = \max\{l(t), 0\}$. It is easily seen that

$$u(b) - \int_a^b l^+(\tau)d\tau \leq u(t) \leq u(a) + \int_a^b l^+(\tau)d\tau, \qquad t \in [a, b],$$

hence u is bounded. Notice that $u \in UAC[a, b]$ if and only if the mapping $t \mapsto u(t) - \int_a^t l(\tau)d\tau$ is nonincreasing for some integrable function $l : [a, b] \to \mathbb{R}$. It follows from the property of monotonic functions and from the continuity of $\int_a^t l(\tau)d\tau$ that u has at most countably many points of discontinuity, only of the first kind. □

Remark 2.2. If $u \in UAC[a, b]$ (resp. $u \in LAC[a, b]$) then u is left-side lower (resp. upper) semicontinuous and right-side upper (resp. lower) semicontinuous, i.e. $u(t) \in [u(t^+), u(t^-)] \neq \emptyset$ (resp. $u(t) \in [u(t^-), u(t^+)] \neq \emptyset$) where

$$u(t^+) = \lim_{s \to t^+} u(s), \qquad u(t^-) = \lim_{s \to t^-} u(s), \quad u(a^-) = u(a),\ u(b^+) = u(b).$$

Let $AC[a, b]$ denote the set of all absolutely continuous scalar function in $[a, b]$. It is clear that $AC[a, b] = UAC[a, b] \cap LAC[a, b]$, and $u \in UAC[a, b]$ if and only if $-u \in LAC[a, b]$. Moreover, if $u, v \in UAC[a, b]$ then $u + v \in UAC[a, b]$.

In the following we will concentrate on UAC functions. Analogous facts can be proved for LAC functions by considering $-u$ instead of u.

A function $u : [a,b] \to \mathbb{R}$ is said to be *generalized lower absolutely continuous* if $[a,b]$ is a union of countably many closed intervals such that u is lower absolutely continuous on each. We write $u \in UACG[a,b]$ (cf. [6,12] and [21] where u is also assumed to be continuous).

A function $u : [a,b] \to \mathbb{R}$ is said to be *lower closed monotone* (simply LCM) if for every $[c,d] \subset [a,b]$ u is nonincreasing on $[c,d]$ whenever it is nonincreasing on (c,d).

It is immediate that if u is nonincreasing then $u \in UAC[a,b]$ and $u'(t) \leq 0$ a.e. in $[a,b]$. The next result follows from [12, Theorem 1].

Theorem 2.3. A function $u : [a,b] \to \mathbb{R}$ is nonincreasing if and only if $u \in LCM \cap UACG[a,b]$ and $\underline{D}u(t) \leq 0$ a.e. in $[a,b]$, where $\underline{D}u$ is a lower derivative of u.

Proposition 2.4. A function $u : [a,b] \to \mathbb{R}$ is nonincreasing if and only if $u \in UAC[a,b]$ and $u' \leq 0$ a.e. in $[a,b]$.

Proof. $UAC \subset LCM$ (see Remark 2.2). □

Corollary 2.5. $u \in UAC[a,b]$ if and only if u' exists a.e., is integrable in $[a,b]$ and

$$u(t) - u(s) \leq \int_s^t u'(\tau)d\tau \quad \text{for } a \leq s \leq t \leq b.$$

Proof. "$\Rightarrow$" Notice that $w(t) = u(t) - \int_0^t u'(\tau)d\tau \in UAC[a,b]$ and $w' = 0$ a.e. in $[a,b]$, hence w is nonincreasing. "$\Leftarrow$" is obvious. □

Remark 2.6. Let $BV^-([0,T])$ ($BV^+([0,T])$) be the set of functions of bounded variation on $[0,T]$ which have nonincreasing (nondecreasing) singular parts. In papers [16,17] the author defines subsolution (supersolution) as a function in $BV^-([0,T])$ ($BV^+([0,T])$) with an additional assumption that its composition with f is integrable. Since we do not assume this, it is not difficult to show that our definition is more general.

Proposition 2.7. If $u_i \in UAC[a,b]$, $i = 1,\ldots,n$, then $w = \max\{u_i : i = 1,\ldots,n\} \in UAC[a,b]$ and $w' \leq \max\{u_i' : i = 1,\ldots,n\}$ a.e. in $[a,b]$.

Proof. For $s \leq t$ in $[a,b]$ and $i = 1,\ldots,n$ we have

$$u_i(t) \leq u_i(s) + \int_s^t u_i'(\tau)d\tau \leq w(s) + \int_s^t l(\tau)d\tau$$

where $l = \max\{u_i' : i = 1,\ldots,n\}$. We complete the proof by taking maximum on the left. □

Proposition 2.8. Let $u \in UAC[a,c]$, $v \in UAC[c,b]$, $c \in (a,b)$ and

$$w(t) = \begin{cases} u(t), & t \in [a,c), \\ \alpha, & t = c, \\ v(t), & t \in (c,b]. \end{cases}$$

Then $w \in UAC[a,b]$ if and only if $\alpha \in [v(c^+), u(c^-)] \neq \emptyset$.

Proof. First we demonstrate "⇐". There exist integrable $l_1 : [a, c] \to \mathbb{R}$, $l_2 : [c, b] \to \mathbb{R}$ such that for $s \le t$

$$u(t) \le u(s) + \int_s^t l_1(\tau)d\tau \quad \text{in } [a, c],$$

$$v(t) \le v(s) + \int_s^t l_2(\tau)d\tau \quad \text{in } [c, b].$$

Define $l : [a, b] \to \mathbb{R}$ by setting $l = l_1$ in $[a, c)$ and $l = l_2$ in $(c, b]$. For $s < c < t$ we have

$$w(c) - w(s) = \alpha - u(s) \le u(c^-) - u(s) \le \lim_{r \to c^-} \int_s^r l_1(\tau)d\tau = \int_s^c l_1(\tau)d\tau$$

$$w(t) - w(c) \le v(t) - \alpha \le v(t) - v(c^+) \le \lim_{r \to c^+} \int_r^t l_2(\tau)d\tau = \int_c^t l_2(\tau)d\tau.$$

Thus for $s \le c \le t$

$$w(t) - w(s) = w(t) - w(c) + w(c) - w(s) \le \int_c^t l_2(\tau)d\tau + \int_s^c l_1(\tau)d\tau = \int_s^t l(\tau)d\tau.$$

The cases $s < t < c$ and $c < s < t$ are obvious. To demonstrate "⇒" we see that $v(c^+) = w(c^+)$ and $u(c^-) = w(c^-)$ and apply Remark 2.2. □

By a similar argument we demonstrate the following.

Proposition 2.9. Let $u \in UAC[a, b]$ and

$$w(t) = \begin{cases} \alpha, & t = a, \\ u(t), & t \in (a, b), \\ \beta, & t = b. \end{cases}$$

Then $w \in UAC[a, b]$ if and only if $\alpha \ge u(a^+)$, $\beta \le u(b^-)$.

Proposition 2.10. If $u \in UAC[a, b]$ and $w(t) \in [u(t^+), u(t^-)]$ for all $t \in (a, b)$, $w(a) \ge u(a^+)$, $w(b) \le u(b^-)$, then $w \in UAC[a, b]$ and $w = u$, $w' = u'$ a.e. in $[a, b]$.

3 Extremal solutions of differential inequalities

We say that that u is a subsolution of (1.1) if $u \in UAC[0, T]$ and

$$u'(t) \le f(t, u(t)) \quad \text{a.e. in } [0, \mathrm{T}], \quad u(0) \le u_0. \tag{3.1}$$

We say that u is a supersolution of (1.1) if $u \in LAC[0, T]$ and (3.1) is satisfied with reversed inequalities. We say that u is a solution of (1.1) if $u \in AC[0, T]$ and (1.1) is satisfied a.e. in $[0, T]$. Clearly, u is a solution of (1.1) if it is both subsolution and supersolution of (1.1).

Notice that the Cantor function satisfies (1.1) ($f \equiv 0$, $u_0 = 0$) a.e. in $[0, 1]$ but it is only a supersolution of (1.1).

Although the equality $u(t) = u(s) + \int_s^t u'(\tau)d\tau$ is generally not satisfied for $u \in UAC[0, T]$, $s \le t$ (only "$\le$" holds) we can replace (3.1) by

$$u(t) \le u(s) + \int_s^t f(\tau, u(\tau))d\tau, \qquad 0 \le s \le t \le T$$

if $f(t, u(t))$ is integrable.

Definition 3.1. We call μ a maximal solution (resp. maximal subsolution) of (1.1) if μ is a solution (resp. subsolution) of (1.1) and $u \leq \mu$ for every solution (resp. subsolution) u of (1.1).

In a similar way we define a minimal solution (resp. supersolution) of (1.1) .

Proposition 3.1. If μ is a maximal subsolution of (1.1) then $\mu(0) = u_0$.

Proof. Suppose that $\mu(0) < u_0$. Then $u(t) = \mu(t)$ for $t \in (0, T]$, $u(0) = u_0$ is a subsolution of (1.1) (see Remark 2.2, Proposition 2.9). Hence μ is not a maximal subsolution. □

Proposition 3.2. If there exists an integrable function $g : [0, T] \to \mathbb{R}$ such that $f(t, u) \geq g(t)$ in $[0, T] \times \mathbb{R}$ and there exists a maximal subsolution μ of (1.1) then $\mu \in AC[0, T]$.

Proof. Since $\mu \in UAC[0, T]$ we need to show that $\mu \in LAC[0, T]$. Fix $s \in [0, T)$. Define $\hat{u} = \mu$ in $[0, s]$ and $\hat{u}(t) = \mu(s) + \int_s^t g(\tau)d\tau$ for $t \in (s, T]$. Of course, $\hat{u}$ is a subsolution of (1.1) hence $\hat{u} \leq \mu$. This yields $\mu(t) \geq \mu(s) + \int_s^t g(\tau)d\tau$ for $t \in [s, T]$. Since s is arbitrary, $\mu \in LAC[0, T]$. □

Proposition 3.3. Suppose that u_i $i = 1, 2 \ldots, n$ are subsolutions of (1.1). Then $w = \max\{u_i : i = 1, \ldots, n\}$ is a subsolution of (1.1).

Proof. We may assume that $n = 2$. Let u_1, u_2 be subsolutions of (1.1) and $w = \max(u_1, u_2)$. In view of Proposition 2.7 we have $w \in UAC[a, b]$. Consider $t \in (0, T)$ such that $u_1'(t)$, $u_2'(t)$, $w'(t)$ exist. By the property of upper absolutely continuous functions the set of such t has a full measure. Suppose that $w(t) = u_1(t)$. Since $u_1(t + h) - u_1(t) \leq w(t + h) - w(t)$ for h satisfying $t + h \in [0, T]$ we obtain $w'(t) = u_1'(t)$, and consequently w satisfies (3.1) at the point t. The case $w(t) = u_2(t)$ we treat similarly. Since $w(0) \leq u_0$, the proof is complete. □

In the following we need a weaker version of Assumption 1.1.

Assumption 3.4. Suppose that

1) there exists an integrable function $h : [0, T] \to \mathbb{R}$ such that $f(t, u) \leq h(t)$ for a.e. $t \in [0, T]$ and for all $u \in \mathbb{R}$,

2) for a.e. $t \in [0, T]$ and for all $u \in \mathbb{R}$ we have

$$\limsup_{v \to u^-} f(t, v) \leq f(t, u).$$

Notice that the condition 2) of Assumption 3.4 is satisfies if $f(t, \cdot)$ is nondecreasing a.e. in t.

Proposition 3.5. Suppose that Assumption 3.4 1) holds and $\mathcal{X} \neq \emptyset$ is the set of all subsolutions of (1.1). Then

1) $\bar{u} = \sup\{u : u \in \mathcal{X}\} < \infty$ and $\bar{u} \in UAC[0, T]$,

2) there exists a nondecreasing sequence $\{u_n\} \subset \mathcal{X}$ such that $u_n \uparrow \bar{u}$ a.e. in $[0, T]$.

Proof. 1) For $u \in \mathcal{X}$ we have $u(t) \leq u(s) + \int_s^t h(\tau)d\tau \leq \bar{u}(s) + \int_s^t h(\tau)d\tau$, $s, t \in [0, T]$, $s \leq t$. This gives $\bar{u}(t) \leq \bar{u}(s) + \int_s^t h(\tau)d\tau$. Setting $s = 0$ we have $u(t) \leq u_0 + \int_0^t h(\tau)d\tau$ and $\bar{u} < \infty$.

2) Let $t_{i,n} = \frac{iT}{n}$ $i = 0, 1, \ldots, n$. We claim that for every $\varepsilon > 0$ and n there exists $u_n \in \mathcal{X}$ such that $0 \leq \bar{u}(t_{i,n}) - u_n(t_{i,n}) \leq \frac{\varepsilon}{3}$ for $i = 0, 1, \ldots, n$.

Indeed, for each $t_{i,n}$ there exists $u_{i,n} \in \mathcal{X}$ such that $0 \le \bar{u}(t_{i,n}) - u_{i,n}(t_{i,n}) \le \varepsilon$. We set $\tilde{u}_n = \max\{u_{i,n} : i = 0, 1 \ldots, n\}$ and $u_1 = \tilde{u}_1$, $u_n = \max(\tilde{u}_n, u_{n-1})$ for $n > 1$. In view of Proposition 3.3 $u_n \in \mathcal{X}$.

Let t be a point of continuity of $\bar{u}$ such that Assumption 3.4 1) holds. Let $\delta > 0$ be such that $|\bar{u}(t) - \bar{u}(s)| \le \frac{\varepsilon}{3}$ and $\int_s^t h(s)ds < \frac{\varepsilon}{3}$ if $|t - s| \le \delta$. For n satisfying $T < n\delta$ and i such that $t \in [t_{i-1,n}, t_{i,n}]$ we have

$$0 \le \bar{u}(t) - u_n(t) \le \bar{u}(t) - \bar{u}(t_{i,n}) + \bar{u}(t_{i,n}) - u_n(t_{i,n}) + u_n(t_{i,n}) - u_n(t) \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

because $u_n(t_{i,n}) - u_n(t) \le \int_t^{t_{i,n}} u_n'(\tau)d\tau \le \int_t^{t_{i,n}} h(\tau)d\tau < \frac{\varepsilon}{3}$. □

Remark 3.6. If $\bar{u}$ (see Proposition 3.5 1)) is continuous then $u_n \uparrow \bar{u}$ uniformly.

The next example shows the role of Assumption 3.4 1) in Proposition 3.5.

Example 3.7. Set $f(t, u) = 1 + u^2$, $t \in [0, \pi/2]$, $u_0 = 0$. For $a \in [0, \pi/2)$ define $u_a(t) = \tan t$, $t \in [0, a]$, $u_a(t) = \tan a$, $t \in (a, \pi/2]$. Since $\{u_a : a \in [0, \pi/2)\} \subset \mathcal{X}$ we see that 1) in Proposition 3.5 fails.

Theorem 3.8. Suppose that Assumption 3.4 holds. Let $\mathcal{X} \ne \emptyset$ be the set of all subsolutions of (1.1). Then $\bar{u} = \sup\{u : u \in \mathcal{X}\}$ is a maximal subsolution of (3.1).

Proof. We will show that $\bar{u} = \sup\{u : u \in \mathcal{X}\} < \infty$ is in $\mathcal{X}$. Indeed, in view of Proposition 3.3 and Proposition 3.5, there exists a nondecreasing sequence $u_n \in \mathcal{X}$ such that $u_n \uparrow \bar{u}$ a.e. in $[0, T]$. Let t be such that $\bar{u}'(t)$ exists, $u_n(t) \to \bar{u}(t)$ and $t \in [0, T]$ is a Lebesgue point of $\limsup_{n\to\infty} u_n'(t)$. The set of such t has a full Lebesgue measure in $[0, T]$. For $0 \le s \le t$ we have

$$u_n(t) - \bar{u}(s) \le u_n(t) - u_n(s) \le \int_s^t u_n'(\tau)d\tau.$$

Since $u_n' \le h$ a.e. in $[0, T]$, by Fatou's lemma (letting $n \to \infty$)

$$\bar{u}(t) - \bar{u}(s) \le \limsup_{n\to\infty} \int_s^t u_n'(\tau)d\tau \le \int_s^t \limsup_{n\to\infty} u_n'(\tau)d\tau.$$

Hence, by the Lebesgue differentiation theorem:

$$\bar{u}'(t) \le \limsup_{n\to\infty} u_n'(t) \le \limsup_{n\to\infty} f(t, u_n(t)) \le \max\{\limsup_{v\to\bar{u}(t)^-} f(t, v), f(t, \bar{u}(t))\} \le f(t, \bar{u}(t)).$$

Since $\bar{u}(0) \le u_0$, $\bar{u} \in \mathcal{X}$ and the proof is complete. □

The following examples show that the assumption $\mathcal{X} \ne \emptyset$ in Theorem 3.8 is important and that one cannot omit Assumption 3.4 2).

Example 3.9. Set $f(t, u) = -1/t$, $t \in (0, T]$ and $f(0, u) = 0$. Then Assumption 3.4 is satisfied and $\mathcal{X} = \emptyset$ for any $u_0 \in \mathbb{R}$.

Example 3.10. Set $f(t, u) = 1$, $u < 0$, $f(t, u) = -1$, $u \ge 0$, $u_0 = 0$. Here $\bar{u} \equiv 0$ is a supremum of all subsolutions, but it is not a subsolution. Here, $u_n(t) = -\frac{t}{n}$ is a sequence that exists in view of Proposition 3.5.

4 Extremal solutions of differential equations

If a maximal subsolution of (1.1) is a solution it is always a maximal solution. The following examples show that Assumption 3.4 does not imply that a maximal subsolution is a solution.

Example 4.1. Set $f(t,u) = 1,\ u \le 0,\ f(t,u) = -1,\ u > 0,\ u_0 = 0$. We see that $\bar{u} \equiv 0$ is a maximal subsolution, but it is not a solution (the solution does not exist).

Example 4.2. Set $T = 2,\ u_0 = 0$

$$f(t,u) = \begin{cases} 2\sqrt{u}, & u \in [0,1], \\ 0, & u \in \mathbb{R} \setminus [0,1]. \end{cases}$$

It is a simple matter to check that

$$\phi(t) = \begin{cases} t^2, & t \in [0,1], \\ 1, & t \in (1,2] \end{cases}$$

is a maximal subsolution of (1.1) and

$$\mu(t) = \begin{cases} 0, & t \in [0,1], \\ (t-1)^2, & t \in (1,2] \end{cases}$$

is a maximal solution of (1.1). Clearly, $\mu \le \phi$ and $\mu \ne \phi$.

Define

$$f^*(t,u) = \liminf_{v \to u^+} f(t,v).$$

It is easily seen that

$$f^*(t,u) = \lim_{k\to\infty} \inf_{u<v<u+\frac{1}{k}} f(t,v). \tag{4.1}$$

Lemma 4.3. Suppose that Assumption 1.1 is satisfies then

(i) $f(t,u) \le f^*(t,u)$ a.e. in $t \in [0,T]$ for all $u \in \mathbb{R}$,

(ii) if $x_1, x_2 : [0,T] \to \mathbb{R}$ are continuous and $x_1 < x_2$ then

$$\psi(t) = \inf_{x_1(t)<u<x_2(t)} f(t,u)$$

is measurable.

(iii) $f^*(\cdot, u(\cdot))$ is Lebesgue measurable for any continuous $u : [0,T] \to \mathbb{R}$.

Proof. (i) is obvious, for (ii) see [9, Lemma 2.1] and its short proof, (iii) follows from (4.1) and (ii). □

Theorem 4.4. Suppose that Assumption 1.1 is satisfied then, there exists a maximal solution of (1.1) such that it is its maximal subsolution.

Proof. Since $u_0 - \int_0^t h(s)ds$ is a subsolution of (1.1) then, by Theorem 3.8, there exists a maximal subsolution $\bar{u}$ of (1.1). In view of Proposition 3.1 and Proposition 3.2 $u(0) = u_0$ and $\bar{u} \in AC[0,T]$. We have to demonstrate that $\bar{u}$ satisfies (1.1) a.e. in $[0,T]$. Suppose, on the contrary, that $\bar{u}$ fails to satisfy (1.1) a.e. in $[0,T]$. Since $\bar{u}'(t) \le f(t,\bar{u}(t)) \le f^*(t,\bar{u}(t))$ a.e. in $[0,T]$, $\bar{u}$ does not satisfy $\bar{u}'(t) = f^*(t,\bar{u}(t))$ a.e. in $[0,T]$. Since $\bar{u}$ is its subsolution, there exists a positive measure set $A \subset [0,T]$ such that $\bar{u}'(t) < f^*(t,\bar{u}(t))$. We may assume without loss of generality that all conditions used in the proof hold in A. It follows by the standard argument, that there exists a positive integer n such that $\bar{u}'(t) < f^*(t,\bar{u}(t)) - \frac{2}{n}$ in some set of positive measure $A_n \subset A$. Define

$$A_{k,n} = \left\{ t \in A_n : \inf_{\bar{u}(t) < v < \bar{u}(t) + \frac{1}{k}} f(t,v) > f^*(t,\bar{u}(t)) - \frac{1}{n} \right\}.$$

By (4.1), $A_n = \cup_{k=1}^{\infty} A_{k,n}$ hence, there exists a positive integer k and a set of positive measure $A_{k,n} \subset A_n$ such that

$$\bar{u}'(t) < \inf_{\bar{u}(t) < v < \bar{u}(t) + \frac{1}{k}} f(t,v) - \frac{1}{n} \quad \text{in } A_{k,n}.$$

Let χ be a characteristic function of $A_{k,n}$. Define $\rho : [0,T] \to \mathbb{R}$

$$\rho(t) = \int_a^t \left[\frac{1}{n}\chi(s) + (\chi(s) - 1)(h(s) + \bar{u}'(s)) \right] ds$$

where $a \in A_{k,n} \cap [0,T)$ is such that $\rho'(a) = \frac{1}{n}$. The existence of such a follows from the Lebesgue differentiation theorem and from the fact that $A_{k,n}$ has a positive measure. Since $\rho'(a) = \lim_{t \to a^+} \frac{\rho(t)}{t-a} = \frac{1}{n} > 0$, there exists $\tilde{\delta} > 0$ such that $0 < \rho(t) < \frac{2}{n}(t-a)$ for $t \in (a, a+\tilde{\delta})$. Putting $\delta = \min(\tilde{\delta}, \frac{n}{2k}, T-a)$ we get $0 < \rho(t) < \frac{1}{k}$ for $t \in (a, a+\delta)$. Define

$$\hat{u}(t) = \begin{cases} \bar{u}(t) + \rho(t), & t \in (a, a+\delta), \\ \bar{u}(t), & t \in [0,T] \setminus (a, a+\delta). \end{cases}$$

We see that, $\bar{u} \le \hat{u}$ in $[0,T]$ and $\bar{u} < \hat{u} < \bar{u} + \frac{1}{k}$ in $(a, a+\delta)$. By Proposition 2.8 $\hat{u} \in UAC[0,T]$. We claim that $\hat{u}$ is a subsolution of (1.1). We only need to check that $\hat{u}$ satisfies (3.1) a.e. in $(a, a+\delta)$. If $t \in A_{k,n} \cap (a, a+\delta)$ is such that $\rho'(t)$ exists we have

$$\hat{u}'(t) = \bar{u}'(t) + \frac{1}{n} < \inf_{\bar{u}(t) < v < \bar{u}(t) + \frac{1}{k}} f(t,v) \le f(t, \hat{u}(t)).$$

If $t \in (a, a+\delta) \setminus A_{k,n}$ is such that $\rho'(t)$ exists we have

$$\hat{u}'(t) = \bar{u}'(t) - h(t) - \bar{u}'(t) = -h(t) \le f(t, \hat{u}(t)).$$

Since the set of all t considered in both cases is a full measure subset of $(a, a+\delta)$ we see that $\hat{u}$ is a subsolution of (1.1). This is a contradiction with the definition of $\bar{u}$. □

Remark 4.5. By considering the problem $v' = -f(t,-v)$, $v(u) = -u_0$ we obtain analogical results for supersolutions. We consider "symmetric" version of Assumption 3.4. Since Assumption 1.1 combines these two cases, in Theorem 4.4 the word "maximal" may be replaced by "minimal" and the word "subsolution" by "supersolution".

Assumption 4.6. Suppose that

1) for every $r > 0$ there exists an integrable function $h_r : [0, T] \to \mathbb{R}$ such that $|f(t, u)| \le h_r(t)$ for $|u| \le r$ and for a.e. $t \in [0, T]$,

2) for a.e. $t \in [0, T]$ and all $u \in \mathbb{R}$

$$\limsup_{v \to u^-} f(t, v) \le f(t, u) \le \liminf_{v \to u^+} f(t, v), \tag{4.2}$$

3) for every $u \in \mathbb{R}$, $f(\cdot, u)$ is Lebesgue measurable.

For $\alpha, \beta : [0, T] \to \mathbb{R}$ such that $\alpha \le \beta$ we define

$$[\alpha, \beta] = \{u : [0, T] \to \mathbb{R} : \alpha \le u \le \beta\}.$$

Theorem 4.7. Suppose that Assumption 4.6 holds. If α is a subsolution and β is a supersolution of (1.1) such that $\alpha \le \beta$, then problem (1.1) has a maximal (minimal) solution in $[\alpha, \beta]$ such that it is a maximal (minimal) subsolution (supersolution) in $[\alpha, \beta]$.

Proof. We concentrate on a maximal solution. Consider the problem

$$u'(t) = \hat{f}(t, u(t)), \quad \text{a.e. } t \in [0, T], \quad u(0) = u_0 \tag{4.3}$$

where

$$\hat{f}(t, u) = \begin{cases} \beta'(t), & \beta(t) < u, \\ f(t, u), & \alpha(t) \le u \le \beta(t), \\ \alpha'(t), & u < \alpha(t). \end{cases} \tag{4.4}$$

Note that a similar method with modified problem (4.3) was first used in [1] (see [1, equation (2.5)]). In view of Theorem 4.4 problem (4.3) has a maximal solution μ. We will show that μ is also a maximal solution of (1.1) in $[\alpha, \beta]$. In order to demonstrate this we have to show that the set of solutions of (1.1) which belong to $[\alpha, \beta]$ and the set of all solutions of (4.3) are equal. Of course, every solution of (1.1) which is in $[\alpha, \beta]$ is a solution of (4.3). We will show that an arbitrary solution u of (4.3) belongs to $[\alpha, \beta]$, hence is a solution of (1.1) in $[\alpha, \beta]$. We will show $u \le \beta$ ($u \ge \alpha$ is similar). Suppose, on the contrary, that there exists $\bar{t} \in (0, T]$ such that $u(\bar{t}) > \beta(\bar{t})$. Since $u - \beta \in USC[0, T]$ is left-side lower semicontinuous (see Remark 2.2), there exists $a \in [0, \bar{t}]$ such that $u > \beta$ in $(a, \bar{t}]$ and $u(a) \le \beta(a)$. This gives, by (4.4) $u'(t) = \beta'(t)$ a.e. in $[a, \bar{t}]$. By Proposition 2.4 $u - \beta$ is nonincreasing in $[a, \bar{t}]$, hence $u(\bar{t}) \le \beta(\bar{t})$, a contradiction. To complete the proof suppose that u is a subsolution of (1.1) such that $u \in [\alpha, \beta]$. Clearly, it is also a subsolution of (4.3) hence, by Theorem 4.4 $u \le \mu$. □

Remark 4.8. If there exist integrable functions $a, b : [0, T] \to \mathbb{R}$ such that $|f(t, u)| \le a(t)|u| + b(t)$ a.e. in $t \in [0, T]$, then Theorem 4.7 gives the existence of global extremal solutions of (1.1). In this case we can easily find a pair of sub- and supersolutions $\alpha \le \beta$ such that all the subsolutions are not greater than β and all supersolutions are not less than α.

Example 4.9. Consider the problem

$$u'(t) = -u^2(t) - u(t) + 2t + 1, \quad t \in [0, 2], \qquad u(0) = 0.$$

Define

$$\alpha(t) = \begin{cases} t, & t \in [0,1], \\ 1, & t \in (1,2], \end{cases}$$

$$\beta(t) = \begin{cases} 1, & t \in [0,1/2), \\ 2, & t \in [1/2,2]. \end{cases}$$

It is easy to check that $\alpha \in AC[0,2]$ is a subsolution and $\beta \in LAC[0,2]$ is a supersolution. The function α is not a solution. In this case, as in the proof of Theorem 4.4, we can find a greater subsolution by increasing α on a small interval without preserving continuity. Indeed, define

$$\bar{\alpha}(t) = \begin{cases} t, & t \in [0, \frac{1}{2}], \\ \frac{9}{8}(t - \frac{1}{2}) + \frac{1}{2}, & t \in (\frac{1}{2}, \frac{9}{16}], \\ t, & t \in (\frac{9}{16}, 1], \\ 1, & t \in (1,2]. \end{cases}$$

An easy computation shows that $\bar{\alpha} \in UAC[0,2]$ and $\bar{\alpha}$ is a subsolution such that $\alpha \le \bar{\alpha}$, $\alpha \ne \bar{\alpha}$. By virtue of Theorem 4.7 the problem has extremal solutions in $[0,2]$, between $\bar{\alpha}$ and β.

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