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On an extension of a recurrent relation from combinatorics

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Abstract. The following recurrent relation/partial difference equation

$$w_{n,k} = aw_{n-1,k-1} + bw_{n-1,k},$$

where $k, n, a, b \in \mathbb{N}$, appears in a problem in combinatorics. Here we show that an extension of the recurrent relation is solvable on, the, so called, combinatorial domain $\mathcal{C} = \{(n,k) \in \mathbb{N}_0^2 : 0 \leq k \leq n\} \setminus \{(0,0)\}$, when its coefficients and the boundary values $w_{j,0}$, $w_{j,j}$, $j \in \mathbb{N}$, are complex numbers, by presenting a representation of the general solution to the recurrent relation on the domain in terms of the boundary values. As a special case we obtain a solution to the problem in combinatorics in an elegant way. From the general solution along with an application of the linear first-order difference equation is also obtained the solution of the recurrent relation in the case $w_{j,j} = c \in \mathbb{C}$, $j \in \mathbb{N}$.

Keywords: partial difference equation, boundary-value problem, combinatorial domain, equation solvable in closed form, method of half-lines.

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1 Introduction

Throughout the paper by C_k^n, $0 \leq k \leq n$, are denoted the binomial coefficients. Recall that

$$C_0^n = C_n^n = 1, \quad n \in \mathbb{N}_0,$$

and

$$C_k^n = C_k^{n-1} + C_{k-1}^{n-1}, \tag{1.1}$$

for every $k, n \in \mathbb{N}$, such that $1 \leq k < n$. Many other properties and relations of the coefficients, can be found, for example, in [11,13,15,23,24,43] (some basic ones can be found in [11] and [13], more complex ones can be found in [15], a list of numerous ones can be found in [23], some

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advanced methods for dealing with the coefficients can be found in [24], for a combinatorial approach see [43]).

The following recurrent relation

$$w_{n,k} = aw_{n-1,k-1} + bw_{n-1,k}, \tag{1.2}$$

where $k, n, a, b \in \mathbb{N}$, appears, among others, in a well-known problem from combinatorics. Namely, if a box contains a white balls and b balls of a different color, then $w_{n,k}$ in (1.2) counts the number of ways of obtaining k white balls in n draws with replacement. It is clear that the problem satisfies the following 'boundary-value' conditions

$$w_{j,j} = a^j \quad \text{and} \quad w_{j,0} = b^j, \quad j \in \mathbb{N}_0, \tag{1.3}$$

(see, for example, [10]).

The fact that recurrent relation (1.2) and many related ones model concrete problems from combinatorics and consequently deal with integer numbers only, seems one of the main reasons why a great majority of mathematicians do not consider them on more general domains.

Our investigations of the solvability of difference equations and systems of difference equations up to 2013 (see, for example, [3, 25, 26, 33–36] and numerous related references therein), along with the facts that many combinatorial problems of this kind have solutions in the form of some closed-form formulas, and that they are frequently presented by some recurrent relations, have suggested us that the closed-form formulas could be special cases of the general solutions to some problems connected to the recurrent relations. Hence, looking at recurrent relation (1.1) and its solution in the closed form, which is given by the formula

$$C_k^n = \frac{n!}{k!(n-k)'},$$

we came up with an idea to try to solve the recurrent relation on its natural domain, which in this case, as well as for the case of many problems in enumerative combinatorics is the following one

$$\mathcal{C} = \{(n,k) \in \mathbb{N}_0^2 : 0 \le k \le n\} \setminus \{(0,0)\}.$$

We call it *the combinatorial domain*. The problem seems classical, but we could not locate the problem and its solution in the literature and later we published our solution to the problem in [27]. A more important fact connected to our paper [27], than the solution to the concrete problem, is that it provides a method for solving related classes of recurrent relations, which was later used in [28] and [32]. So, it turned out that the method is a general one.

The recurrent relation (1.2) with two independent variables, which is a generalization of (1.1), is, in fact, a partial difference equation. Unlike difference equations with one independent variable these ones are not studied so much, and many books contain just a few of them or, as usual, appear as models in combinatorics (see, for example, [11, 13, 15, 24, 43]). Some classical methods for finding solutions to partial difference equations can be found, for example, in [9, Chapter 12] and [12, Chapter 8] (see also [10]). For some results up to 2003, see [5] (see also [16]). A problem connected to solving partial difference equations is that formulas for their general solutions cannot be easily used to get solutions on concrete domains. Hence, an equation should be considered on a domain, and tried to use some of its special features which can help in finding its solutions. For example, for the case of the combinatorial domain it should be used its boundary which consists of two discrete half-lines. So, the general problem is to find the solution to a partial difference equation in terms of the values on the boundary on a given domain, that is, to solve a boundary-value problem on the domain.

Since many of the partial difference equations in combinatorics are linear, our idea in [27] is to consider the equations on some one-dimensional domains and treat them as linear difference equations of first-order, that is, as special cases of the following equation

$$x_n = \alpha_n x_{n-1} + \beta_n, \quad n \in \mathbb{N}, \tag{1.4}$$

where x_0, $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are real or complex numbers.

The importance of equation (1.4) is in its solvability (in [4, 9, 10, 15] can be found several methods for solving the equation; book [15] contains a nice presentation of three methods for solving the equation). No doubt it is one of the most important solvable equations. It is only enough to note that all the nonlinear equations and systems in the following papers: [3,21,25,26,33–39] (see also the references therein) are essentially solved by transforming them to one or several special cases of equation (1.4). Some recent papers consider product-type equations and systems on the complex domain (see, for example, [29,40–42] and the references therein). They cannot be easily dealt with by equation (1.4), although at the final steps some linear difference equations decide their solvability, and essentially the solvability of equation (1.4) is hidden behind their solvability. In fact, papers [29, 40–42] use, among others, the solvability of the product-type analog of equation (1.4), that is, of the equation

$$z_n = b_n z_{n-1}^{a_n}, \quad n \in \mathbb{N}_0.$$

For some recent results and applications of equation (1.4) on various subdomains of $\mathbb{Z}$ see also [30] and [31], which are partly motivated by a problem in [6]. Many other classical solvable difference equations and systems can be found, for example, in [4, 9, 10, 12, 14, 15], while numerous related classical problems can be found in [6, 11, 14, 22]. For some related topis, such as is finding concrete type of solutions or invariants of difference equations and systems or some applications of solvable difference equations, see, for example, [2,7,8,17–20], as well as the references therein.

Our aim here is to solve the following extension of equation (1.2)

$$w_{n,k} = a w_{n-1,k-1} + b_n w_{n-1,k}, \tag{1.5}$$

on domain $\mathcal{C}$, where $(b_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers. Unlike the equation treated in [27], where equation (1.4) is used in one of its most simplest forms, that is, when $\alpha_n = 1$, $n \in \mathbb{N}$, so that essentially was used the method of telescoping summation, here we will have a real application of equation (1.4) in solving equation (1.5). Regarding the case $\alpha_n = 1$, $n \in \mathbb{N}$, recall only that the behavior of the solutions to equation (1.4) can be quite complex (see [14,22], and [1] for the case of metric spaces), but here we will not deal with the behavior of the solutions to equation (1.5). As usual, when dealing with finite sums, we will use the convention $\sum_{j=m}^{l} c_j = 0$, when $l < m$. If $k, l \in \mathbb{Z}$, are such that $k \le l$, then $j = \overline{k, l}$ is a notation for the set of all $j \in \mathbb{Z}$ such that $k \le j \le l$.

2 Main result

In this section we state and prove the main results in this paper. Two cases are considered separately: a) $a = 0$; b) $a \neq 0$.

Case $a = 0$. Since $a = 0$, then equation (1.5) becomes

$$w_{n,k} = b_n w_{n-1,k}, \tag{2.1}$$

from which it is not difficult to see that

$$w_{n,k} = w_{0,k} \prod_{j=1}^{n} b_j, \tag{2.2}$$

which is the solution to equation (1.5) in this case.

Formula (2.2) shows that special form of equation (1.5) in this case produces solutions which depend only on the boundary values $w_{0,k}$, $k \in \mathbb{N}$. Hence, the natural domain for equation (2.1) is not the combinatorial one, but the following one

$$\mathcal{Q} := \{(n,k) \in \mathbb{N}_0^2 : k \in \mathbb{N}_0, n \in \mathbb{N}\}.$$

Case $a \in \mathbb{C} \setminus \{0\}$. To solve equation (1.5) in this case on domain $\mathcal{C}$, we use our *method of half-lines*. A detailed explanation of the method and ideas behind it can be found in our original source [27] (see, also [28] and [32]). Recall that, as usual, the domain will be sliced by the lines $n = k + s$ for each (fixed) $s \in \mathbb{N}$ (the name of the method has been motivated by the obvious fact that the intersection of the domain with the lines produces discrete half-lines).

Let $s = 1$, then $n = k + 1$ and

$$w_{k+1,k} = a w_{k,k-1} + b_{k+1} w_{k,k}, \tag{2.3}$$

for $k \in \mathbb{N}$.

Following the idea in [27], we regard (2.3) as an equation of the form in (1.4), more precisely, as the one where

$$x_{k-1} = w_{k,k-1}, \quad \alpha_k = a \quad \text{and} \quad \beta_k = b_{k+1} w_{k,k}, \quad k \in \mathbb{N},$$

and then we 'solve' the equation by one of known methods (see, for example, the ones in [15]).

Namely, multiplying the following equality

$$w_{j+1,j} = a w_{j,j-1} + b_{j+1} w_{j,j}, \tag{2.4}$$

by a^{k-j} for each $j \in \{1, \ldots, k\}$, and summing up such obtained k equalities, it follows that

$$w_{k+1,k} = a^k w_{1,0} + \sum_{j=1}^{k} a^{k-j} b_{j+1} w_{j,j}, \tag{2.5}$$

for $k \in \mathbb{N}_0$.

Now, let $s = 2$, then $n = k + 2$ and

$$w_{k+2,k} = a w_{k+1,k-1} + b_{k+2} w_{k+1,k}, \tag{2.6}$$

for $k \in \mathbb{N}$.

By, multiplying the following equality

$$w_{l+2,l} = a w_{l+1,l-1} + b_{l+2} w_{l+1,l}, \tag{2.7}$$

by a^{k-l} for each $l \in \{1, \ldots, k\}$, and summing up such obtained k equalities, it follows that

$$w_{k+2,k} = a^k w_{2,0} + \sum_{l=1}^{k} a^{k-l} b_{l+2} w_{l+1,l}, \tag{2.8}$$

for $k \in \mathbb{N}_0$.

Combining (2.5) and (2.8), it follows that

$$\begin{aligned} w_{k+2,k} &= a^k w_{2,0} + \sum_{l=1}^{k} a^{k-l} b_{l+2} \Big(a^l w_{1,0} + \sum_{j=1}^{l} a^{l-j} b_{j+1} w_{j,j} \Big) \\ &= a^k w_{2,0} + a^k w_{1,0} \sum_{l=1}^{k} b_{l+2} + \sum_{l=1}^{k} a^{k-l} b_{l+2} \sum_{j=1}^{l} a^{l-j} b_{j+1} w_{j,j}, \end{aligned} \tag{2.9}$$

for $k \in \mathbb{N}_0$.

Motivated by (2.5) and (2.9), one can assume that for a fixed $r \in \mathbb{N}$

$$\begin{aligned} w_{k+r,k} = a^k w_{r,0} &+ a^k w_{r-1,0} \sum_{j_r=1}^{k} b_{j_r+r} + \cdots \\ &+ a^k w_{1,0} \sum_{j_r=1}^{k} b_{j_r+r} \sum_{j_{r-1}=1}^{j_r} b_{j_{r-1}+r-1} \cdots \sum_{j_2=1}^{j_3} b_{j_2+2} \\ &+ \sum_{j_r=1}^{k} a^{k-j_r} b_{j_r+r} \sum_{j_{r-1}=1}^{j_r} a^{j_r-j_{r-1}} b_{j_{r-1}+r-1} \cdots \sum_{j_1=1}^{j_2} a^{j_2-j_1} b_{j_1+1} w_{j_1,j_1}, \end{aligned} \tag{2.10}$$

for $k \in \mathbb{N}_0$.

Let $s = r+1$, then $n = k+r+1$ and

$$w_{k+r+1,k} = a w_{k+r,k-1} + b_{k+r+1} w_{k+r,k}, \quad k \in \mathbb{N}. \tag{2.11}$$

By, multiplying the following equalities

$$w_{j_{r+1}+r+1,j_{r+1}} = a w_{j_{r+1}+r,j_{r+1}-1} + b_{j_{r+1}+r+1} w_{j_{r+1}+r,j_{r+1}}, \quad j_{r+1} = \overline{1,k}, \tag{2.12}$$

by $a^{k-j_{r+1}}$, $j_{r+1} = \overline{1,k}$, and summing up such obtained k equalities, it follows that

$$w_{k+r+1,k} = a^k w_{r+1,0} + \sum_{j_{r+1}=1}^{k} a^{k-j_{r+1}} b_{j_{r+1}+r+1} w_{j_{r+1}+r,j_{r+1}}, \tag{2.13}$$

for $k \in \mathbb{N}_0$.

Using the hypothesis (2.10) in (2.13) and some simple calculation, we have

$$\begin{aligned} w_{k+r+1,k} = a^k w_{r+1,0} & \\ &+ \sum_{j_{r+1}=1}^{k} a^{k-j_{r+1}} b_{j_{r+1}+r+1} \Big(a^{j_{r+1}} w_{r,0} + a^{j_{r+1}} w_{r-1,0} \sum_{j_r=1}^{j_{r+1}} b_{j_r+r} + \cdots \\ &+ a^{j_{r+1}} w_{1,0} \sum_{j_r=1}^{j_{r+1}} b_{j_r+r} \sum_{j_{r-1}=1}^{j_r} b_{j_{r-1}+r-1} \cdots \sum_{j_2=1}^{j_3} b_{j_2+2} \\ &+ \sum_{j_r=1}^{j_{r+1}} a^{j_{r+1}-j_r} b_{j_r+r} \sum_{j_{r-1}=1}^{j_r} a^{j_r-j_{r-1}} b_{j_{r-1}+r-1} \cdots \sum_{j_1=1}^{j_2} a^{j_2-j_1} b_{j_1+1} w_{j_1,j_1} \Big) \end{aligned}$$

$$
\begin{aligned}
&= a^k w_{r+1,0} + a^k w_{r,0} \sum_{j_{r+1}=1}^{k} b_{j_{r+1}+r+1} + \cdots \\
&\quad + a^k w_{1,0} \sum_{j_{r+1}=1}^{k} b_{j_{r+1}+r+1} \sum_{j_r=1}^{j_{r+1}} b_{j_r+r} \cdots \sum_{j_2=1}^{j_3} b_{j_2+2} \\
&\quad + \sum_{j_{r+1}=1}^{k} a^{k-j_{r+1}} b_{j_{r+1}+r+1} \sum_{j_r=1}^{j_{r+1}} a^{j_{r+1}-j_r} b_{j_r+r} \cdots \sum_{j_1=1}^{j_2} a^{j_2-j_1} b_{j_1+1} w_{j_1,j_1},
\end{aligned}
\tag{2.14}
$$

for $k \in \mathbb{N}_0$. Hence, by induction we have shown that (2.10) holds.

By changing the order of summation, (2.10) can be also written in the following form

$$
\begin{aligned}
w_{k+r,k} &= a^k w_{r,0} + a^k w_{r-1,0} \sum_{j_r=1}^{k} b_{j_r+r} + \cdots \\
&\quad + a^k w_{1,0} \sum_{j_r=1}^{k} b_{j_r+r} \sum_{j_{r-1}=1}^{j_r} b_{j_{r-1}+r-1} \cdots \sum_{j_2=1}^{j_3} b_{j_2+2} \\
&\quad + \sum_{j_1=1}^{k} a^{k-j_1} b_{j_1+1} w_{j_1,j_1} \sum_{j_2=j_1}^{k} b_{j_2+2} \cdots \sum_{j_r=j_{r-1}}^{k} b_{j_r+r},
\end{aligned}
\tag{2.15}
$$

for $k \in \mathbb{N}_0$.

Due to the above considerations, we are now in a position to formulate and prove the main result in this paper.

Theorem 2.1. *Consider equation (1.5). Assume that $a \neq 0$, and that $(u_k)_{k\in\mathbb{N}}$, $(v_k)_{k\in\mathbb{N}}$ are given sequences of complex numbers. Then the solution to the equation on domain C with the following boundary-value conditions*

$$
w_{k,0} = u_k \quad \text{and} \quad w_{k,k} = v_k, \quad k \in \mathbb{N},
\tag{2.16}
$$

is given by

$$
\begin{aligned}
w_{n,k} &= a^k u_{n-k} + a^k u_{n-k-1} \sum_{j_{n-k}=1}^{k} b_{j_{n-k}+n-k} + \cdots \\
&\quad + a^k u_1 \sum_{j_{n-k}=1}^{k} b_{j_{n-k}+n-k} \sum_{j_{n-k-1}=1}^{j_{n-k}} b_{j_{n-k-1}+n-k-1} \cdots \sum_{j_2=1}^{j_3} b_{j_2+2} \\
&\quad + \sum_{j_1=1}^{k} a^{k-j_1} b_{j_1+1} v_{j_1} \sum_{j_2=j_1}^{k} b_{j_2+2} \cdots \sum_{j_{n-k}=j_{n-k-1}}^{k} b_{j_{n-k}+n-k}.
\end{aligned}
\tag{2.17}
$$

Proof. If we put $r = n-k$ in (2.15), we get

$$
\begin{aligned}
w_{n,k} &= a^k w_{n-k,0} + a^k w_{n-k-1,0} \sum_{j_{n-k}=1}^{k} b_{j_{n-k}+n-k} + \cdots \\
&\quad + a^k w_{1,0} \sum_{j_{n-k}=1}^{k} b_{j_{n-k}+n-k} \sum_{j_{n-k-1}=1}^{j_{n-k}} b_{j_{n-k-1}+n-k-1} \cdots \sum_{j_2=1}^{j_3} b_{j_2+2} \\
&\quad + \sum_{j_1=1}^{k} a^{k-j_1} b_{j_1+1} w_{j_1,j_1} \sum_{j_2=j_1}^{k} b_{j_2+2} \cdots \sum_{j_{n-k}=j_{n-k-1}}^{k} b_{j_{n-k}+n-k}.
\end{aligned}
\tag{2.18}
$$

Using (2.16) in (2.18), the formula (2.17) follows. □

Now we are going to see what Theorem 2.1 gives in the case of equation (1.2), that is, when

$$b_n = b, \quad n \in \mathbb{N}, \tag{2.19}$$

in equation (1.5).

By using (2.19) in formula (2.18) and some simple calculation, we get

$$\begin{aligned} w_{n,k} &= a^k w_{n-k,0} + a^k b w_{n-k-1,0} \sum_{j_{n-k}=1}^{k} 1 + \cdots \\ &\quad + a^k b^{n-k-1} w_{1,0} \sum_{j_{n-k}=1}^{k} \sum_{j_{n-k-1}=1}^{j_{n-k}} \cdots \sum_{j_2=1}^{j_3} 1 \\ &\quad + b^{n-k} \sum_{j_1=1}^{k} a^{k-j_1} w_{j_1,j_1} \sum_{j_2=j_1}^{k} \cdots \sum_{j_{n-k}=j_{n-k-1}}^{k} 1. \end{aligned} \tag{2.20}$$

On the other hand, it is known that

$$\sum_{j_m=1}^{k} \sum_{j_{m-1}=1}^{j_m} \cdots \sum_{j_1=1}^{j_2} 1 = C_m^{k+m-1}, \tag{2.21}$$

for $k, m \in \mathbb{N}$, (see, for example, [15]).

By changing the order of summation and by using equality (2.21), we obtain

$$\begin{aligned} \sum_{j_2=j_1}^{k} \sum_{j_3=j_2}^{k} \cdots \sum_{j_{n-k}=j_{n-k-1}}^{k} 1 &= \sum_{j_{n-k}=j_1}^{k} \sum_{j_{n-k-1}=j_1}^{j_{n-k}} \cdots \sum_{j_2=j_1}^{j_3} 1 \\ &= \sum_{i_{n-k}=1}^{k-j_1+1} \sum_{i_{n-k-1}=1}^{j_{n-k}-j_1+1} \cdots \sum_{i_2=1}^{j_3-j_1+1} 1 \\ &= C_{n-k-1}^{n-j_1-1}. \end{aligned} \tag{2.22}$$

From (2.20)–(2.22), is obtained

$$\begin{aligned} w_{n,k} &= a^k w_{n-k,0} + a^k b w_{n-k-1,0} C_1^k + \cdots + a^k b^{n-k-1} w_{1,0} C_{n-k-1}^{n-2} \\ &\quad + b^{n-k} \sum_{j=1}^{k} a^{k-j} C_{n-k-1}^{n-j-1} w_{j,j}, \end{aligned} \tag{2.23}$$

for $0 \le k < n$.

From this we obtain the following corollary of Theorem 2.1, which we formulate as a theorem.

Theorem 2.2. *Consider equation (1.2). Assume that $a \neq 0$, and that $(u_k)_{k\in\mathbb{N}}$, $(v_k)_{k\in\mathbb{N}}$ are given sequences of complex numbers. Then the solution to the equation on domain $\mathcal{C}$ satisfying the boundary-value conditions in (2.16) is given by*

$$w_{n,k} = a^k u_{n-k} + a^k b u_{n-k-1} C_1^k + \cdots + a^k b^{n-k-1} u_1 C_{n-k-1}^{n-2} + b^{n-k} \sum_{j=1}^{k} a^{k-j} C_{n-k-1}^{n-j-1} v_j, \tag{2.24}$$

for $(n, k) \in \mathcal{C}$.

Proof. By using boundary-value conditions (2.16) in formula (2.23) is obtained (2.24). □

Corollary 2.3. *The solution to partial difference equation* (1.2) *on domain $\mathcal{C}$ satisfying the following boundary-value conditions*

$$w_{j,j} = ca^j \quad \text{and} \quad w_{j,0} = cb^j, \quad j \in \mathbb{N}, \tag{2.25}$$

for some $a, b, c \in \mathbb{C}$, is given by

$$w_{n,k} = ca^k b^{n-k} C_k^n. \tag{2.26}$$

Proof. Using boundary value conditions (2.25) in (2.24), it follows that

$$w_{n,k} = ca^k b^{n-k} \left(\sum_{l=0}^{n-k-1} C_l^{k+l-1} + \sum_{j=1}^{k} C_{n-k-1}^{n-j-1} \right), \tag{2.27}$$

for $0 \le k < n$.

Noticing that $C_0^{k-1} = C_0^k$, and using (1.1), we get

$$\sum_{l=0}^{n-k-1} C_l^{k+l-1} = C_0^k + C_1^k + C_2^{k+1} + \cdots + C_{n-k-1}^{n-2} = C_{n-k-1}^{n-1}. \tag{2.28}$$

Further, we have

$$C_{n-k-1}^{n-1} + \sum_{j=1}^{k} C_{n-k-1}^{n-j-1} = \sum_{j=0}^{k} C_{n-k-1}^{n-j-1} = \sum_{i=0}^{k} C_{n-k-1}^{i+n-k-1}. \tag{2.29}$$

By using recurrent relation (1.1), we have

$$\sum_{i=0}^{k} C_{n-k-1}^{i+n-k-1} = \sum_{i=0}^{k} \left(C_{n-k}^{i+n-k} - C_{n-k}^{i+n-k-1} \right) = C_{n-k}^{n} - C_{n-k}^{n-k-1} = C_{n-k}^{n}, \tag{2.30}$$

for $0 \le k < n$.

Employing (2.28)–(2.30) in (2.27), and the following well-known fact

$$C_{n-k}^n = C_k^n,$$

is obtained (2.26), as desired. □

If in Corollary 2.3 is chosen $c = 1$, we obtain the following result which solves the combinatorial problem mentioned in introduction.

Corollary 2.4. *The solution to partial difference equation* (1.2) *on domain $\mathcal{C}$ satisfying the boundary-value conditions in* (1.3) *is given by*

$$w_{n,k} = a^k b^{n-k} C_k^n. \tag{2.31}$$

Remark 2.5. Formula (2.31) presents the well-known solution to the combinatorial problem mentioned in introduction, which is usually obtained by some combinatorial arguments.

One of the interesting cases of equation (1.2) is obtained for the following conditions

$$w_{k,k} = c \in \mathbb{C}, \quad k \in \mathbb{N}. \tag{2.32}$$

Note that they are only given on the diagonal half-line.

Namely, then from (2.24) with $n = k + r$, is get

$$w_{k+r,k} = a^k w_{r,0} + a^k b w_{r-1,0} C_1^k + \cdots + a^k b^{r-1} w_{1,0} C_{r-1}^{k+r-2} + cb^r \sum_{s=0}^{k-1} C_{r-1}^{s+r-1} a^s, \tag{2.33}$$

for $k \in \mathbb{N}_0$ and $r \in \mathbb{N}$.

Hence, of some interest is to know the exact value of the sum

$$S_k^{(r)}(z) = \sum_{s=0}^{k-1} C_r^{s+r} z^s, \tag{2.34}$$

for $r \in \mathbb{N}_0$.

It is clear that

$$S_k^{(0)}(z) = \frac{1 - z^k}{1 - z}, \tag{2.35}$$

for $k \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{1\}$.

One way to calculate the sum (2.34) is to note that

$$S_k^{(r)}(z) = \frac{1}{r!} \sum_{s=0}^{k-1} (s + r) \cdots (s + 1) z^s = \frac{1}{r!} \left(\sum_{j=0}^{r+k-1} z^j \right)^{(r)} = \frac{1}{r!} \left(\frac{1 - z^{r+k}}{1 - z} \right)^{(r)},$$

and then to calculate the derivative.

Another way for calculating the sum $S_k^{(r)}(z)$ is to note that the following relations hold

$$\begin{aligned} (1 - z) S_k^{(r)}(z) &= \sum_{s=0}^{k-1} C_r^{s+r} z^s - \sum_{s=0}^{k-1} C_r^{s+r} z^{s+1} \\ &= C_r^r + \sum_{s=1}^{k-1} (C_r^{r+s} - C_r^{r+s-1}) z^s - C_r^{r+k-1} z^k \\ &= C_{r-1}^{r-1} + \sum_{s=1}^{k-1} C_{r-1}^{r+s-1} z^s - C_r^{r+k-1} z^k \\ &= S_k^{(r-1)}(z) - C_r^{r+k-1} z^k, \end{aligned} \tag{2.36}$$

for $z \neq 1$.

From the point of view of difference equations the way is nicer. Namely, from equality (2.36), we see that sequence $(S_k^{(r)}(z))_{r \in \mathbb{N}}$, for the case $z \neq 1$, satisfies the following difference equation

$$S_k^{(r)}(z) = \frac{1}{1 - z} S_k^{(r-1)}(z) - \frac{C_r^{r+k-1} z^k}{1 - z}, \tag{2.37}$$

which for a fixed k is a special case of equation (1.4). So, this is another of many existing examples where equation (1.4) appears, which shows a huge influence and applicability of the equation, as well as its importance, which was also pointed out in the introduction of this paper.

Let us solve equation (2.37) by the above method. By multiplying the following equality

$$S_k^{(j)}(z) = \frac{1}{1-z} S_k^{(j-1)}(z) - \frac{C_j^{j+k-1} z^k}{1-z},$$

by $(1-z)^{-(r-j)}$ for each $j \in \{1, \dots, r\}$, and summing up such obtained r equalities, we get

$$S_k^{(r)}(z) = \frac{S_k^{(0)}(z)}{(1-z)^r} - z^k \sum_{j=0}^{r-1} \frac{C_{j+1}^{k+j}}{(1-z)^{r-j}}. \tag{2.38}$$

From (2.35) and (2.38), it follows that

$$\begin{aligned} S_k^{(r)}(z) &= \frac{1-z^k}{(1-z)^{r+1}} - z^k \sum_{j=0}^{r-1} \frac{C_{j+1}^{k+j}}{(1-z)^{r-j}} \\ &= \frac{1 - z^k \sum_{j=-1}^{r-1} (1-z)^{j+1} C_{j+1}^{k+j}}{(1-z)^{r+1}}, \end{aligned} \tag{2.39}$$

for $k, r \in \mathbb{N}_0$ and $z \in \mathbb{C} \setminus \{1\}$.

On the other hand, if $a = 1$, then by using (1.1), we have

$$\sum_{s=0}^{k-1} C_{r-1}^{s+r-1} = \sum_{s=0}^{k-1} \left(C_r^{s+r} - C_r^{s+r-1} \right) = C_r^{k+r-1}. \tag{2.40}$$

From (2.33), (2.39) and (2.40), is obtained the following result.

Corollary 2.6. *The solution to partial difference equation* (1.2) *on domain* $\mathcal{C}$ *satisfying the boundary-value conditions in* (2.32) *is given by*

$$w_{n,k} = a^k \sum_{j=1}^{n-k} w_{j,0} C_{n-k-j}^{n-1-j} b^{n-k-j} + c b^{n-k} \frac{1 - a^k \sum_{j=-1}^{n-k-2} (1-a)^{j+1} C_{j+1}^{k+j}}{(1-a)^{n-k}},$$

when $a \neq 1$, *while it is given by*

$$w_{n,k} = a^k \sum_{j=1}^{n-k} w_{j,0} C_{n-k-j}^{n-1-j} b^{n-k-j} + c b^{n-k} C_{n-k}^{n-1},$$

when $a = 1$.

Remark 2.7. The trick/method in (2.36) appears in the literature for calculating the sums of the form

$$\widetilde{S}_n^{(i)}(z) := \sum_{j=1}^{n} j^i z^{j-1}, \tag{2.41}$$

for concrete values of $i \in \mathbb{N}_0$ ([11, 14, 15]). In fact, there is a recurrent formula for calculating the sums in (2.41) (see [29]), which has motivated us to get the recurrent relation in (2.37).

Remark 2.8. By suitable choosing of sequences $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$, it can be obtained many sums of the following forms

$$\sum_{j=0}^{n-k-1} d_j C_j^{k-1+j} \quad \text{and} \quad \sum_{j=1}^{k} \hat{d}_j C_{n-k-1}^{n-j-1},$$

which can be summated in closed-form [15,24]. We leave the choice of suitable sequences to the imagination of the reader.

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