Zoltán Buczolich,* Department of Analysis, Pázmány Péter sétány 1/c, 1117 Budapest, Hungary. email: buczo@cs.elte.hu

Stéphane Seuret,† Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, F-94010, Créteil, France. email: seuret@u-pec.fr

HOMOGENEOUS MULTIFRACTAL MEASURES WITH DISJOINT SPECTRUM AND MONOHÖLDER MONOTONE FUNCTIONS

Abstract

We proved in an earlier paper that the support of the multifractal spectrum of a homogeneously multifractal (HM) measure within $[0,1]$ must be an interval. In this paper we construct a homogeneously multifractal measure with spectrum supported by $[0,1] \cup \{2\}$. This shows that there can be a different behaviour for exponents exceeding one. We also provide details of the construction of a strictly monotone increasing monohőlder (and hence HM) function which has exact Hölder exponent one at each point. This function was also used in our paper about measures and functions with prescribed homogeneous multifractal spectrum.

1 Introduction

In this paper we provide details of the construction of two examples announced in [1, Propositions 1.7 and 1.16]. Before stating them in Propositions 7 and 8 we recall some definitions and earlier results.

We denote by $B(x_0, r)$ the open ball with center x_0 and radius r. Recall that the support $\mathrm{Supp}(\mu)$ of a positive Borel measure is the smallest closed set E such that $\mu(E^c) = 0$.

Mathematical Reviews subject classification: Primary: 28A80, 26A16; Secondary: 28A78

Key words: Hölder exponent, multifractal spectrum, homogeneously multifractal

*Research supported by the Hungarian National Foundation for Scientific research K075242 and K104178.

†The second author is partially supported by the grant ANR MUTADIS ANR-11-JS01-0009.

Definition 1. *The local exponent (or local dimension) of a positive Borel measure μ on $\mathbb{R}$ at a given $x_0 \in Supp(\mu)$ is defined as*

$$h_\mu(x_0) = \liminf_{r\to 0^+} \frac{\log \mu(B(x_0,r))}{\log r}. \tag{1}$$

In this paper we will consider only diffuse (non-atomic) measures. The definition of the local regularity exponent of functions is slightly different and we recall it now.

Definition 2. *Let Z be a locally bounded function on $\mathbb{R}$, $x_0 \in \mathbb{R}$ and $\alpha > 0$. The function Z belongs to $C^\alpha(x_0)$ if there is a polynomial P of degree less than $[\alpha]$ and a constant $K > 0$ such that*

$$\text{for every } x \text{ in a neighborhood of } x_0, \;\; |Z(x) - P(x-x_0)| \leq K|x-x_0|^\alpha. \tag{2}$$

The pointwise Hölder exponent of Z at x_0 is $h_Z(x_0) = \sup\{\alpha \geq 0 : \ f \in C^\alpha(x_0)\}$.

There exists a relation between the local exponent of a measure μ at x_0 and the pointwise Hölder exponent of the primitive of μ at x_0, that we will explain later.

Definition 3. *The multifractal spectrum of a measure μ is the mapping d_μ defined as*

$$h \geq 0 \;\longmapsto\; d_\mu(h) := \dim E_\mu(h), \;\text{ where } E_\mu(h) := \{x : h_\mu(x) = h\}. \tag{3}$$

Here, by definition, $\dim \emptyset = -\infty$.

Definition 4. *We call the support of the multifractal spectrum of μ the set*

$$Support\,(d_\mu) = \{h \geq 0 : d_\mu(h) \geq 0\}. \tag{4}$$

Analogous definitions hold for the multifractal spectrum and its support for functions Z.

Homogeneously multifractal measures, or functions, have the same spectrum in any neighborhood of any point of their support. The precise definition is the following:

Definition 5. *A measure μ supported on $[0,1]$ is homogeneously multifractal (or HM) if for any non-empty subinterval $U \subset [0,1]$,*

$$\text{for any } h \geq 0, \;\; \dim\{x \in U : h_\mu(x) = h\} = \dim\{x \in [0,1] : h_\mu(x) = h\} = d_\mu(h).$$

Homogeneously multifractal functions $Z : [0,1] \to \mathbb{R}$ are defined similarly.

In [1, Theorem 1.5] we proved the following Darboux-like theorem about the connectedness in $[0,1]$ of the support of the multifractal spectra of HM measures.

Theorem 6. *For any diffuse HM measure μ supported on $[0,1]$, Support $(d_\mu) \cap [0,1]$ is necessarily an interval of the form $[\alpha,1]$, where $0 \le \alpha \le 1$.*

This theorem holds for HM measures, but not for HM functions. Indeed, there are (necessarily non-monotone) HM functions with disconnected spectra. The most famous example of such a function is the non-differentiable Riemann function

$$\sum_{n\ge 1} \frac{\sin \pi n^2 x}{n^2},$$

which, as shown by S. Jaffard in [5], is HM, and whose support of its multifractal spectrum is $[1/2,3/4] \cup \{3/2\}$. In this paper we prove that for exponents greater than one, Theorem 6 does not hold, that is, HM measures can also have disconnected spectra.

Proposition 7. *There is a HM measure μ on $[0,1]$ such that Support $(d_\mu) = [0,1] \cup \{2\}$.*

The other result of this paper is the construction of a monohölder function with exponent one everywhere. Here is some motivation for this example.

For exponents less than 1 the Hölder exponents of a diffuse measure μ and the monotone increasing function $F_\mu(x) = \mu([0,x])$ coincide. For higher exponents these values may differ. For example if μ is the uniform distribution on $[0,1]$ then $F_\mu(x) = x$ belongs to $C^\infty(\mathbb{R})$, while $h_\mu(x) = 1$ everywhere in $[0,1]$. The function constructed in the proof of the next proposition (which was stated as Proposition 1.16 in [1]) was a tool used in some arguments of [1] to eliminate points where $\{x \in [0,1] : h_Z(x) > 1\}$ for some strictly monotone increasing HM function Z.

Proposition 8. *There exists $Z : [0,1] \to [0,1]$ a strictly monotone increasing HM function with $h_Z(x) = 1$ for all $x \in [0,1]$.*

We will need an upper bound for multifractal spectra of monotone functions. Recall part of Proposition 2.2 of [4] (see also Proposition 4.9 of [3]):

Proposition 9. *Let $E \subset \mathbb{R}^n$ be a Borel set, let μ be a finite Borel measure on $\mathbb{R}^n$ and $0 < c < \infty$. If $\limsup_{r\to 0+} \mu(B(x,r))/r^s \ge c$ for all $x \in E$ then $\mathcal{H}^s(E) \le 2^s \mu(E)/c$.*

Here $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure. From Proposition 9 one can easily deduce the following upper bound, the details can be found for example at the beginning of the proof of assertion (ii) in [2].

Proposition 10. *For any monotone continuous function $Z : [0,1] \to \mathbb{R}$ for every $h \in [0,1]$,*

$$d_Z(h) = \dim E_Z(h) = \dim\{x : h_Z(x) = h\} \leq h. \tag{5}$$

In [2] it is also verified that for the typical monotone continuous function we have equality everywhere in (5).

2 An HM measure with a spectrum gap when $h > 1$

The function $\mathbf{1}^*_{[\alpha_0,\beta_0]}(h)$ is an indicator function of the interval $[\alpha_0, \beta_0]$, i.e. it equals 1 if $h \in [\alpha_0, \beta_0]$ and equals $-\infty$ otherwise. The oscillation of Z on $B(x,r)$ is denoted by $\omega_{B(x,r)}(Z)$:

$$\omega_{B(x,r)}(Z) = \sup_{x \in B(x,r)} Z(x) - \inf_{x \in B(x,r)} Z(x).$$

We recall [1, Theorem 4.1], which allows one to construct measures with arbitrary increasing affine multifractal spectrum:

Theorem 11. *Let $0 < \alpha_0 \leq \beta_0 < 1$. Let $0 < d < \alpha_0$ and $\eta > 0$ satisfy*

$$d(1 + \eta\beta_0) \leq \beta_0 \text{ and } d(1 + \eta\alpha_0) \leq \alpha_0. \tag{6}$$

Then there exists a monotone continuous function Z with the following properties: $Z(x) = 0$ when $x \leq 0$, $Z(x) = 1$ when $x \geq 1$, $d_Z(+\infty) = 1$ and

$$d_Z(h) = d(1 + \eta h)\mathbf{1}^*_{[\alpha_0,\beta_0]}(h) \text{ for } h \in [0, \infty). \tag{7}$$

Moreover, Z can be constructed with the additional properties:

(i) $\{x : h_Z(x) < +\infty\} = \{x : h_Z(x) < 1\} = \{x : h_Z(x) \leq \beta_0\}$ *is located on a Cantor set $\mathcal{C}$, strictly included in* $[0,1]$,

(ii) $[0,1] \setminus \mathcal{C}$ *consists of a countable number of open intervals whose maximal length is less than 1/10,*

(iii) there exists $0 < r_0 < 1$ such that for every $x \in [0,1]$ and $0 < r < r_0$,

$$\omega_{B(x,r)}(Z) = |Z(x + r) - Z(x - r)| \leq (2r)^{\alpha_0}. \tag{8}$$

Next we prove Proposition 7. Our construction is quite technical, but we do not know any other "easy" construction of such a measure. Actually, we will rather build a continuous increasing HM function.

Definition 12. *Given a non-degenerate closed interval $J = [a, b]$ we call $\varphi : J \to \mathbb{R}$ a Cantor-type interpolating function if φ is continuous, non-decreasing and there exists a closed set E_φ of zero Lebesgue measure such that φ is constant on the intervals contiguous to E_φ. Moreover, it has a multifractal spectrum as bad as possible, that is $d_\varphi(h) = h$, for all $h \in [0, 1]$.*

Remark 13. *It is not too difficult to provide direct constructions of such functions, but using Theorem 11 we can easily verify their existence. Indeed, consider sequences $\alpha_{0,n}$, $\beta_{0,n}$, d_n and η_n such that all possible rational values of parameters satisfying the assumptions of Theorem 11 appear among them. Denote by Z_n the sequence of monotone continuous functions which we obtain from Theorem 11 by using these parameter values. Set $\varphi(a) = 0$ and*

$$\varphi(x) = \frac{1}{2^{n-1}} Z_n\Big(\Big(\frac{b-a}{n} - \frac{b-a}{n+1}\Big)^{-1}\Big(x - \frac{b-a}{n+1}\Big)\Big)$$
$$\text{for } x \in \Big(\frac{b-a}{n+1}, \frac{b-a}{n}\Big],\ n = 1, 2, ...$$

Definition 14. *A finite set of real numbers $S = \{s_1, ..., s_n\} \subset (0, 1)$ is said to be δ-discrete (with $\delta > 0$) if*

$$\begin{aligned}&\text{the distance between any two intervals } [s_i - \delta, s_i + \delta],\ i = 1, ..., n,\\ &\text{is larger than } 2\delta, \text{ and the distance of these intervals from } 0 \text{ or}\\ &\text{from } 1 \text{ is also larger than } 2\delta.\end{aligned} \tag{9}$$

Now we are turning to the proof of Proposition 7. We are going to construct the HM measure μ by defining its Borel integral $F(x) = \int_0^x d\mu$. The function F will be an infinite sum of monotone increasing continuous functions F_n.

2.1 First part of the definition of the functions F_n by induction

We introduce several sequences of numbers, sets and functions, which will be the basis of our induction.

First, we fix a sequence of intervals $(I_n := [a_n, b_n])_{n\geq 1}$ satisfying $I_n \subset I_1 := (0, 1)$, $b_n - a_n \searrow 0$ and $\{a_n : n = 1, ...\}$ is dense in $[0, 1]$. The sequence of intervals (I_n) is thus also dense.

We start with $\delta_0 = 1$, $S_0 = \emptyset$, $H_0 = \emptyset$, $I'_1 = [0, 1]$, $T_0 = \emptyset$, $\widetilde{I}_1 = I_1$.

We assumme that for some integer $n \geq 0$, we have built the following:

(i) $n+1$ real numbers $\delta_0 > ... > \delta_n > 0$, satisfying for $1 \le p \le n$

$$(2^{-p}+p)\delta_p^2 < \frac{\delta_{p-1}^2}{2}. \tag{10}$$

(We remark that for the case $n=0$ here and later there is no p satisfying $1 \le p \le n$, which means that in the $n=0$ case these assumptions are not needed.)

(ii) for $1 \le p \le n$, an increasing sequence of finite sets of points $S_p = \{s_1, ..., s_p\} \subset (0,1)$ that are δ_p-discrete and such that $s_p \in I_p$. We put

$$H_p = \bigcup_{i=1}^{p} [s_i - \delta_p, s_i + \delta_p]. \tag{11}$$

(iii) closed intervals $(\widetilde{I}_p)_{p=1,...,n+1}$, such that $\widetilde{I}_p \subset I_p$ and for $1 \le p \le n$,

$$\widetilde{I}_{p+1} \setminus H_p \text{ and } I_{p+2} \setminus H_p \text{ contain closed intervals of length larger than } 2\delta_p. \tag{12}$$

(iv) monotone non-decreasing continuous functions $F_p : [0,1] \to [0,1]$, $1 \le p \le n$ satisfying:

(a) F_p is constant on the intervals $[s_i - \delta_p, s_i + \delta_p/2]$, $i = 1, ..., p$,

(b) on the intervals $[s_i + \delta_p/2, s_i + \delta_p]$, $i = 1, ..., p$, the function F_p coincides with a Cantor-type interpolating function whose increment on this interval is given by

$$F_p(s_i + \delta_p) - F_p(s_i + \delta_p/2) = \delta_p^2 \text{ for } i = 1, ..., p.$$

We call $T_{p,i}$ the nowhere dense closed set associated with the restriction of F_p to $[s_i+\delta_p/2, s_i+\delta_p]$, and $T_p = \bigcup_{i=1}^{p} T_{p,i}$. By construction, F_p is constant on the intervals contiguous to T_p.

(c) for all $x \notin H_p$ there exists an interval $I_{x,p} = [a_{x,p}, b_{x,p}]$ such that

$$x \in I_{x,p},\ b_{x,p} - a_{x,p} < \delta_p \ \text{ and } \ F_p(b_{x,p}) - F_n(a_{x,p}) \ge (b_{x,p} - a_{x,p})^{1+\frac{1}{p}}. \tag{13}$$

(d) for $1 \le p < i \le n$, F_p is constant on the intervals of H_i (see (11)). In other words, H_n "avoids" the Cantor sets T_p, $p \le n-1$ of zero Lebesgue measure where the functions F_p increase.

(e) finally, for $1 \leq p \leq n$,

$$F_p(1) - F_p(0) \leq p\delta_p^2 + \frac{\delta_p^2}{2^p}. \tag{14}$$

Observe that part (iii) implies that $S = \cup_n S_n$ is dense in $[0,1]$.

2.2 The next step of the induction

Suppose $n \geq 0$. We need to define s_{n+1}, δ_{n+1}, $\widetilde{I}_{n+2}$, T_{n+1} and F_{n+1}. We assume that

$$\frac{\delta_{n+1}^2}{2^{n+1}} + (n+1)\delta_{n+1}^2 < \frac{\delta_n^2}{2}.$$

Using (12) we select a closed subinterval $\widetilde{I}_{n+2} \subset I_{n+2} \setminus H_n$ of length $2\delta_n$. Then,

$$\text{in the interior of } \widetilde{I}_{n+1} \setminus H_n, \text{ we select a point } s_{n+1} \notin T_n. \tag{15}$$

Hence, by choosing a sufficiently small $0 < \delta_{n+1} < \delta_n/2$, we can ensure that for all $1 \leq p \leq n$ the functions F_p are constant on $[s_{n+1} - \delta_{n+1}, s_{n+1} + \delta_{n+1}] \subset [0,1] \setminus T_n$, and thus on H_{n+1}.

By (15), we suppose that δ_{n+1} is also so small that

$$[s_{n+1} - \delta_{n+1}, s_{n+1} + \delta_{n+1}] \subset \operatorname{int}(\widetilde{I}_{n+1} \setminus H_n). \tag{16}$$

Now we build F_{n+1}. Set $F_{n+1}(0) = 0$.

Next we define F_{n+1} on $[0,1] \setminus H_{n+1}$, which is constituted by finitely many intervals contiguous to $H_{n+1} \cup \{0\} \cup \{1\}$. Observe that by choosing a sufficiently small δ_{n+1}, we can ensure that for all intervals $[\alpha, \beta]$ contiguous to $H_{n+1} \cup \{0\} \cup \{1\}$, $\beta - \alpha > 2\delta_{n+1}$. In addition, we can also suppose that both $\widetilde{I}_{n+2} \setminus H_{n+1}$ and $I_{n+3} \setminus H_{n+1}$ contain a closed interval of length $2\delta_{n+1}$.

Now fix an interval $J = [\alpha, \beta]$ contiguous to $H_{n+1} \cup \{0\} \cup \{1\}$. We pick an integer $\kappa \in \mathbb{N}$ such that

$$\frac{\beta - \alpha}{\kappa} \leq \frac{\delta_{n+1}^{2(n+1)}}{2^{(n+1)^2}} \tag{17}$$

and subdivide J into subintervals

$$J_l = \left[\alpha + (l-1)\frac{\beta - \alpha}{\kappa}, \alpha + l\frac{\beta - \alpha}{\kappa}\right] = [\alpha_l, \beta_l], \ l = 1, ..., \kappa. \tag{18}$$

We define F_{n+1} so that:

- the increments on J_l are

$$F_{n+1}(\beta_l) - F_{n+1}(\alpha_l) = (\beta_l - \alpha_l)\frac{\delta_{n+1}^2}{2^{n+1}}, \text{ for all } l = 1, ..., \kappa. \tag{19}$$

- On the interior of J_l, $F_{n+1}(x) - F_{n+1}(\alpha_l)$ is a Cantor type interpolating function φ_l and (19) is also satisfied.

From (17) it follows that

$$(\beta_l - \alpha_l)^{\frac{1}{n+1}} = \left(\frac{\beta - \alpha}{\kappa}\right)^{\frac{1}{n+1}} \leq \frac{\delta_{n+1}^2}{2^{n+1}}.$$

By (19) we obtain

$$F_{n+1}(\beta_l) - F_{n+1}(\alpha_l) \geq (\beta_l - \alpha_l)^{1+\frac{1}{n+1}}.$$

By construction, if $x \notin H_{n+1}$, then with the above notation, x belongs to an interval $J_x = [\alpha^x_{l_x}, \beta^x_{l_x}]$, for some interval $[\alpha^x, \beta^x]$ and some suitable integer l_x. We put $a_{x,n+1} = \alpha^x_{l_x}$, $b_{x,n+1} = \beta^x_{l_x}$ and this choice yields part (iv)(c) of the induction.

Finally, it remains to impose the behavior of F_{n+1} on H_{n+1}. We impose that for every $i = 1, ..., n+1$, the function F_{n+1} coincides with a Cantor type interpolating function on $[s_i+\delta_{n+1}/2, s_i+\delta_{n+1}]$, is constant on $[s_i-\delta_{n+1}, s_i+\delta_{n+1}/2]$ and $F_{n+1}(s_i+\delta_{n+1})-F_{n+1}(s_i+\delta_{n+1}/2) = F_{n+1}(s_i+\delta_{n+1})-F_{n+1}(s_i-\delta_{n+1}) = \delta_{n+1}^2$. Hence, the increment of F_{n+1} is defined on all components of H_{n+1}, and thus on $[0, 1]$.

Since H_{n+1} consists of $n + 1$ many component intervals, (19) gives

$$F_{n+1}(1) - F_{n+1}(0) \leq (n + 1)\delta_{n+1}^2 + \frac{\delta_{n+1}^2}{2^{n+1}}. \tag{20}$$

The attentive reader has checked that all the parts of the induction are verified for $n + 1$ instead of n.

Definition 15. *Iterating the procedure, we construct a sequence of functions $(F_n)_{n\geq 1}$ and define the continuous strictly increasing function*

$$F = \sum_{n=1}^{+\infty} F_n.$$

The continuity follows from the uniform convergence guaranteed by (20), and the strict monotonicity from the density of the I_n and the fact that the F_n do not increase at the same locations.

2.3 Multifractal properties of F

Proposition 16. *For every $x \in [0,1]$ which is not one of the s_n, one has $h_F(x) \leq 1$.*

PROOF. Consider a real number $x \in [0,1]$ which is not one of the s_n.

Lemma 17. *There exists an infinite number of integers n such that $x \notin H_n$.*

PROOF. Assume that there exists n_0 such that $x \in H_{n_0}$. Choose $n_1 \geq n_0$ such that for $n_0 \leq n < n_1$,

$$x \in \bigcup_{i=1}^{n_0} [s_i - \delta_n, s_i + \delta_n] \subset H_n \quad \text{but} \quad x \notin \bigcup_{i=1}^{n_0} [s_i - \delta_{n_1}, s_i + \delta_{n_1}]. \tag{21}$$

This integer n_1 exists, because the sequence δ_n converges to zero.

Since $x \in H_{n_1-1}$, $x \notin \widetilde{I}_{n_1} \setminus H_{n_1-1}$. Moreover, (9), (15) and (21) imply that $x \notin \cup_{i=n_0+1}^{n_1-1} [s_i - \delta_{n_1-1}, s_i + \delta_{n_1-1}]$. By (16), $x \notin [s_{n_1} - \delta_{n_1}, s_{n_1} + \delta_{n_1}]$, which is included in the interior of $\widetilde{I}_{n_1} \setminus H_{n_1-1}$. Therefore $x \notin H_{n_1}$.

Since this argument can be repeated, there are infinitely many n_j's, $j = 1, 2, ...$ such that $x \notin H_{n_j}$. □

Now, if $x \notin H_n$, then item (iv)(c) of the induction provides us with an interval $I_{x,n} = [a_{x,n}, b_{x,n}]$ such that (13) holds with n. Since by Lemma 17, this holds for infinitely many intervals whose size goes to zero, this implies $h_F(x) = h_\mu(x) \leq 1$. □

We take care of the points s_n, where F is more regular.

Proposition 18. *For every $n \geq 1$, $h_F(s_n) = h_\mu(s_n) = 2$. Hence, $d_F(2) = d_\mu(2) = 0$.*

PROOF. Consider one of the points s_{n_0}, and $n \geq n_0$. Then $s_{n_0} \in H_n$ and by part (iv)(d) of the induction, the functions F_p with $1 \leq p < n$ are constant on $[s_{n_0} - \delta_n, s_{n_0} + \delta_n]$.

By parts (iv)(a) and (iv)(b) of the induction, we have

$$F_n(s_{n_0} + \delta_n) - F_n(s_{n_0}) = F_n(s_{n_0} + \delta_n) - F_n(s_{n_0} - \delta_n) = \delta_n^2. \tag{22}$$

Suppose $y \in [s_{n_0} - \delta_n, s_{n_0} + \delta_n] \setminus [s_{n_0} - \delta_{n+1}, s_{n_0} + \delta_{n+1}]$. Then

$$\begin{aligned} &|F(y) - F(s_{n_0})| \leq \\ &\sum_{k=1}^{n-1} |F_k(y) - F_k(s_{n_0})| + |F_n(y) - F_n(s_{n_0})| + \sum_{k=n+1}^{\infty} |F_k(y) - F_k(s_{n_0})| \\ &= 0 + \Delta_n + \Sigma_{n+1}. \end{aligned}$$

If $\delta_n/2 \le |y - s_{n_0}| \le \delta_n$, then $|\Delta_n| \le \delta_n^2 \le 4|y - s_{n_0}|^2$ and $|\Sigma_{n+1}| \le \delta_n^2 \le 4|y - s_{n_0}|^2$.

If $\delta_{n+1} < |y - s_{n_0}| < \delta_n/2$ then $\Delta_n = 0$. By using the definition of H_{n+1} and the choice of δ_n and δ_{n+1}, we obtain that $|\Sigma_{n+1}| < 3\delta_{n+1}^2 + \sum_{k=n+2}^{\infty}(k\delta_k^2 + \frac{\delta_k^2}{2^k}) < 4|y - s_{n_0}|^2$.

We have used that (10) and (14) hold for all $n \ge n_0$. This combined with (22) implies that $h_F(s_{n_0}) = h_\mu(s_{n_0}) = 2$. Since all the other points have an exponent less than 1, this concludes the proof. □

Proposition 19. *For every $h \in [0,1]$, $d_F(h) = h$.*

PROOF. Obviously, by Proposition 10, only the lower bound needs to be proved.

By construction, in each interval $J_l = [\alpha_l, \beta_l]$ (recall (18)), if $\widetilde{F_l}$ stands for the restriction of F onto J_l, one has $d_{\widetilde{F_l}}(h) \ge h$ for all $h \in [0,1]$. Hence $d_{\widetilde{F_l}}(h) = h$ for $h \in [0,1]$. The fact that F is homogeneously multifractal follows from the density of the intervals I_n (which implies the density of the J_l). □

3 Construction of a monotone function with Hölder exponents 1

In this section we prove Proposition 8. The function Z we obtain is a sort of monotone increasing Weierstrass-like function.

PROOF. First we need positive sequences $(a_n)_{n \ge N}$ and $(b_n)_{n \ge N}$. The sequence (a_n) will tend to zero sufficiently fast, while (b_n) will consist of integers growing to infinity. The sequences are built inductively. Set $a_1 = 1$, $b_1 = 4$. Suppose that $n > 1$ and the terms of $a_{n'}$ and $b_{n'}$ have already been selected for $n' < n$. We select $a_n \in (0,1)$ and

$$b_n \in \mathbb{N} \text{ is congruent to 3 modulo 4} \tag{23}$$

$$\text{such that} \qquad \frac{a_{n-1}}{300} > a_n > (4b_n)^{\frac{-1}{n}} \quad \text{and} \quad \frac{a_n b_n}{100} > \sum_{n'=1}^{n-1} a_{n'} b_{n'}. \tag{24}$$

Iterating the scheme gives the sequences $(a_n)_{n \ge N}$ and $(b_n)_{n \ge N}$. From the left handside inequality in (24), it follows that

$$\frac{a_n}{100} > \sum_{n'=n+1}^{\infty} a_{n'}. \tag{25}$$

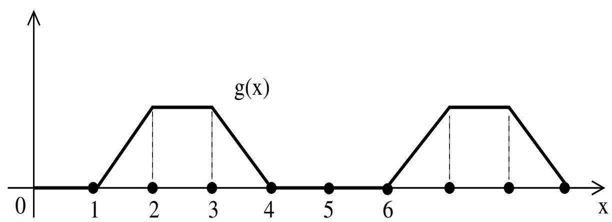

Figure 1: The function g

Put $g(x) = 0$ on $[0, 1]$, $g(x) = 1$ on $[2, 3]$, $g(4) = 0$ and suppose that g is linear on $[1, 2]$ and $[3, 4]$, moreover to define g on $\mathbb{R}$ we also assume that it is periodic by 4 (see Figure 1). Further, we set

$$G(x) = \sum_{n=1}^{\infty} a_n g(b_n x) \text{ and } Z(x) = \int_0^x G(t)dt.$$

Then one can easily see that G is continuous as uniform limit of continuous functions, and Z is continuously differentiable with $Z'(x) = G(x)$ for all $x \in [0, 1]$. In particular, we have $h_Z(x) \geq 1$ for all $x \in [0, 1]$. If we can verify that for every $x_0 \in [0, 1]$ and for every sufficiently large $n_1 \in \mathbb{N}$ we can find x_1, depending on n_1, such that $|x_1 - x_0| \leq 4/b_{n_1}$ and

$$|Z(x_1) - Z(x_0) - Z'(x_0)(x_1 - x_0)| > \frac{1}{16^2}|x_1 - x_0|^{1+\frac{1}{n_1}}, \tag{26}$$

then $h_Z(x) \leq 1$ for all $x \in [0, 1]$. Combining this with the differentiability of Z, we obtain that $h_Z(x) = 1$ for all $x \in [0, 1]$, proving Proposition 8.

Observe that g is a Lipschitz function with Lipschitz constant one, hence

$$|g(b_{n_1}x) - g(b_{n_1}x)| \leq b_{n_1}|x - y| \text{ for all } x, y \in \mathbb{R}. \tag{27}$$

Fix $x_0 \in [0, 1]$. It remains us to check that (26) holds true. If $x_0 = 0$, then $g(b_{n_1}x_0) = 0$ and if $x_0 = 1$, then $g(b_{n_1}x_0) = 1$, by (23).

From the definition of g, if n_1 is sufficiently large, we can find $x_1 \in [0, 1]$ such that:

- $|x_1 - x_0| \leq 4/b_{n_1}$,
- $g(b_{n_1}x) - g(b_{n_1}x_0)$ is of constant sign for x in the interval $I_{0,1}$ with endpoints x_0 and x_1,

- there exists a subinterval $I'_{0,1} \subset I_{0,1}$ of length $\frac{1}{b_{n_1}}$ such that $|g(b_{n_1}x) - g(b_{n_1}x_0)| \geq 1/2$ for all $x \in I'_{0,1}$.

Without limiting generality, we suppose that $x_1 > x_0$, $[x_0, x_1] = I_{1,0}$, and

$$|x_1 - x_0| \leq 4/b_{n_1}, \quad g(b_{n_1}x) - g(b_{n_1}x_0) \geq 0 \text{ for } x \in I_{1,0} \tag{28}$$
$$g(b_{n_1}x) - g(b_{n_1}x_0) \geq 1/2 \quad \text{for all } x \in [x', x' + 1/b_{n_1}] = I'_{0,1} \subset I_{1,0}.$$

Then

$$\begin{aligned} I &:= |Z(x_1) - Z(x_0) - Z'(x_0)(x_1 - x_0)| = \Big| \int_{x_0}^{x_1} G(t)dt - G(x_0)(x_1 - x_0) \Big| \\ &= \Big| \sum_{n=1}^{\infty} a_n \Big(\int_{x_0}^{x_1} g(b_n t)dt - g(b_n x_0)(x_1 - x_0) \Big) \Big| \end{aligned}$$

by the uniform convergence of the series. Dividing the sum into three parts, and using successively (27), (28), and the fact that $|g| \leq 1$, we obtain

$$\begin{aligned} I &= \Big| \sum_{n=1}^{n_1-1} \Big(a_n \int_{x_0}^{x_1} g(b_n t) - g(b_n x_0)dt \Big) + a_{n_1} \int_{x_0}^{x_1} g(b_{n_1}t) - g(b_{n_1}x_0)dt \\ &\quad \sum_{n=n_1+1}^{\infty} \Big(a_n \int_{x_0}^{x_1} g(b_n t) - g(b_n x_0)dt \Big) \Big| \\ &\geq a_{n_1} \int_{x_0}^{x_1} g(b_{n_1}t) - g(b_{n_1}x_0)dt - \sum_{n=1}^{n_1-1} \Big(a_n \int_{x_0}^{x_1} |g(b_n t) - g(b_n x_0)|dt \Big) \\ &\quad - \sum_{n=n_1+1}^{\infty} \Big(a_n \int_{x_0}^{x_1} |g(b_n t) - g(b_n x_0|dt \Big) \\ &\geq \frac{a_{n_1}}{2b_{n_1}} - \sum_{n=1}^{n_1-1} \Big(a_n \int_{x_0}^{x_1} b_n |t - x_0| dt \Big) - \sum_{n=n_1+1}^{\infty} a_n |x_1 - x_0| \end{aligned}$$

Then, by (24), (25) and (28), one finally has

$$\begin{aligned} I &\geq \frac{a_{n_1}}{2b_{n_1}} - \sum_{n=1}^{n_1-1} a_n b_n \frac{16}{2b_{n_1}^2} - \sum_{n=n_1+1}^{\infty} a_n \frac{4}{b_{n_1}} \geq \frac{a_{n_1}}{4b_{n_1}} \\ &\geq (4b_{n_1})^{-1-\frac{1}{n_1}} \geq \Big(\frac{|x_1 - x_0|}{16} \Big)^{1+\frac{1}{n_1}}. \end{aligned}$$

This implies (26). $\square$

References

[1] Z. Buczolich and S. Seuret, Measures and functions with prescribed homogeneous multifractal spectrum, Journal of Fractal Geometry, 1, No. 3, 295-333 (2014).

[2] Z. Buczolich and J. Nagy, Hölder spectrum of typical monotone continuous functions, *Real Anal. Exchange* 26, (2000/01), 133–156.

[3] K. J. Falconer, *Fractal Geometry,* John Wiley & Sons, (1990).

[4] K. J. Falconer, *Techniques in Fractal Geometry,* Wiley, New York (1997).

[5] S. Jaffard, The spectrum of singularities of Riemann's function. *Rev. Mat. Iberoamericana,* 12(2), (1996), 441–460.