On embeddings of finite metric spaces

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***Abstract*—Let $\langle X, \varrho\rangle$ be a finite metric space, and for a natural number d, let $\mathbb{R}^d$ be the real d-dimensional vector space endowed with its usual Euclidean metric. We interested in estimations for d such that $\langle X, \varrho\rangle$ can be "embedded" in some sense into $\mathbb{R}^d$. This classical topic of functional analysis recently has received renewed impetus motivated by several problems of theoretical computer science. We will recall some of these problems which also help us to find the "good" notion of embeddings and announce some recently obtained related results.**

I. Introduction

Studies of embedding metric spaces into Hilbert-spaces originates back to the 1980's when Bourgain [2] as well as Johnson and Lindenstrauss [6] published their papers. Their motivation came from functional analysis, and was purely mathematical, although since then their work have been cited frequently by applied mathematicians. In order to motivate our investigations, in this section we recall some research directions in computer science based on embedding finite metric spaces into Euclidean spaces.

In general, an arbitrary finite metric space cannot be isometrically embedded into an Euclidean space, the "embedding" has a certain distortion.

Definition 1.1: Let (X, δ) and (Y, ϱ) be metric spaces and $f : X \to Y$ an injective function. We say that f is an embedding with *distortion* at most α, if α is a real number for which there exists $r > 0$ such that for each $x_1, x_2 \in X$

$$r\delta(x_1, x_2) \leq \varrho(f(x_1), f(x_2)) \leq \alpha r\delta(x_1, x_2).$$

The main results of the paper are Theorems 3.5 and 3.6. In Theorem 3.6 we give a sufficient condition implying that a finite metric space can be embedded into $\mathbb{R}^d$ with distortion at most 3, for some d. The condition is rather technical, but it essentially claims, that if all "small" subspaces can be embedded, then the whole space can be embedded. Theorem 3.5 contains the necessary technical preliminaries and it can be regarded as an analogous statement when distortion has been replaced with "additive inaccuracy" introduced in Definition 3.3 below.

The rest of the present section is devoted to provide motivation for related investigations. Low-distortion metric embeddings has numerous algorithmic and hence real-life applications. Linial, London and Rabinovich [9] gave an actual approximation algorithm that embeds an n-point metric space into an Eucledian space with $O(\log n)$ distortion. We recall the following algorithm-theoretical applications of this result.

First Application. A multicommodity flow problem consists of a finite graph $G = (V, E)$ and s_i, t_i source-sink pairs of vertices for each commodity i. There is a demand d_i for each commodity i that needs to be delivered from s_i to t_i respecting each edge's limited capacity c_j. In the case of only one commodity, the optimal logistics can be found by partitioning the vertices into two sets A, B where A contains s_1 and B contains t_1 and the edges between A and B has minimal sum of capacity amongst such partitions. Thus, the min-cut is equal to the max-flow. This is not true, in general, for multicommodity flow problems; although the ratio between them is proven to be $O(\log k)$ with k many sources and sinks.

Using the result presented in [9] about embeddings of finite metric spaces, Linial, London and Rabinovich were able to prove the already-known $O(\log n)$ bound for the max-flow min-cut problem of multicommodity flows as well as to gave answer to open questions of this area. Tighter – $O(\log k)$ for k number of sources and sinks – bounds were proven by Aumann and Rabani [1] by embedding the graph into $\mathbb{R}^k$. Multicommodity max-flow min-cut theorems serve as an initial step of approximation algorithms for various NP-hard problems in graph partitioning, VLSI layout and network design, to name a few. See [8] and [7] for further details.

Second Application. The theory of embedding finite metric spaces into normed or Hilbert-spaces is utilized in data-mining as well. For example, in bioinformatics, searching in large data sets of proteins or DNA's is problematic. One way to index these proteins (or DNA sequences) is based on to embed them into an Eucledian space of a reasonably low dimension. There are various methods and approaches in this direction, see for example [4], [5] or the work of Faloutsos and Kim on multimedia databases in [3].

II. Some elementary properties of Metric Spaces

In this section we shortly recall some well known facts and notions from the theory of metric spaces we will use later.

Definition 2.1: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a metric space, $a \in X$ and let γ be a non-negative real number. As usual, the open γ-ball $B(\gamma, a)$ at a is defined to be

$$B(\gamma, a) = \{x \in X : \varrho(a, x) < \gamma\}.$$

A family $\{B_i : i \in I\}$ of γ-balls is defined to be a γ-net iff it covers X, that is,

$$X = \bigcup_{i \in I} B_i.$$

Recall from elementary topology, that $\mathcal{X}$ is compact iff for all positive $\gamma \in \mathbb{R}$ there exists a finite γ-net in $\mathcal{X}$. Moreover, every finite metric space is compact. If $\mathcal{X}$ is a compact metric space, then $\nu(\mathcal{X}, \gamma)$ denotes the cardinality of the smallest γ-net of $\mathcal{X}$.

III. Technical details

Let $\mathcal{X} = \langle X, \varrho \rangle$ be a metric space. We associate a relational structure to $\mathcal{X}$ as follows. If d is a distance of $\mathcal{X}$, that is, $d \in ran(\varrho)$ then the binary relation R_d is defined to be

$$R_d = \{\langle a, b \rangle \in X^2 : \varrho(a, b) \leq d\}.$$

Thus, the relational structure $\langle X, R_d \rangle_{d \in ran(\varrho)}$ completely describes $\mathcal{X}$ and, at the same time, it can be treated as a model for an appropriate first order language. In this note we do not make strict distinction between a metric space $\mathcal{X}$ and the relational structure associated to it.

We define types as usual in model theory.

Definition 3.1: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a metric space, let $A \subseteq X$ and let $b \in X$. Then the Δ-type of b over A in $\mathcal{X}$ is defined to be

$$tp^{\mathcal{X}}_{\Delta}(b/A) = \{R_d(v, a) : \mathcal{X} \models R_d(b, a)\}.$$

Thus, the Δ-type of b is just the set (of atomic formulas with parameters from A describing the) distances of b from elements of A.

By a Δ-type over A we mean a set of atomic formulas (with parameters from A) which is of the form $tp^{\mathcal{X}}_{\Delta}(b/A)$ for some $b \in X$.

Keeping the notation introduced so far, we say, that $c \in X$ realizes the Δ-type p in $\mathcal{X}$ iff $p = tp^{\mathcal{X}}(c/A)$.

The following notion is an approximate version of splitting introduced in [12].

Definition 3.2: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a metric space, let $B \subseteq A \subseteq X$, let p be a Δ-type over A in $\mathcal{X}$ and let ε, δ be non-negative real numbers. Then we say, that p is (ε, δ)-splitting over B iff there exist $c_0, c_1 \in A$ such that for all $b \in B$ we have

$$|\varrho(b, c_0) - \varrho(b, c_1| \leq \delta$$

but whenever a realizes p, we have

$$|\varrho(a, c_0) - \varrho(a, c_1| \geq \varepsilon.$$

The following theorems will be essential in this paper. Some variants of them (in different contexts) had been utilized e.g. in [10] and in [11].

Theorem 3.1: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a compact metric space, let $a \in X$, let $\varepsilon \in \mathbb{R}^+_0$ and let $\delta < \frac{\varepsilon}{5}$ be arbitrary. Suppose

$$A_0 \subseteq A_1 \subseteq ... \subseteq A_M$$

is a strictly increasing sequence of subsets of $X - \{a\}$ such that for all $n < M$ the type

$$tp^{\mathcal{X}}_{\Delta}(a/A_{n+1})$$

is (ε, δ)-splitting over A_n. Then $M \leq \nu(\mathcal{X}, \delta)$.

Proof. Let $\{B_i : i < \nu(\mathcal{X}, \delta)\}$ be a δ-net of $\mathcal{X}$ with smallest possible cardinality. By our assumption on splitting, for each $n < M$ there exist $c_n, d_n \in A_{n+1} - A_n$ such that for all $b \in A_n$ we have

$$|\varrho(b, c_n) - \varrho(b, d_n)| \leq \delta$$

but

$$(*) \qquad |\varrho(a, c_n) - \varrho(a, d_n)| \geq \varepsilon.$$

Assume, seeking a contradiction, that $M > \nu(\mathcal{X}, \delta)$. By the pigeon-hole principle, there exists $N \leq \nu(\mathcal{X}, \delta)$ such that B_N (the N^{th} δ-ball in our net) contains at least two c_n's; more precisely, there exist $n_0 < n_1 \leq 1 + \nu(\mathcal{X}, \delta)$ with $c_{n_0}, c_{n_1} \in B_N$. Since B_N is a δ-ball, it follows, that $\varrho(c_{n_0}, c_{n_1}) \leq 2\delta$. Therefore, by the triangle inequality,

$$\varrho(a, c_{n_1}) \leq \varrho(a, c_{n_0}) + \varrho(c_{n_0}, c_{n_1}) \leq \varrho(a, c_{n_0}) + 2\delta,$$

and by symmetry,

$$\varrho(a, c_{n_0}) \leq \varrho(a, c_{n_1}) + 2\delta.$$

It follows, that

$$|\varrho(a, c_{n_0}) - \varrho(a, c_{n_1})| \leq 2\delta.$$

By, construction,

$$|\varrho(c_{n_0}, c_{n_1}) - \varrho(c_{n_0}, d_{n_1})| \leq \delta,$$

particularly,

$$\varrho(c_{n_0}, d_{n_1}) \leq 3\delta.$$

Therefore,

$$\varrho(a, c_{n_0}) \leq \varrho(a, d_{n_1}) + \varrho(d_{n_1}, c_{n_0}) \leq \varrho(a, d_{n_1}) + 3\delta$$

and similarly,

$$\varrho(a, d_{n_1}) \leq \varrho(a, c_{n_0}) + \varrho(c_{n_0}, d_{n_1}) \leq \varrho(a, c_{n_0}) + 3\delta.$$

It follows, that

$$|\varrho(a, c_{n_0}) - \varrho(a, d_{n_1})| \leq 3\delta.$$

Combining these, we get

$$|\varrho(a, c_{n_1}) - \varrho(a, d_{n_1})| \leq$$

$$|\varrho(a, c_{n_1}) - \varrho(a, c_{n_0})| + |\varrho(a, c_{n_0}) - \varrho(a, d_{n_1})| \leq 5\delta.$$

Since $5\delta < \varepsilon$, this contradicts to (ii) (more concretely, to $(*)$ above) and the proof is complete.

Theorem 3.2: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a compact metric space, let $a \in X$ and let $\varepsilon \in \mathbb{R}^+$. Then there exist $\delta \in \mathbb{R}^+$ and

$A(a) \subseteq X - \{a\}$ such that for any $B \subseteq X - \{a\}$ with $A \subseteq B$, the type

$$tp^{\mathcal{X}}(a/A(a) \cup B)$$

does not (ε, δ)-split over $A(a)$. In fact, arbitrary $\delta < \frac{\varepsilon}{5}$ is suitable and $A(a)$ can be chosen so, that $|A(a)| \leq 2(\nu(\mathcal{X}, \delta))$ is satisfied, as well.

Proof. Let $\delta < \frac{\varepsilon}{5}$ be an arbitrary positive real number.

Suppose, seeking a contradiction, that the consequence of the theorem is not true. By recursion, we define finite subsets $A_n \subseteq X - \{a\}$ for every natural number n, such that the following stipulations are satisfied:

(i) $A_n \subseteq A_{n+1}$, in fact, $A_{n+1} - A_n = \{c_n, d_n\}$;

(ii) $tp^{\mathcal{X}}_{\Delta}(a/A_{n+1})$ is (ε, δ)-splitting over A_n.

Let $A_0 = \emptyset$ and suppose A_m has already been defined for all $m \leq n$ such that stipulations (i) and (ii) are satisfied. Then, by our indirect assumption, there exists $B \subseteq X - \{a\}$ with $A_n \subseteq B$ such that $tp^{\mathcal{X}}_{\Delta}(a/B)$ is (ε, δ)-splitting over A_n. This means, that there exist $c_n, d_n \in B$ such that for all $b \in A_n$ we have

$$|\varrho(b, c_n) - \varrho(b, d_n)| \leq \delta$$

but

$$|\varrho(a, c_n) - \varrho(a, d_n)| \geq \varepsilon.$$

Let $A_{n+1} = A_n \cup \{c_n, d_n\}$. Then stipulations (i),(ii) remain true. In this way, one can define A_n, c_n, d_n for all $n \leq 1 + \nu(\mathcal{X}, \delta)$; this contradicts to Theorem 3.1. Thus, the proof is complete: $A(a)$ can be chosen to be some A_n (note, that an inspection shows, that each A_n has cardinality at most $2n$ and $2n \leq 2\nu(\mathcal{X}, \delta)$).

Theorem 3.3: Suppose $X \subseteq \mathbb{R}^d$ is such that $|X| \geq d + 1$ and X generates $\mathbb{R}^d$ as a vector space. Then each Δ-type p over X has a unique realization.

Proof. Let p be any Δ-type over X and let a be a realization of it. Let $S(x, r)$ be the sphere with center x and radius r, that is,

$$S(x, r) = \{y \in \mathbb{R}^d : ||x - y||_2 = r\}.$$

Observe, that p describes the distances between a and the elements of X. With another words, a should lie in the intersection of the spheres

$$\{S(x, r) : x \in X, R_r(v, x) \in p\}.$$

(Using induction on the dimension d) it is a routine exercise in geometry to check that such an intersection may contain at most one element.

Theorem 3.4: Suppose $X \subseteq \mathbb{R}^d$ is such that $|X| \geq d + 1$ and X generates $\mathbb{R}^d$ as a vector space. Let $a \in \mathbb{R}^d - X$ and let $Y \subseteq \mathbb{R}^d - \{a\}$. Let $\delta < \varepsilon \in \mathbb{R}^+_0$ be arbitrary. Then the type

$$tp^{\mathbb{R}^d}_{\Delta}(a/X \cup Y)$$

does not (ε, δ)-split over X.

Proof. Suppose, seeking a contradiction, that $Y \subseteq \mathbb{R}^d - \{a\}$ is such, that

$$tp^{\mathbb{R}^d}_{\Delta}(a/X \cup Y)$$

is (ε, δ)-splitting over X. Then, there exists $a_0, a_1 \in Y - X$ such that for all $b \in X$ we have

$$|\ ||a_0 - b||_2 - ||a_1 - b||_2\ | \ \leq\ \delta$$

but

$$|\ ||a_0 - a||_2 - ||a_1 - a||_2\ | \ \geq\ \varepsilon.$$

Hence, by continuity, and by Theorem 3.3 we have

$$||a_0 - a_1||_2 \leq \delta.$$

Thus,

$$||a_0 - a||_2 \leq ||a_0 - a_1||_2 + ||a_1 - a||_2 \leq ||a_1 - a||_2 + \delta$$

and similarly,

$$||a_1 - a||_2 \leq ||a_1 - a_0||_2 + ||a_0 - a||_2 \leq ||a_0 - a||_2 + \delta.$$

It follows, that

$$|\ ||a_0 - a||_2 - ||a_1 - a||_2\ | \ \leq\ \delta.$$

Since $\varepsilon > \delta$, this contradiction completes the proof.

Definition 3.3: Let $\langle X, \varrho \rangle$ and $\langle Y, \sigma \rangle$ be metric spaces and let $\gamma \in \mathbb{R}^+_0$. An injective function $f : X \to Y$ is defined to be an embedding with additive inaccuracy γ iff for all $a, b \in X$ we have

$$\varrho(a, b) - \gamma \quad \leq \quad \sigma(f(a), f(b)) \quad \leq \quad \varrho(a, b) + \gamma.$$

Definition 3.4: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a metric space and let $k, d \in \mathbb{N}, \gamma \in \mathbb{R}^+_0$. Then $\mathcal{X}$ is defined to be (k, γ)-locally embeddable into $\mathbb{R}^d$ iff the following holds. For any k-element subspace $\mathcal{Y}$ of $\mathcal{X}$, for any $a \in X$ and for any embedding

$$f : Y \to \mathbb{R}^d$$

of $\mathcal{Y}$ with additive inaccuracy γ, there exists an embedding $f^* : Y \cup \{a\} \to \mathbb{R}^d$ with additive inaccuracy γ such that $f \subseteq f^*$ (that is, f^* extends f).

Let $\mathcal{X} = \langle X, \varrho \rangle$ be a finite metric space and let $Y \subseteq X$. Then $s^{\mathcal{X}}(Y)$ is defined to be the cardinality of the set of all Δ-types over Y, in addition, for $k \in \mathbb{N}$ we define

$$s^{\mathcal{X}}(k) = max\{s^{\mathcal{X}}(Y) : Y \subseteq X, |Y| \leq k\}.$$

Now we are able to state and prove the main results of the paper.

Theorem 3.5: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a finite metric space with $n = |X|$ and let $d \in \mathbb{N}$. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary, let $0 < \delta < \frac{\varepsilon}{5}$ and let

$$k \geq max\{d + 1, s^{\mathcal{X}}(2\nu(\mathcal{X}, \delta))\}.$$

Suppose

$(*)$ $\mathcal{X}$ is $(k, 3^m\varepsilon)$-locally embeddable into $\mathbb{R}^d$ for all $m \leq n - 1$.

Then $\mathcal{X}$ is embeddable into $\mathbb{R}^d$ with an additive inaccuracy $3^n\varepsilon$.

Proof. We apply induction on $|X|$ (that is, on n). If $|X| \leq d+1$, then, according to our assumption $(*)$, $\mathcal{X}$ can be embedded into $\mathbb{R}^d$ with an additive inaccuracy $3^{n-1}\varepsilon$.

Now suppose, that $n > d+1$ and the theorem is true for all finite metric spaces with fewer than n elements. Let $a \in X$ be arbitrary. By the induction hypothesis, there exists an embedding

$$f : X - \{a\} \to \mathbb{R}^d$$

with an additive inaccuracy $3^{n-1}\varepsilon$. Let

$$p = tp^{\mathcal{X}}_{\Delta}(a/dom(f)).$$

By Theorem 3.2 there exists $A(a) \subseteq X - \{a\}$ such that p does not (ε, δ)-split over $A(a)$ and

$$|A(a)| \leq 2\nu(\mathcal{X}, \delta).$$

Then, there exists $B \subseteq A - \{a\}$ such that every Δ-type over $A(a)$ can be realized in B and $|B| \leq s^{\mathcal{X}}(2\nu(\mathcal{X}, \delta))$. After a small perturbation, if necessary, we may assume, that $ran(f|_B)$ generates $\mathbb{R}^d$ as a vector space. Observe, that $f|_B$ remains an embedding with additive inaccuracy $3^{n-1}\varepsilon$, whose domain is of size at most k. Hence, by $(*)$, there exists an embedding $g : B \cup \{a\} \to \mathbb{R}^d$ with additive inaccuracy $3^{n-1}\varepsilon$. Define the function

$$f^* : X \to \mathbb{R}^d$$

to be

$$f^*(x) = \begin{cases} f(x) & \text{if } x \neq a, \\ g(a) & \text{if } x = a. \end{cases}$$

To complete the proof, we shall show, that f^* is an embedding of X with additive inaccuracy $3^n\varepsilon$. To do so, it is enough to check, that for all $x \in X - \{a\}$ we have

$$\varrho(x,a) - 3^n\varepsilon \;\leq\; ||f^*(x) - f^*(a)||_2 \;\leq\; \varrho(x,a) + 3^n\varepsilon$$

or equivalently,

$$\varrho(x,a) - 3^n\varepsilon \;\leq\; ||f(x) - g(a)||_2 \;\leq\; \varrho(x,a) + 3^n\varepsilon.$$

So let $x \in X - \{a\}$ be arbitrary. Then, by construction, there exists $y \in B$ which realizes $tp_{\Delta}(a/A(a))$. Since p does not (ε, δ)-split over $A(a)$, we have

$$|\varrho(a,x) - \varrho(a,y)| \leq \varepsilon,$$

that is,

$$\varrho(a,x) - \varepsilon \;\leq\; \varrho(a,y) \;\leq \varrho(a,x) + \varepsilon.$$

Now, because g is an embedding with additive inaccuracy

$$3^{n-1}\varepsilon,$$

we have

$$\varrho(a,y) - 3^{n-1}\varepsilon \leq$$
$$||g(a) - g(y)||_2 \leq$$
$$\varrho(a,y) + 3^{n-1}\varepsilon.$$

Observe, that $f(y) = g(y)$, so we also have

$$\varrho(a,y) - 3^{n-1}\varepsilon \leq$$
$$||g(a) - f(y)||_2 \leq$$
$$\varrho(a,y) + 3^{n-1}\varepsilon.$$

Finally, by Theorem 3.4, the type

$$tp^{\mathbb{R}^d}_{\Delta}(g(a)/ran(f))$$

does not $(3^{n-1}\varepsilon, (3^{n-1}+1)\varepsilon)$-split over $ran(f|_B)$. It follows, that

$$|\;\; ||g(a) - f(x)||_2 - ||g(a) - f(y)||_2 \;\; | \leq (3^{n-1}+1)\varepsilon,$$

particularly,

$$||g(a) - f(y)||_2 - (3^{n-1}+1)\varepsilon \;\leq$$
$$||f(x) - g(a)||_2 \;\leq$$
$$||g(a) - f(y)||_2 + (3^{n-1}+1)\varepsilon.$$

Combining these, we get

$$\varrho(a,x) - ((3^{n-1}+1) + 3^{n-1})\varepsilon \leq$$
$$||f(x) - g(a)||_2 \leq$$
$$\varrho(a,x) + ((3^{n-1}+1) + 3^{n-1})\varepsilon.$$

Since

$$(3^{n-1}+1) + 3^{n-1} = 2 \cdot 3^{n-1} + 1 \leq 3^n$$

we also have

$$\varrho(a,x) - 3^n\varepsilon \;\leq\; ||f(x) - g(a)||_2 \;\leq\; \varrho(a,x) + 3^n\varepsilon$$

as desired, and the proof is complete.

Theorem 3.6: Let $\mathcal{X} = \langle X, \varrho \rangle$ be a finite metric space. Let $s = min(ran(\varrho) - \{0\})$ and let

$$0 < \varepsilon \leq \frac{s}{2 \cdot 3^{|X|}}.$$

Suppose, that the conditions of Theorem 3.5 are satisfied. Then $\mathcal{X}$ can be embedded into $\mathbb{R}^d$ with distortion at most 3.

Note, that the distortion does not depend on $n := |X|$; however, condition $(*)$, which is depend on n, should be satisfied for all small (at most k-element) subspaces of $\mathcal{X}$.

Proof. By Theorem 3.5 there exists an embedding

$$f : X \to \mathbb{R}^d$$

with additive inaccuracy $3^{|X|}\varepsilon$. We will show, that this f has distortion at most 3. To do so, observe, that for all $a \neq b \in X$ we have

$$\varrho(a,b) - 3^{|X|}\varepsilon \geq \varrho(a,b) - \frac{s}{2} \geq \frac{\varrho(a,b)}{2}$$

and similarly,

$$\varrho(a,b) + 3^{|X|}\varepsilon \leq \varrho(a,b) + \frac{s}{2} \leq \frac{3\varrho(a,b)}{2}.$$

Thus, with $r = 1/2$ and $\alpha = 3$, for all $a \neq b \in X$ we have

$$r \cdot \varrho(a,b) \leq \varrho(a,b) - 3^{|X|}\varepsilon$$

and

$$\varrho(a,b) + 3^{|X|}\varepsilon \leq \alpha \cdot r \cdot \varrho(a,b).$$

This completes the proof.

Conclusion

In Theorem 3.6 we proved, that if $\mathcal{X}$ is a finite metric space, k and d are natural numbers satisfying further technical conditions (detailed in Theorem 3.6) and each k-element subspace of $\mathcal{X}$ satisfy a further technical condition (detailed in $(*)$ of Theorem 3.5), then $\mathcal{X}$ can be embedded into $\mathbb{R}^d$ with distortion at most 3. In certain cases k may be very small compared to the number of points of $\mathcal{X}$. A further interesting aspect of this result is, that the distortion does not depend on the number of points of $\mathcal{X}$; however, the technical condition which should be satisfied by the k-element subspaces of $\mathcal{X}$ depends on $|\mathcal{X}|$.

Special metric spaces, whose elements are finite sequences and the distance is a variant of Hamming distance (or depends on the length of the longest common initial segments) may satisfy our conditions; so it seems, that in some special cases, some of these spaces can be embedded into $\mathbb{R}^d$ with constant distortion.

This investigation has been intended to carry out later.

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