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**Multidimensional global games
and some applications**

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Multidimensional global games and some applications

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Abstract

I extend the standard global games framework by introducing an additional target on which agents can coordinate on. I compare this multidimensional case to the standard global games problem. Furthermore, I investigate the effects of consolidating the multiple targets. I find that introducing an additional option generates a negative strategic correlation between the options and thus weakens the coordination. However, unifying the options eliminates the endogenous correlation and thus restores the coordination. I also show two potential applications to be modeled by these kinds of games.

Keywords: global games, coordination

JEL: C72, D84

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Több dimenziós globális játékok és néhány alkalmazásuk

Vonnák Dzsamila

Összefoglaló

A tanulmányban kiterjesztem a sztenderd globális játékok keretét egy újabb koordinációs célponttal. Ezt a többdimenziós esetet összehasonlítom a sztenderd globális játék modellel. Ezen kívül azt is megvizsgálom, hogy milyen hatása van a koordinációs célpontok egyesítésének. Azt találom, hogy egy újabb koordinációs opció megjelenése negatív stratégiakorrelációt generál a célpontok között és így gyengíti a koordinációt. A célpontok egyesítése viszont kiküszöböli ezt az endogén korrelációt és így visszaállítja a koordinációt. A tanulmányban két olyan alkalmazást is bemutatok, amelyek modellezésére alkalmas lehet ez a játékelméleti keret.

Tárgyszavak: globális játékok, koordináció

JEL kód: C72, D84

Multidimensional global games and some applications*

Dzsamila Vonnák[†]

January 25, 2018

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I extend the standard global games framework by introducing an additional target on which agents can coordinate on. I compare this multidimensional case to the standard global games problem. Furthermore, I investigate the effects of consolidating the multiple targets. I find that introducing an additional option generates a negative strategic correlation between the options and thus weakens the coordination. However, unifying the options eliminates the endogenous correlation and thus restores the coordination. I also show two potential applications to be modeled by these kinds of games.

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1 Introduction

Global games are coordination games with incomplete information. This class of game is appropriate to model economic situations where agents have incentive to coordinate on some action, but due to incomplete information perfect coordination fails. Global games have been applied to several economic situations, such as bank runs (Goldstein and Pauzner (2004), Goldstein and Pauzner (2005)), currency crisis (Morris and Shin (1998), Cukierman et al. (2004)), debt crisis (Morris and Shin (2004)), and technology adoption (Chamley (1999), Heidhues and Melissas (2006)).

In this paper I extend the standard global games framework by introducing an additional target on which agents can coordinate on. I compare this multidimensional case to the standard global games problem. Furthermore, I investigate the effects of consolidating the multiple targets. I find that introducing an additional option generates a negative strategic correlation between the options and thus weakens the coordination. However, unifying the options eliminates the endogenous correlation and thus restores the coordination. I also show two potential applications to be modeled by these kinds of games.

I build a model with two risky options. There is a continuum of agents who can choose between a safe and the two risky options. The payoff of the agents choosing a risky option is increasing with the number of agents choosing the same outcome. This provides an incentive for the agents to take coordinated actions. However, as the agents have imperfect information, perfect coordination is not possible. I investigate two scenarios: One, in which the two risky options are available separately, and another, in which the two are unified.

The unified-risky-options case is formally equivalent to a usual one dimensional global game. Therefore, in the unique equilibrium, agents choose the risky option only if their signal is above some constant threshold. However, in case of separate risky options multidimensionality results an important difference: the threshold is not a constant but a function of the agents' signal about the risky outcome. In the equilibrium agents choose a certain option if their signal on that outcome exceeds the value taken by the cutoff function at their signal on the other option. I prove the existence and the uniqueness of such an equilibrium to a certain range of parameters by using Banach fixed point theorem. I have no closed-form solution for the threshold functions, instead I construct them by using numerical methods.

Multidimensionality has an important consequence for the power of coordination. When there are multiple options, coordination weakens. This is due to strategic motives of agents. Agents have incentives to make mutually consistent actions. Since there are a fixed number of agents, when there are multiple options, their power is split. The more people coordinate on one option the less people there are who can potentially coordinate on the other. This generates a negative correlation between the two options which I call strategic correlation.

The key element of the model is the interaction of the coordination motives of agents to move together and the substitutability of the options. When there are multiple options, each potential object of coordination, they are in fact substitutes. Thus, with multiple op-

tions the coordination disperses. However, unifying the options eliminates the coordination split and thus strengthens the power of coordination.

I show two applications which can be modeled by the multidimensional global games framework. The first application is the choice of invoicing currency of oil. In the oil market the historically established currency is the US Dollar. I show that there are situations when an agent would switch to the usage of a new currency if there were one new currency besides the US Dollar, however, would not switch if there were two other currencies. The second application is the introduction of common European bond. A common argument for joint issuance is that it smooths out idiosyncratic risk. While this argument is present in my model, there is an extra layer: joint bond issuance can make participating countries more vulnerable to speculative attacks.

To my knowledge, this is the first paper showing how the consolidation of multiple coordination targets can increase coordination. This paper belongs to the stream of the global games literature which extends the dimension of the standard setup. Oury (2013) deals with global games where the state space is multidimensional. She focuses on the sufficient conditions for equilibrium uniqueness in a general class of multidimensional global games. However, her result does not apply to my model as the action space she defines differs from the one I use in my model.

Some of the theoretical models of contagion of self-fulfilling crises (e.g. Goldstein and Pauzner (2004), Keister (2009)) also employ global games techniques with multidimensional state space, thus let the payoffs be influenced by not only one single, but multiple economic variables. In particular these models show that when two markets have the same group of agents, however independent fundamentals, contagion of crises from one market to the other is likely to occur. However, my model differs from these papers in both the choice set and the driving force. In these papers the decisions related to the two markets are not mutually exclusive, agents can choose both of the investment options. The main mechanism is driven by wealth effect: the crisis in one country influences the wealth of the agents which changes their behavior toward the other country. In contrast, in my model agents have to decide among different options. Because there exist multiple options agents cannot coordinate on the same action, thus the power of their aggregate move is dispersed.

He et al. (2016) also build a global games model with two risky options to investigate what makes an asset a safe asset. Similar to this paper they also apply their model to the common European bond issuance. Their paper differs from mine in both the main focus and the driving mechanism. In general they use their model to investigate the determinants and features of safe assets. Meanwhile the emphasis of my model is on the coordination of agents and how it is influenced by the available coordination targets. The key elements of their model is the trade-off between the strategic complementarity and the strategic substitutability of the agents' action. In my paper there is only strategic complementarity, however there is not only two risky option but also a safe outside option which induce that the coordination is not only splits between the two options but also weakens.

The closest paper to mine is Fujimoto (2014). He explores the similar extension of global games as I do, namely the introduction of multiple mutually exclusive options. He proves equilibrium uniqueness and existence for such games in a quite general setup and also examines the consequences of multidimensionality. My work differs from his in three important aspects. First, he considers regime change models and thus discrete outcomes, while in my model the aggregate outcome is continuous. Second, because of the different setup his mathematical proofs do not apply directly to my model and thus I provide different proofs to show the existence and uniqueness of the equilibrium in my setup. Third, I concentrate on the coordination issues of multidimensional global games in general, while he focuses on speculative attacks.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the equilibrium behaviour of agents in case of separate issuance, Section 4 deals with the case of joint issuance. In Section 5 I compare the outcomes in the different scenarios. In Section 6 I show some comparative statics. Section 7 shows two applications. Finally, Section 8 concludes.

2 The Model

There are two uncertain economic fundamentals, θ_A and θ_B . There is a continuum of agent with measure one, indexed by $i \in I = [0, 1]$. There are two periods. Each agent is born in period 1 with an endowment e . Consumption occurs only in period 2 and each agent obtains utility of $u(c)$, where c is her consumption in period 2. Function u is increasing, implying that in period 2 agents consume all their wealth. In period 1 agents have to decide between the available options. There are two scenarios. In the first scenario there are two risky options each related to one of the uncertain economic fundamentals and a safe outside option. That is the set of available actions for each agent is $\Omega = \{0, A, B\}$, where the two risky options are denoted by A and B , while the safe action is represented by 0. In the second scenario the two economic fundamentals are unified and thus agents can choose either a risky option related to the unification of the fundamentals or a safe outside action. Thus the set of available actions for the agents is $\Omega = \{0, C\}$, where C means the unified risky option and 0 is the outside option. Hereafter the superscripts stand for the agents, while the subscripts take the same values as the actions. I use $a \in \{0, A, B, C\}$ to denote the actions in general. When I consider only the two risky options I use $r \in \{A, B\}$ to represent one of them, while $-r$ denotes the other, i.e. $-r = \{A, B\}/r$.

Settlement takes place in period 2. Agents who chose the safe outside option get a risk free payment o . Agents who chose a risky option $a \in \{A, B, C\}$ realize payoff $p(\theta_a + L_a)$, where $p' > 0$. The fundamental values θ_A and θ_B are independently and randomly drawn from the real line (i.e. the common priors are independent¹ improper uniform over \mathbb{R}^2).

¹I relax this assumption and derive the model with correlated fundamentals in the Appendix.

While θ_C represents the fundamental value of the unified option and is equal to the average² of the individual fundamental values, that is $\theta_C = \frac{1}{2}(\theta_A + \theta_B)$. Furthermore L_A , L_B and L_C denote the mass of agents choosing option A, B or C, respectively. The mass of agents taking action $a \in \Omega$ is given by the aggregate actions $L_a = \int_0^1 1_{[a^i=a]} di$, where a^i is the action taken by agent i , while $1_{[a^i=a]}$ is the indicator function which takes the value of one if $a^i = a$ and zero otherwise. The assumption that θ_a and L_a enters the payoff function in an additive way is not essential for the results but simplifies the model.

Agents have incomplete information about the economic fundamentals. Each of them receives a noisy signal about both fundamentals. The private signal of agent $i \in [0, 1]$ about fundamental $r \in \{A, B\}$ is $x_r^i = \theta_r + \varepsilon_r^i$, where ε_r^i is an idiosyncratic noise. The noise term consists of two parts: $\varepsilon_r^i = e^i + e_r^i$. The first component, e^i , is the systemic part of the noise which is common in both signals received by an agent. The second component, e_r^i , is the fundamental specific part. The components e^i , e_A^i and e_B^i are distributed independently and normally with mean 0 and standard deviation s , s_A and s_B , respectively, and are independent across agents. Thus, by standard properties of the normal distribution, ε_r^i also has a normal distribution with mean 0 and standard deviation $\sigma_r = \sqrt{s^2 + s_r^2}$, furthermore ε_A^i and ε_B^i are correlated with a correlation coefficient $\rho = \frac{s^2}{\sqrt{(s^2+s_A^2)}\sqrt{(s^2+s_B^2)}}$. The parameter distribution and the noise technology is common knowledge among the agents.

3 Separate Options

3.1 Equilibrium

In the first scenario agents can choose among the two risky and one safe option, that is $\Omega = \{0, A, B\}$. I consider symmetric Bayesian Nash Equilibria. A Bayesian pure strategy is a map $s : \mathbb{R}^2 \rightarrow \Omega$, where $s(x^i)$ is the action chosen if the agent receives the pair of signals $x^i = (x_A^i, x_B^i)$.

In equilibrium each agent chooses a risky action if the expected payoff from this option given her own pair of signals and others' strategy is higher than both the expected payoff from the other risky action and the payoff from the safe option.

An agent prefers the risky option r to the safe option if $p(\theta_r + L_r) > o$. As p is strictly increasing, there exists a constant $n \equiv p^{-1}(o)$, such that the latter is equivalent to $\theta_r + L_r > n$. Between the two risky actions agents prefer to choose the one with higher expected payoff, thus, given that p is strictly increasing, agents prefer action A on action B if $\theta_A + L_A > \theta_B + L_B$, and prefer action B otherwise. Given these preference rankings, the equilibrium is such that each agent chooses the risky option r if the expected value of $\theta_r + L_r$ given her own pair of signals (x^i) and others' strategy ($\times_{j \in I/i} s(x^j)$) is higher than both n and the expected value of $\theta_{-r} + L_{-r}$. That is, for each $x^i \in \mathbb{R}^2$

²This assumption is not essential for the main result, but simplifies the model.

$$s(x^i) = \begin{cases} A & \text{if } E[\theta_A + L_A | x^i, \times_{j \in I/i} s(x^j)] > \max\{E[\theta_B + L_B | x^i, \times_{j \in I/i} s(x^j)], n\} \\ B & \text{if } E[\theta_B + L_B | x^i, \times_{j \in I/i} s(x^j)] > \max\{E[\theta_A + L_A | x^i, \times_{j \in I/i} s(x^j)], n\} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

According to the signal generating process, the posterior cumulative distribution function about the fundamental value θ_r is increasing in the agent's related private signal x_r^i and - as the fundamentals are assumed to be uncorrelated - does not change with her other private signal x_{-r}^i . Corollary I consider monotone Bayesian Nash equilibria in which agents' strategies are increasing in own signals and non-increasing in cross signals. Proposition 1 states that such a strategy is coherent with the equilibrium (see the proof in the Appendix).

Proposition 1 *If all agents have monotone strategies increasing in related own signal and non-increasing in cross signal then the best response of any agent is to also have such a strategy.*

There have to be some cutoff values, such that an agent would choose a given strategy if and only if her signal about the underlying value exceeds this cutoff value. In the usual case when the state space is one dimensional the cutoff is given by a constant (see for instance Morris and Shin (2003)). But here each cutoff is conditional on the signal received by the agent about the other fundamental, thus the cutoffs are not constants, but functions.³ As the action space consists of three elements, I define cutoffs between each possible pair of actions. The cutoff function between actions $r \in \{A, B\}$ and $q \in \Omega \setminus r = \{-r, 0\}$ is a map $k^{rq} : \mathbb{R} \rightarrow \mathbb{R}$, where $k^{rq}(x_{-r}^i)$ prescribes a private signal about r (i.e. a value for x_r^i)⁴ such that an agent with pair of signals $(x_{-r}^i, k^{rq}(x_{-r}^i))$ is indifferent between choosing option r and q . The monotonicity of the strategies implies that the above defined k^{rq} cutoff functions are indeed functions, i.e. for each element of their domain associate one single value. Altogether 4 cutoff functions are defined: k^{A0} , k^{B0} , k^{AB} , k^{BA} , such that they solve the following equations:

$$E[\theta_A + L_A | (x_A^i, k^{B0}(x_A^i))] = n \quad (2)$$

$$E[\theta_B + L_B | (k^{A0}(x_B^i), x_B^i)] = n \quad (3)$$

³There is an identical formulation of the problem where some function of the two signals is set against a constant cutoff value. This identical formulation is closer to the logic of the standard one dimensional global games, though the solution concept I apply better matches with the formulation I use in the paper.

⁴Note that the superscript of the functions sets out the two actions that the function separates. For example $k^{A0}(x_B^i)$ is the cutoff function between actions A and 0. The first digit of the superscript shows which signal is set as a function of the other signal. That is $k^{A0}(x_B^i)$ gives the value of x_A^i for a given x_B^i making the agent independent between choosing risky option A or choosing 0, the safe outside option.

$$E [\theta_A + L_A | (k^{AB}(x_B^i), x_B^i)] = E [\theta_B + L_B | (k^{AB}(x_B^i), x_B^i)] \quad (4)$$

$$E [\theta_A + L_A | (x_A^i, k^{BA}(x_A^i))] = E [\theta_B + L_B | (x_A^i, k^{BA}(x_A^i))] \quad (5)$$

Note that from (4) and (5) follows that k^{AB} and k^{BA} are inverse functions. Thus a monotone equilibrium is defined by a joint solution $(k^{A0}(x_B^i), k^{B0}(x_A^i), k^{AB}(x_B^i))$ to equations (2)-(4).

Indeed, agents prefer the risky option $r \in \{A, B\}$ over the other two options, if and only if both $x_r^i > k^{r0}(x_{-r}^i)$ and $x_r^i > k^{r(-r)}(x_{-r}^i)$. This gives the equilibrium strategies:

Proposition 2 (*strategy profile*). *If strategies are monotone increasing in the related own signal and non-increasing in the cross signal, the strategy for $\forall i \in [0, 1]$ is as follows:*

$$s(x^i) = \begin{cases} A & \text{if } x_A^i > K^A(x_B^i) \\ B & \text{if } x_B^i > K^B(x_A^i) \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

where $K^A(x_B^i) \equiv \max \{k^{A0}(x_B^i), k^{AB}(x_B^i)\}$, $K^B(x_A^i) \equiv \max \{k^{B0}(x_A^i), k^{BA}(x_A^i)\}$ and $k^{BA}(x_A^i) = \text{inv}(k^{AB}(x_B^i))$.

Suppose these functions indeed exist, thus they should be such as Proposition 3 shows (see the proof in the Appendix).

Proposition 3 (*cutoff functions*). *The cutoff functions can be characterized by the following equations:*

$$k^{A0}(x_B^i) = n - 1 + \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^A(x_B^i + \sqrt{2}\sigma_B z) - k^{A0}(x_B^i) - \sqrt{2}\sigma_A \rho z}{\sqrt{2}\sigma_A \sqrt{1 - \rho^2}} \right) dz \quad (7)$$

$$k^{B0}(x_A^i) = n - 1 + \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^B(x_A^i + \sqrt{2}\sigma_A z) - k^{B0}(x_A^i) - \sqrt{2}\sigma_B \rho z}{\sqrt{2}\sigma_B \sqrt{1 - \rho^2}} \right) dz \quad (8)$$

$$k^{AB}(x_B^i) = x_B^i + \int_{-\infty}^{\infty} \phi(z) \left[\begin{array}{c} \Phi \left(\frac{K^A(x_B^i + \sqrt{2}\sigma_B z) - k^{AB}(x_B^i) - \sqrt{2}\sigma_A \rho z}{\sqrt{2}\sigma_A \sqrt{1 - \rho^2}} \right) \\ - \Phi \left(\frac{K^B(k^{AB}(x_B^i) + \sqrt{2}\sigma_A z) - x_B^i - \sqrt{2}\sigma_B \rho z}{\sqrt{2}\sigma_B \sqrt{1 - \rho^2}} \right) \end{array} \right] dz \quad (9)$$

where $\phi(z)$ and $\Phi(z)$ denote the pdf and the cdf, respectively, of the univariate standard normal distribution.

It can be shown that if there is enough noise in the signal generating process, there exists a unique equilibrium in monotone strategies. This is stated in Proposition 4 (see the proof in the Appendix).

Proposition 4 (*existence, uniqueness*). *If $\frac{2\sigma_r + \sigma_{-r}}{\sigma_r \sigma_{-r}} < 2\sqrt{\pi}\sqrt{1 - \rho^2}$ for all $r \in \{A, B\}$, there exists an essentially⁵ unique Bayesian equilibrium described by the cutoff functions given in Proposition 3.*

3.2 Implications

Figure 1 provides the geometrical representation of the cutoff lines that are characterized by Proposition 3 in the space of private signals. On the figure x_A^i is measured on the horizontal axis, while x_B^i on the vertical axis. If the pair of signals received by an agent falls into the bottom left area enclosed by k^{B0} and k^{A0} she chooses the safe action. In this case her signals about both fundamentals are so low that she rather chooses the outside option. The top left area enclosed by k^{B0} and k^{BA} shows the case when the agent picks action B. In this case her signal on the fundamental value of B is high enough to have an expected gain from choosing action B higher than both from choosing the outside option and from choosing action A. Similarly, the bottom right area enclosed by k^{A0} and k^{BA} shows the case when the agent chooses option A.

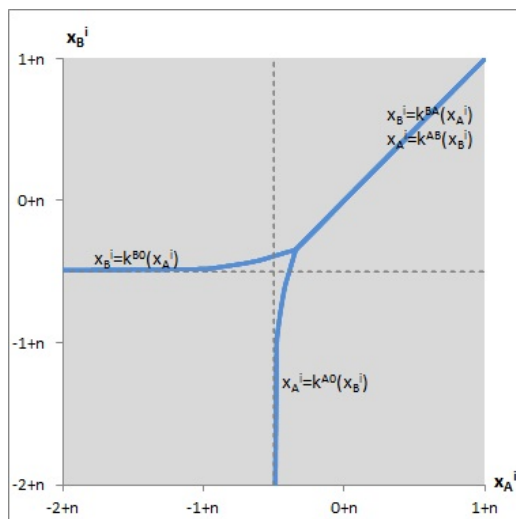


Figure 1: Cutoffs for the Separate-risky-option Case in the Space of Private Signals

⁵The equilibrium is not unique but essentially unique because at the cutoff the agent is indifferent between the concerned actions.

The mass of agents taking a particular action is given by the share of agents falling in the different domains. Thus the aggregate number of agents choosing risky action r in case of the separate risky options is given by the following equation:

$$E(L_r | \theta) = P(x_r^i > K^r(x_{-r}^i) | \theta) = \int_{-\infty}^{\infty} \int_{K^r(u_{-r})}^{\infty} f(u_A, u_B) du_r du_{-r} \quad (10)$$

where $f(u_A, u_B)$ is the joint pdf of $x_A^i | \theta$ and $x_B^i | \theta$. Given the signal generating process, $f(u_A, u_B)$ is in fact a bivariate normal distribution with mean vector $[\theta_A \ \theta_B]$ and covariance matrix $\begin{bmatrix} \sigma_A^2 & \rho \\ \rho & \sigma_B^2 \end{bmatrix}$.

Figure 2 shows the share of agents picking option A. The number of agents choosing action A increases in θ_A and decreases in θ_B . If θ_A increases the distribution of signals on fundamental value A (x_A^i) shifts to the right. Thus the higher θ_A is, the more agents get signal x_A^i high enough (relative to the other signal, x_B^i) to pick action A. While when θ_B decreases the distribution of signals on fundamental value B (x_B^i) shifts to the left, hence the cutoff for choosing option A decreases and thus more agents opt for action A.

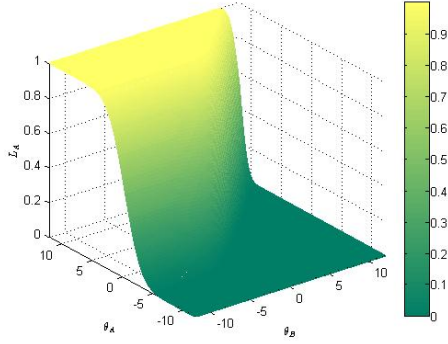


Figure 2: Share of Agents Choosing Option A in the Separate-risky-option Case in the Space of Fundamental Values

4 Unified Options

4.1 Equilibrium

In the second scenario agents either choose the safe option or a risky option depending on the unified fundamentals, that is $\Omega = \{0, C\}$. Again, I consider symmetric Bayesian Nash Equilibria with Bayesian pure strategy $s : \mathbb{R}^2 \rightarrow \Omega$, where $s(x^i)$ is the action chosen if the agent receives the pair of signals $x^i = (x_A^i, x_B^i)$. In the equilibrium each agents pick the

the risky unified option if its expected payoff given her own pair of signals (x^i) and others' strategy ($\times_{j \in I/i} s(x^j)$) is higher than the payoff from the outside option.

An agent prefers the unified risky option to the safe option if $p(\theta_C + L_C) > o$. As p is strictly increasing, there exists a constant $n \equiv p^{-1}(o)$, such that the latter is equivalent to $\theta_C + L_C > n$.

Hence the strategy should be such that an agent opts for the unified option if and only if she expects $\theta_C + L_C$ conditional on her own pair of signals (x^i) and others' strategy ($\times_{j \in I/i} s(x^j)$) to be higher than n . That is, for each $x^i \in \mathbb{R}^2$

$$s(x^i) = \begin{cases} C & \text{if } E[\theta_C + L_C | x^i, \times_{j \in I/i} s(x^j)] > n \\ 0 & \text{otherwise} \end{cases}$$

Let us define $x_C^i \equiv \frac{1}{2}(x_A^i + x_B^i)$. Given the signal generating processes for x_A^i and x_B^i this can be rearranged to $x_C^i = \frac{1}{2}(\theta_A + \varepsilon_A^i + \theta_B + \varepsilon_B^i) = \theta_C + \varepsilon_C^i$, with $\varepsilon_C^i = \frac{1}{2}(\varepsilon_A^i + \varepsilon_B^i) = e^i + \frac{1}{2}(e_A^i + e_B^i)$. As ε_A^i and ε_B^i are normal random variables ε_C^i is also distributed normally with mean 0 and standard deviation $\sigma_C = \sqrt{\sigma^2 + \frac{1}{4}\sigma_A^2 + \frac{1}{4}\sigma_B^2}$. One can show that the case of unified option is equivalent to the standard one dimensional case with fundamental value θ_C and private signal $x_C^i = \theta_C + \varepsilon_C^i$. This implies⁶ that there is always a unique equilibrium in switching strategies, such that agents choose the unified risky option if and only if $x_C^i > n - \frac{1}{2}$.

By using that $x_C^i \equiv \frac{1}{2}(x_A^i + x_B^i)$, we can express the cutoff in terms of the original signals. Thus we get a cutoff function $k^{C0}(x_A^i) = -x_A^i + 2n - 1$, implying the strategy

$$s(x^i) = \begin{cases} C & \text{if } x_B^i > k^{C0}(x_A^i) = -x_A^i + 2n - 1 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

4.2 Implications

The cutoff line characterized by Equation 11 is shown in Figure 3. It is in fact a straight line. Agents with a pair of signals falling on the area top right to this line opt for the unified option while others choose the safe option.

The aggregate number of agents choosing the risky joint option is given by the following equation:

$$E(L_C | \theta) = P(x_B^i > k^{C0}(x_A^i) | \theta) = \int_{-\infty}^{\infty} \int_{k^{C0}(u_A)}^{\infty} f(u_A, u_B) du_B du_A \quad (12)$$

where again, $f(u_A, u_B)$ is the joint pdf of $x_A^i | \theta$ and $x_B^i | \theta$.

⁶For a proof see for instance Morris and Shin (2003).

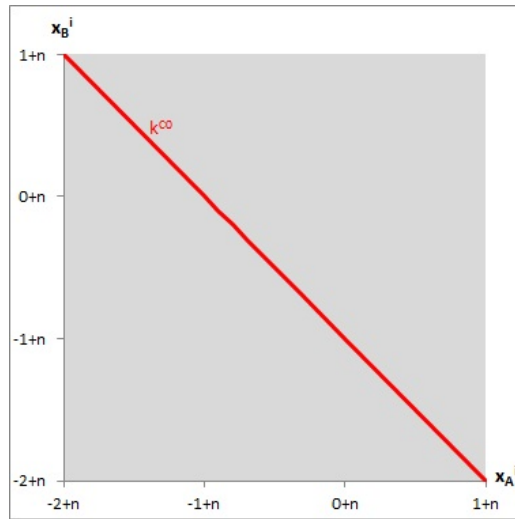


Figure 3: Cutoffs for the Unified-risky-option Case in the Space of Private Signals

Figure 4 shows the share of agents choosing the unified risky option. The share of agents picking the unified option is increasing in both θ_A and θ_B . The higher any of the two fundamental values the more agent gets such signals that their sum is high enough to choose the unified risky option.

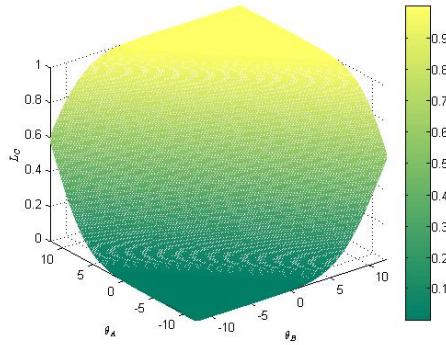


Figure 4: Share of Agents Choosing the Unified Risky Option in the Space of Fundamental Values

5 Comparison

In this section I compare different scenarios. In Subsection 5.1 I set the standard benchmark case when there is a single risky option with its own fundamental value against the case of two separate risky options. Then in Subsection 5.2 I compare the two-separate-risky-actions case with the unified-risky-options case, that is when there is one risky action depending on both fundamental values. In both subsections I first study the difference in the individual decision of agents, then I analyze the aggregate behavior of the agents.

5.1 One Single Risky Option Versus Two Separate Risky Options

Suppose there is a single risky option with its own fundamental value. In this case the cutoff is given by a constant, in our setup $k = -0.5 + n$ is the cutoff value (see Morris and Shin (2003)). An agent chooses the safe option if her signal is smaller than the cutoff value and opt for the risky one if higher.

How does the decision of the agent change if instead of a single risky option there are two risky possibilities? Figure 5 compares the outcomes regarding action A as a function of the private signals.⁷ The lines on the graph are the cutoff functions. The areas determined by the lines are denoted by two letters of which the first indicates the choice in the single-risky-action case, while the second indicates the outcome when there are two risky options.

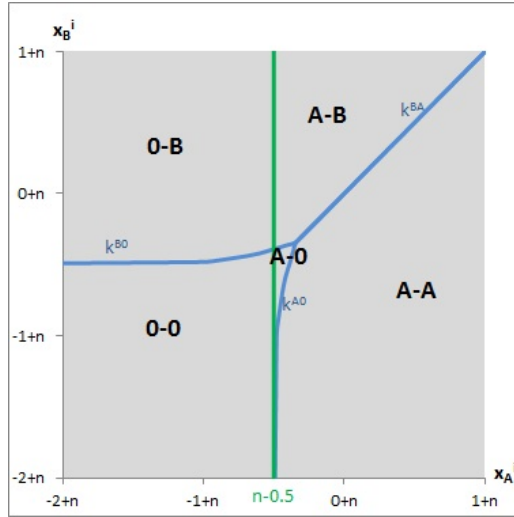


Figure 5: Comparison of the Individual Decisions in the Single-risky-option and Two-separate-risky-option Cases

⁷Note that x_B^i does not influence the decision of the agents when they can only choose action A, still I present the result in the x_A^i - x_B^i space to be comparable with the two-risky-action case.

The vertical line at $n - 0.5$ is the cutoff in the single-risky-option case. With a private signal smaller than $n - 0.5$ (the area left to this line) the agents do not pick the single risky option. Note that this decision is not influenced by the availability of another risky option. Indeed, with such signals she picks risky action A neither in the single-risky-option, nor in the two-separate-risky-option case. The only difference is that when there is a second risky possibility that the agent expects to be attractive enough (her private signal on action B is higher than k^{B0}) she pick that option (see the area denoted by 0-B), which she cannot in the single-risky-option case. Otherwise she keeps choosing the outside option (area 0-0).

When the agent receives a signal above $n - 0.5$ (area right to the line) she picks the single risky option. This action may change when there is another risky possibility. First, she may prefer to take the other risky alternative. With private signals falling on the domain A-B the agent has an expected gain from choosing option B higher than both the outside option and the gain from picking option A. Second, she may also prefer to choose the outside option. Area A-0 shows the pair of signals with which an agent does not choose neither of the two risky options though opt for the single risky option. This domain is enclosed by k^{B0} , k^{A0} and $-0.5 + n$ lines. Proposition 5 states that this area indeed exists (see the proof in the Appendix).

Proposition 5 (*inert area*). *The cutoff functions are above the $-1/2 + n$ line, that is $k^{r0}(x_{-r}^i) > -0.5 + n$ for $r \in \{A, B\}$ and $x_{-r}^i \in \mathbb{R}$.*

This area reveals that when there is a second option, however, not attractable for an agent, she is less willing to choose the first option. This is because agents have incentives to make mutually consistent actions. Given that the number of agents is fixed, when there are multiple options, their power is split. The more agent coordinates on one option the less people there are who can potentially coordinate on the other. This generates a negative correlation between the two options. I refer to this endogenous correlation as strategic correlation. Given that the fundamentals are uncorrelated, without the strategic correlation the willingness of an agent to pick a risky option should not be influenced by her expectations about the fundamental value of the other risky option that she will not choose for sure. Still, due to the endogenous strategic correlation, the availability of a second option, even when not attractable for an agent, makes the agent less willing to choose the first option.

Let us turn to the aggregates and compare the overall number of agents choosing option A under the two scenarios. Equation (10) and Figure 2 show the share of agents choosing option A in the two-separate-risky-option case. In the single-risky-option case the share of agents picking the risky option can be simply calculated by using the properties of a one dimensional normal distribution (see Morris and Shin (2003)). For a given θ_A fundamental value the share of agents taking action A is given by the following equation

$$E(L_A | \theta_A) = P(x_A^i > -1/2 + n) = \frac{1}{\sigma_A} \Phi \left(\frac{\theta_A + 0.5 - n}{\sigma_A} \right) \quad (13)$$

where again $\Phi(z)$ denotes the cdf of the univariate standard normal distribution.

Figure 6 compares the share of agents who choose action A if there are two separate risky and a safe option versus if besides the safe option only risky action A is available ($L_A - L$). The outcomes are plotted in the space of fundamentals⁸. The left panel shows the 3-dimensional surface while the right panel shows its contour projection on the XY plane.

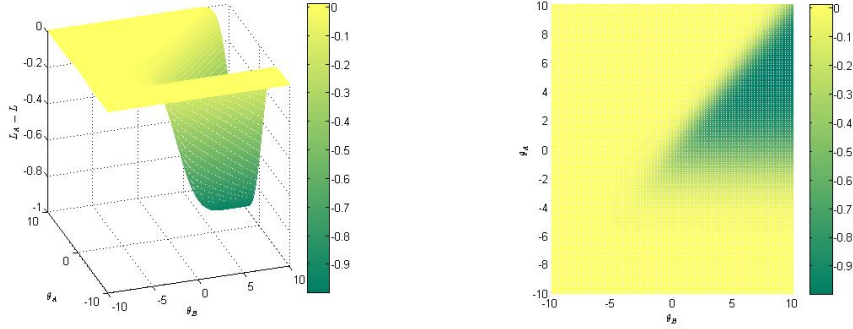


Figure 6: Comparison of the Aggregate Number of Agents Choosing the Single Risky Option and Either of the Two Separate Risky Options

When θ_A is low enough the agents would not pick action A in neither case, thus the presence of the second risky option does not influence the outcome. However, when θ_A is higher the availability of risky option B matters. In particular the higher θ_B is relative to θ_A , the less agents pick option A as they rather choose option B.

5.2 Two Separated Versus Two Unified Risky Options

In this subsection I compare the cases when the two risky options are separated and when they are unified. Figure 7 shows the individual decisions as a function of private signals in both cases. The lines on the graph are the cutoff functions. Similarly as before, the k^{A0} , k^{AB} and k^{B0} lines show the cutoff lines in the two-separate-risky-option case. While k^{C0} shows the cutoff line for the unified-risky-option case.

The areas determined by the lines are denoted by two letters of which the first indicates the choice when there are two separated risky possibilities, while the second indicates the outcome when they are unified. Agents with pair of signals falling on the 0-0 domain choose the outside option in both cases. Then, agents in domains A-C and B-C pick one of the separate risky options (at A-C they opt for action A, while at B-C they choose option B) and also the unified risky option. At the same time, agents with pair of signals on area A-0 or B-0 choose the risky option A or B, respectively, but does not choose their union.

⁸Note that θ_B does not influence the outcome in the single-risky-option case, still I present the result in the θ_A - θ_B space to be comparable with the two-risky-option case.

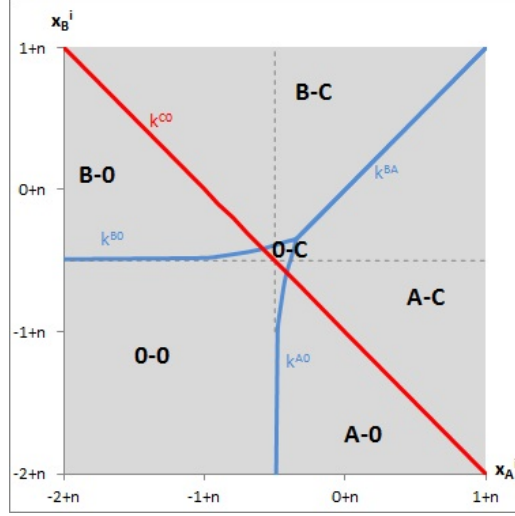


Figure 7: Comparison of Individual Decisions in the Separate-risky-option and the Unified-risky-option Cases

These areas include agents who receive high signal about one of the fundamentals and low about the other. Hence, these agents expect one of the options to be valuable enough for choosing of its own, however the low expected value of the other option makes unattractive the unified option. Finally, agents with private signals on the area 0-C choose neither of the separate risky options but pick their union. This is due to the fact that the negative strategic correlation is present in case of separate options but not in case of unification. Indeed, when there are two separate risky possibilities there are two potential targets to coordinate on, hence the coordination disperses and thus weakens its power. While in case of a unified risky option agents coordinate on a single target, thus the negative strategic correlation does not arise. It is easy to show that the 0-C area indeed exists by using Proposition 5 and the observation that the line k^{CO} crosses the $(n - 0.5, n - 0.5)$ point.

Let us turn to the aggregate behavior of the agents. Equation (10) and Figure 2 show L_A , the share of agents choosing option A in the case of two separate risky options. While if the risky options are unified the aggregate number of agents choosing it is shared, that is $L_A = L_B = \frac{L_C}{2}$, where Equation (12) and Figure 4 show L_C , the share of agents choosing the unified option.

Consider first the overall number of agents choosing any of the risky possibilities. Figure 8 shows the difference between the total number of agents choosing a risky option in case of joint and separate risky options ($L_C - L_A - L_B$). The outcomes are shown in the space of the fundamental values of the two options. The top left panel shows the 3-dimensional surface while the top right panel shows its contour projection on the XY plane while the bottom panel shows the sign of the difference.

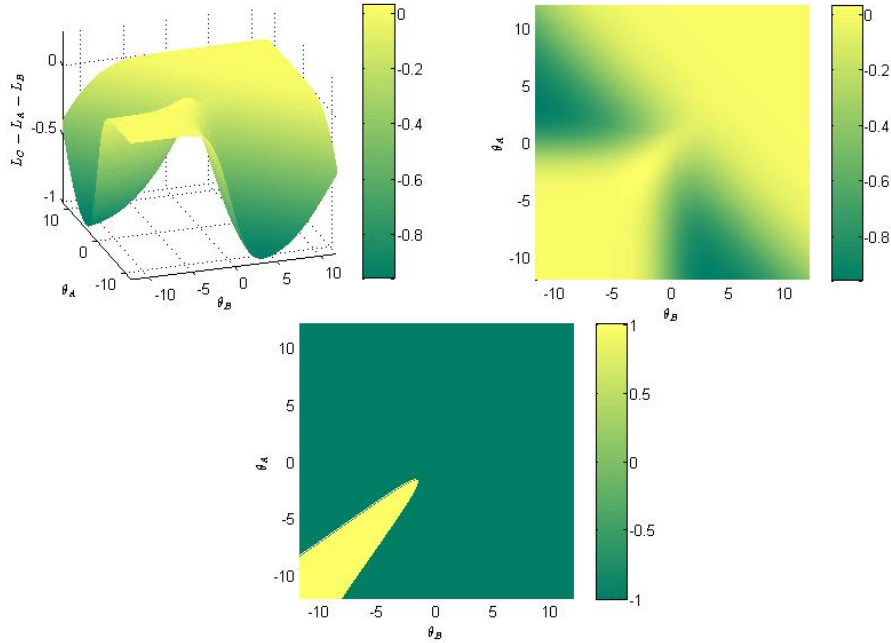


Figure 8: Comparison of the Aggregate Number of Agents Choosing the Unified Risky Option and Either of the Risky Options

The sign is positive when more agents pick the unified option than the two separate risky options and negative in the reverse case. The difference is around zero when the unification does not influence the total number of agents choosing them. There are two such typical situations: first, when one of the fundamentals is so high that it outweighs the other fundamental value (see the northeast part on the top right panel); second, when both fundamentals are low (see the southwest part on the top right panel). In the former case, the option with high fundamental value dominates the union, which is thus chosen with similar intensity, while the other option alone is not selected. In the latter case, none of the separate risky options and neither their union is picked. More agents choose either of the independent risky options than their union (see the dark areas on the top right panel) when one of the risky option has high fundamental value and the other has low, but in absolute value the one with low is greater. In this case the option with high fundamental value is chosen, but not the union. However, there are situations when slightly more agents opt for the joint option than in either of the separate risky options (see the light area on the bottom panel). This is the case when the value of the two fundamentals are close to each other and both are low. In this situation, because of the negative strategic correlation, a

single target is more attractive than multiple targets, this is why more agents choose the union of the risky options than the distinct options.

Consider next how the form of the risky options affects, say, option A. Figure 9 shows the difference between the number of agents assigned to option A in the unified and in the separated case ($L_C/2 - L_A$). The outcomes are shown in the space of fundamentals. The left panel shows the 3-dimensional surface while the right panel shows its contour projection on the XY plane.

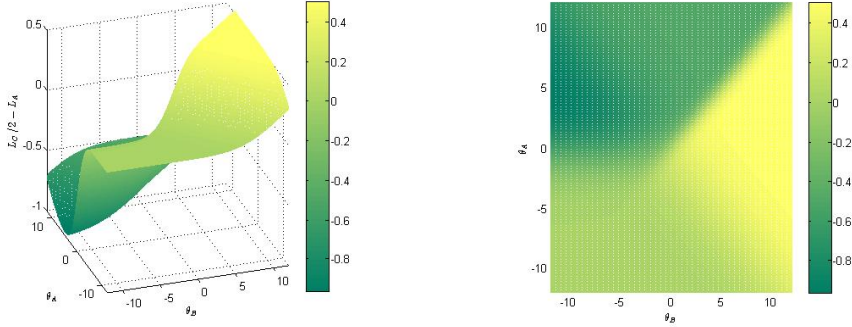


Figure 9: Comparison of the Aggregate Number of Agents Assigned to Option A in the Separate-risky-option and the Unified-risky-option Cases

The figure reveals that when both θ_A and θ_B are low (see the southwest part on the right panel) the form of the risky options does not make a difference, since neither fundamental A individually, nor the alliance of the two fundamentals is chosen. However, when θ_A is high compared to θ_B (see the north and the northwest part on the right panel) less agent picks the union as fundamental B counteracts the strength of fundamental A, so in these cases option A is less popular individually. Meanwhile, when θ_B is high compared to θ_A (see the east and the southeast part on the right panel) more agents pick option A singly as the lower fundamental value of B makes the union less preferred.

6 Information Accuracy

This section shows how the outcomes depend on the information precision. I investigate what happens if the standard deviation, either the systematic part (s) or the fundamental specific parts (s_A and s_B), of the noise term changes. Higher standard deviation means that there is lower information in the signals. The information precision affects the individual decisions when there are separate risky options, but does not affect when they are unified. However, the aggregate behavior of the agents changes in both scenarios.

First, I consider changes in the standard deviation of the systemic part of the noise term (s). Figure 10 provides a geometrical representation of how the cutoff lines separating the

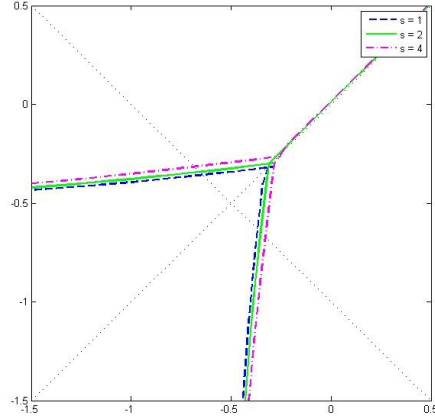


Figure 10: Cutoff Lines in the Two-separate-risky-option Case at Various Standard Deviations of the Systemic Part of the Noise Term ($s = \{1, 2, 4\}$ and $s_A = s_B = 0.7$)

potential choices of an agent shifts for different s values. The fundamental specific variances are fixed at $s_A = s_B = 0.7$.

The plot reveals that when s increases, the cutoff lines k^{AO} and k^{BO} shift equally outwards from the -0.5 lines. So the higher the uncertainty, the higher signal on a given option is needed for an agent to choose that option. The intuition is the following: the higher dispersion of the expectation of the fundamentals makes the agents to expect that a larger share of their fellow agents would pick the other risky option. In other words, higher uncertainty makes coordination harder, thus strengthens the negative strategic correlation, and therefore enlarges the inert area.

Figure 11 shows how the value of s influences the aggregate number of agents choosing option A in the separate-risky-option case (L_A , see Subfigure 11a) and in the unified-risky-option case (L_C , see Subfigure 11b).

As s increases both curves become flatter, so the higher uncertainty reduces the potency of the fundamentals. This is also reflected in the difference between the aggregate behavior of the agents under the two scenarios, which is also plotted on Figure 11. Subfigure 11c plots the difference between the number of agents choosing option A in case of unified and separate risky options ($L_C/2 - L_A$). Subfigure 11d shows the difference between the total number of agents who opt for the unified risky option and who pick either of the separate risky options ($L_C - L_A - L_B$).

Both flatten and drift towards zero as s increases. In other words, when the agents' private information is more accurate, the difference between the agents' aggregate behavior under the various scenarios decreases.

Note that all the effects caused by changes in s are symmetric in the two risky options.

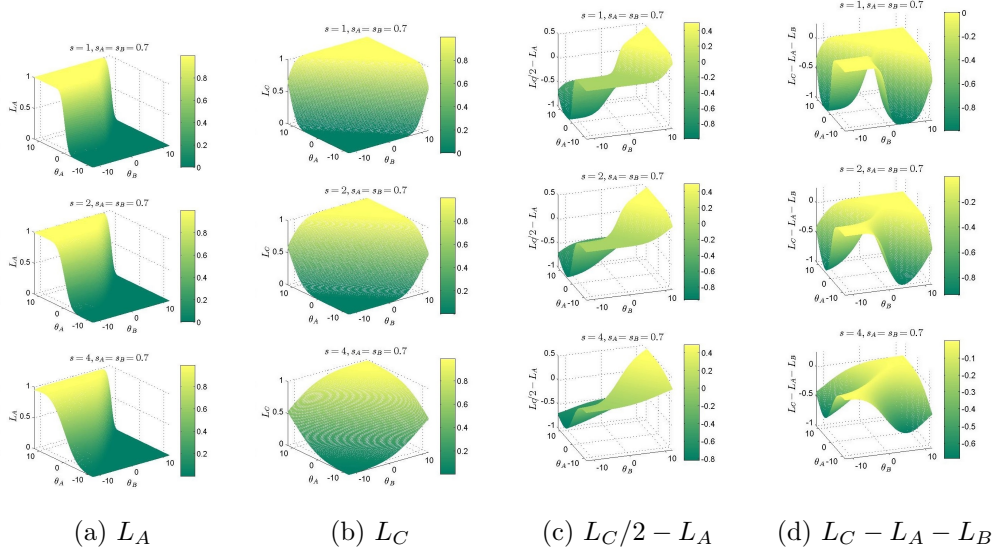


Figure 11: Aggregate Number of Agents and Differences Between the Aggregate Number of Agents Choosing the Different Options at Various Standard Deviations of the Systemic Part of the Noise Term ($s = \{1, 2, 4\}$ and $s_A = s_B = 0.7$)

Now, I bring in asymmetry and consider variation in the standard deviation of the option specific part of the noise. Figure 12 shows how the cutoff lines in the separate-risky-option case vary depending on the value of s_B . The other parameters are fixed at $s_A = 0.7$ and $s = 2$.

When information on option B is less accurate (that is s_B increases), the three cutoff lines move to higher θ_B and lower θ_A values. This is in line with the previous finding that the cutoff lines shifts upward when there is higher uncertainty. Indeed, k^{BO} shifts upward as higher s_B means bigger uncertainty regarding option B. Meanwhile, an increase in s_B decreases the relative (compared to option B) uncertainty about option A, that is why k^{AO} shifts downward.

Figure 13 plots the agents aggregate behavior for various s_B values. Subfigure 13a shows the aggregate number of agents picking option A in the instance of separate risky options (L_A), while Subfigure 13b shows the share of agents choosing the unified risky option (L_C).

Changes in s_B twist the L_A surface, but not L_C . This is because due to the unification all effects are divided equally, thus even the option specific changes have symmetric effects and affect only the steepness but not the curvature of L_C .

But, given that L_A rotates, the difference of the share of agents assigned to a risky option under the different scenarios rotates as well. This is shown on Figure 13. Subfigure 13c

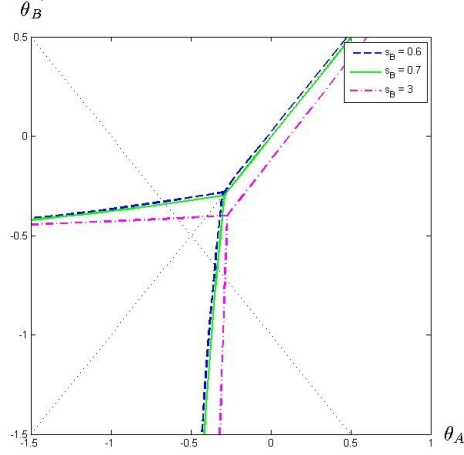


Figure 12: Cutoff Lines in the Two-separate-risky-option Case at Various Standard Deviations of the Option Specific Part of the Noise Term ($s = 2$, $s_A = 0.7$ and $s_B = \{0.6, 0.7, 3\}$)

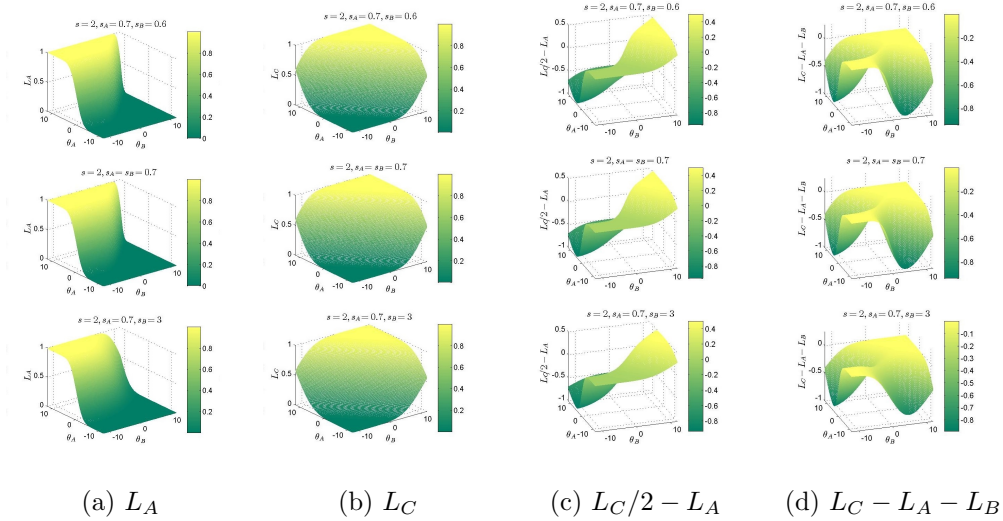


Figure 13: Aggregate Number of Agents and Differences Between the Aggregate Number of Agents Choosing the Different Options at Various Standard Deviations of the Option Specific Part of the Noise Term ($s = 2$, $s_A = 0.7$ and $s_B = \{0.6, 0.7, 3\}$)

plots the difference between the agents assigned to option A in case of joint and separate options ($\frac{L_C}{2} - L_A$). Subfigure 13d compares the total number of agents who choose the joint risky option and who pick either of the separate risky options ($L_C - L_A - L_B$).

7 Applications

In this section I show two potential applications of the multidimensional global games model. First, in Subsection 7.1 I describe a model for the choice of oil invoicing currency. Second, in Subsection 7.2 I present a model for the issuance of the European common bond.

7.1 Choice of Currency for Oil Invoicing

In this subsection I introduce a model that describes the choice of invoicing currency in the oil market. The model is an extended, but partially simplified version of the model developed by Mileva and Siegfried (2007).⁹ There is a continuum of crude oil seller with measure one, indexed by $i \in I = [0, 1]$. Oil sellers have to decide which currency to use for invoicing their oil contracts. Suppose there are three currencies, the US Dollar, the Euro and the British Pound, which can be used, that is $j \in \{USD, EUR, GBP\}$. Each seller can use only a single currency. In time $t = 1$ sellers decide on the currency, while at time $t = 2$ trade takes place and sellers realize their income. The price of oil is independent of the invoicing currency, however the cost varies depending on the currency.

The cost of using currency j is C_j . It contains the transaction cost, the liquidity cost and the information cost. Information cost arises only for the Euro and for the British Pound. In the oil market the historically established invoicing currency is the US Dollar. However, switching to a different currency have information cost as the traders have to learn the usage of the new unit of account. The more trader uses the new currency the lower the information cost is. I assume that the transaction and the liquidity cost do not depend on the number of traders using the given currency,¹⁰ hence the cost of usage is the function of the number of agents who use the currency for the Euro and for the British Pound but not for the US Dollar. The aggregate number of agents using currency j is denoted by L_j , while c_j is the part of the cost of using currency j which does not depend on the aggregate number of users. For simplicity, I assume that L_j enters the cost function in an additive way. Thus the cost functions are $C_{USD} = c_{USD}$, $C_{EUR} = c_{EUR} - L_{EUR}$ and $C_{GBP} = c_{GBP} - L_{GBP}$.

⁹Their emphasis is on the network effects which arise from the assumption that currency choice of crude oil sellers determine the currency distribution of other goods. I exclude this assumption and rather build on the learning element of the model.

¹⁰Contrary to the model in Mileva and Siegfried (2007) I suppose that the denomination of oil producers' expenses is not influenced by the composition of the invoicing currency of oil as the oil market is small compared to the non-oil market.

At $t = 1$ sellers get noisy signals $X_j^i = c_j + \epsilon_j^i$ for each $j \in \{USD, EUR, GBP\}$, where ϵ_j^i are the noise terms which are distributed independently and normally with mean 0 and standard deviation ς_j , and are independent across agents. Given her signal triplet each seller decides on her invoicing currency choice. A seller prefers the Euro over the other two currencies if she expects $c_{EUR} - c_{USD} - L_{EUR}$ to be negative and smaller than $c_{GBP} - c_{USD} - L_{GBP}$. Similarly, a seller prefers the most the British Pound if she expects $c_{GBP} - c_{USD} - L_{GBP}$ to be negative and smaller than $c_{EUR} - c_{USD} - L_{EUR}$. Otherwise, she prefers the most the US Dollar.

Let me introduce the notations $\theta_r \equiv c_{USD} - c_r$ and $x_r^i \equiv X_{USD}^i - X_r^i$, $\varepsilon_r^i = \epsilon_{USD}^i - \epsilon_r^i$ for $r \in \{EUR, GBP\}$. Such we have the same model as described in Section 2. In particular the two risky options are the Euro and the British Pound and the US Dollar is the outside option. The two fundamental values are θ_{EUR} and θ_{GBP} on which oil sellers get signals x_{EUR}^i and x_{GBP}^i , where ϵ_{USD}^i is the systematic part and $-\epsilon_r^i$ is the fundamental specific part of the noise terms. Thus the standard deviations are $\sigma_{EUR} = \sqrt{\varsigma_{USD}^2 + \varsigma_{EUR}^2}$ and $\sigma_{GBP} = \sqrt{\varsigma_{USD}^2 + \varsigma_{GBP}^2}$, while the correlation coefficient is $\rho = \frac{\varsigma_{USD}^2}{\sqrt{(\varsigma_{USD}^2 + \varsigma_{EUR}^2)}\sqrt{(\varsigma_{USD}^2 + \varsigma_{GBP}^2)}}$. Finally, one can get from the payments after some algebra that $n = 0$.

Let me compare the individual decisions when only the Euro and when both the Euro and the British Pound are available besides the US Dollar for invoicing oil contracts. Figure 5 is suitable for the comparison. Option 0 represents the US Dollar, option A is the Euro and option C is the British Pound. The line at -0.5 (since $n = 0$, $n - 0.5 = -0.5$) separate the traders decision when only the US Dollar and the Euro are usable. In the three-currency-case the k^{B0} , k^{BA} and k^{A0} lines separate the traders' decisions. A trader with $x_r^i \equiv < -0.5$ (left to the line at -0.5), or equivalently $X_{USD}^i < X_r^i - 0.5$, switches to the Euro, otherwise continues to use the US Dollar in the two-currencies case. How does the availability of another currency (in our example the British Pound) affects the traders' decision on the invoicing currency? Oil sellers using the US Dollar in the two-currencies case either continue to use the US Dollar (0-0 area) or switches to the British Pound (0-B area). Traders who switch to the usage of the Euro when this is the only new currency besides the US Dollar either choose again the Euro (A-A area) or switch to the British Pound (A-B area) or after all use the US Dollar (A-0 area). Hence there are situations when an oil seller would switch to the usage of a new currency if there were one new currency besides the US Dollar, however would not switch if there were two other currencies.

7.2 Introduction of the Common European Bond

In this subsection I present a model to describe the introduction of a joint bond that would replace the national issuance by member states of the Eurozone. Here I concentrate on the case when symmetric countries issue the common bond. For this, the model with uncorrelated fundamentals is suitable. To assess the case of asymmetric countries a model with

correlated fundamentals is required, which I sketch in Appendix A.2. The multidimensional global games framework is suitable for evaluating the effect of joint bond issuance on the stability of the participating countries.

A and B are two countries with similar economic strength. Both countries borrow from investors by issuing bonds. There are two scenarios. The first is when the two countries issue bond separately. The second is when the two countries do not issue national bonds, instead they together issue a common bond.¹¹

There is a continuum of speculators with measure one, indexed by $i \in I = [0, 1]$. There are two periods. In period 1 speculators can decide whether to short some bonds. Because of short-sale constraints, each trader can short sell exactly 1 unit. If countries issue bonds separately speculators can trade any of the two (but because of the short-sale constraints not both at the same time). That is the set of available actions for each speculator is $\Omega = \{0, A, B\}$, where not trading is represented by 0, taking short position in one of the two national bonds are denoted by A and B . If countries issue bonds jointly the set of available actions for the traders is $\Omega = \{0, C\}$, where C means shorting the common bond.

Settlement takes place in period 2. Speculators choosing the outside option get a risk free interest rate r_f , thus their payment is $1 + r_f$. Speculators going short in bond $a \in \{A, B, C\}$ realizes payoff $p(\theta_a + L_a)$. The fundamental values θ_A and θ_B represent the vulnerability of the two countries, they are independently and randomly drawn from the real line. While $\theta_C = \frac{1}{2}(\theta_A + \theta_B)$ represents the vulnerability of the alliance of the countries and is equal to the average of the individual fundamental values. Furthermore L_A , L_B and L_C denote the mass of speculators shorting the bond of country A, the bond of country B and the joint bond, respectively.

Each trader receives a noisy signal about both countries' fundamentals. The private signal of investor $i \in [0, 1]$ about the fundamental of country $r \in \{A, B\}$ is $x_r^i = \theta_r + \varepsilon_r^i$, where ε_r^i is an idiosyncratic noise. The noise term consists of two parts: $\varepsilon_r^i = e^i + e_r^i$. The first component of the noise term, e^i , is the systemic part of the noise, while the second component, e_r^i , is the country specific part. The components e^i , e_A^i and e_B^i are distributed independently and normally with mean 0 and standard deviation s , s_A and s_B , respectively, and are independent across speculators.

The welfare in country $r \in \{A, B\}$ is a decreasing function of the fundamental vulnerability of the country and the mass of speculators attacking the country: $W_r = w_r(\theta_r, L_r)$, where $w_r' < 0$. Similarly, the global welfare is a decreasing function of the sum of the two countries' vulnerability and the overall number of shorting traders: $W = w(\theta_A + \theta_B, L_A + L_B)$, where $w' < 0$. In case of joint issuance the countries share the cost of attack, that is $L_A = L_B = \frac{L_C}{2}$.

I compare the welfare in the separate and in the joint bond issuance scenarios. The

¹¹Several implementation approaches to common bond issuance have been suggested. Though there are proposals with a mix of national bonds and jointly issued common bonds, in this subsection I only concentrate on the limiting case when there is a full degree of substitution of joint issuance for national issuance.

former scenario is equivalent to the separate-risky-options case, while the latter is identical to the unified-risky-options case. Given that the welfare functions w_r and w are decreasing and the value of the fundamental is independent of the type of issuance, the aggregate number of attackers is the key ingredient of the welfare comparison. In particular the one with higher number of attackers results lower welfare. Hence Figure 8 and Figure 9 capture the results.

Figure 8 shows the difference between the total number of speculators attacking in case of joint and separate issuance. The sign is positive when more speculators attack the common bond than the two separate countries and negative in the reverse case. More agents attack either of the independent countries than their alliance (see the dark areas on the top right panel) when one of the countries has high vulnerability and the other has low, but in absolute value the one with low is greater. In this case the country which is vulnerable alone is attacked, but not the common bond. This reveals that the joint issuance can smooth out idiosyncratic risk, which is a common argument for Eurobond. However, when the vulnerability of the two countries are similarly low, slightly more agents speculate in the joint bond than in either of the national bonds (see the light are on the bottom panel). In this case the negative strategic correlation makes the single target more attractive than the multiple targets.

Figure 9 shows the difference between the number of speculators impairing country A in case of joint and separate issuance. The figure shows that when the vulnerability of both countries are low (both θ_A and θ_B are low, see the southwest part on the right panel) the form of bond issuance does not make a difference, since neither country A individually, nor the alliance of the two countries is attacked. However, when country A is vulnerable compared to country B (high θ_A compared to θ_B , see the north and the northwest part on the right panel) the joint bond is less attacked since country B counteracts the vulnerability of country A, so in these cases country A is worse off individually. Meanwhile, when the country B is more vulnerable than country A (high θ_B compared to θ_A , see the east and the southeast part on the right panel) country A is better off alone as the vulnerability of country B harms also country A in case of their alliance.

8 Conclusion

In this paper I analyze the coordination aspect of multidimensional global games. Global games are coordination games with incomplete information, they have been applied to several economic situations, such as bank runs, currency crisis, and technology adoption. I extend the standard global games framework by introducing and additional coordination target.

Multidimensionality has an important consequence for the power of coordination. When there are multiple options, coordination weakens. This is due to strategic motives of agents. Agents have incentives to make mutually consistent actions. Since there are a fixed number

of agents, when there are multiple options, their power is split. The more people coordinate on one option the less people there are who can potentially coordinate on the other. This generates a negative correlation between the two options which I call strategic correlation.

The key element of the model is the interaction of the coordination motives of agents to move together and the substitutability of the options. When there are multiple options, each potential object of coordination, they are in fact substitutes. Thus, with multiple options the coordination disperses. However, unifying the options eliminates the coordination split and thus strengthens the power of coordination.

I show two applications which can be modeled by the multidimensional global games framework. The first application is the choice of invoicing currency of oil. In the oil market the historically established currency is the US Dollar. I show that there are situations when an agent would switch to the usage of a new currency if there were one new currency besides the US Dollar, however, would not switch if there were two other currencies. The second application is the introduction of common European bond. A common argument for joint issuance is that it smooths out idiosyncratic risk. While this argument is present in my model, there is an extra layer: joint bond issuance can make participating countries more vulnerable to speculative attacks.

References

- Chamley, C. (1999). Coordinating regime switches. *Quarterly Journal of Economics*, 114(3):869–905.
- Cukierman, A., Goldstein, I., and Spiegel, Y. (2004). The choice of exchange-rate regime and speculative attacks. *Journal of the European Economic Association*, 2(6):1206–1241.
- Fujimoto, J. (2014). Speculative attacks with multiple targets. *Economic Theory*, (57):89–132.
- Goldstein, I. and Pauzner, A. (2004). Contagion of self-fulfilling financial crises due to diversification of investment portfolios. *Journal of Economic Theory*, 119(1):151–183.
- Goldstein, I. and Pauzner, A. (2005). Demand–deposit contracts and the probability of bank runs. *Journal of Finance*, 60(3):1293–1327.
- He, Z., Krishnamurthy, A., and Milbradt, K. (2016). A model of safe asset determination. Working paper.
- Heidhues, P. and Melissas, N. (2006). Equilibria in a dynamic global game: the role of cohort effects. *Economic Theory*, 28(3):531–557.
- Keister, T. (2009). Expectations and contagion in self-fulfilling currency attacks. *International Economic Review*, 50(3):991–1012.
- Mileva, E. and Siegfried, N. (2007). Oil market structure, network effects and the choice of currency for oil invoicing. *ECB Occasional Paper*, (77).
- Morris, S. and Shin, H. S. (1998). Unique equilibrium in a model of self-fulfilling currency attacks. *The American Economic Review*, 88(3):587–597.
- Morris, S. and Shin, H. S. (2003). Global games: theory and applications. In M. Dewatripont, L. H. and Turnovsky, S., editors, *Advances in Economics and Econometrics, 8th World Congress of the Econometric Society*. Cambridge University Press, Cambridge, UK.
- Morris, S. and Shin, H. S. (2004). Coordination risk and the price of debt. *European Economic Review*, 48(1):133–153.
- Oury, M. (2013). Noise-independent selection in multidimensional global games. *Journal of Economic Theory*, 148(6):2638–2665.

A Appendix

A.1 Complete Information

In case of complete information there are multiple equilibria for a certain range of fundamentals. The following domains can be separated in case of separate risky options (see Figure 14):

- i) if $\theta_B \geq \max(n, \theta_A + 1)$, each agent has a dominant strategy to choose option B
- ii) if $\theta_A \geq \max(n, \theta_B + 1)$, each agent has a dominant strategy to choose option A
- iii) if $\theta_A < n - 1$ and $\theta_B < n - 1$, each agent has a dominant strategy to choose the outside option
- iv) if $n - 1 \leq \theta_B < n$ and $\theta_A < n - 1$, there are two pure strategy Nash equilibria: either everybody choose option B or everybody choose the outside option
- v) if $n - 1 \leq \theta_A < n$ and $\theta_B < n - 1$, there are two pure strategy Nash equilibria: either everybody choose option A or everybody choose the outside option
- vi) if $\theta_1 - 1 < \theta_B < \theta_A + 1$ and either $n - 1 < \theta_A$ or $n - 1 < \theta_B$, there are two pure strategy Nash equilibria: either everybody choose option A or everybody choose option B
- vii) if $n - 1 \leq \theta_A < n$ and $n - 1 \leq \theta_B < n$, there are three pure strategy Nash equilibria: either everybody choose option A or everybody choose option B or everybody choose the outside option

While in case of joint issuance the following domains can be distinguished (see Figure 15):

- i) if $\theta_B \geq 2n - \theta_A$, each agent has a dominant strategy to choose the unified risky option
- ii) if $2n - \theta_A > \theta_B \geq 2n - \theta_A - 2$, there are two pure strategy Nash equilibria: either everybody choose the unified risky option or everybody choose the outside option
- iii) if $2n - \theta_A - 2 > \theta_B$, each agent has a dominant strategy to choose the outside option

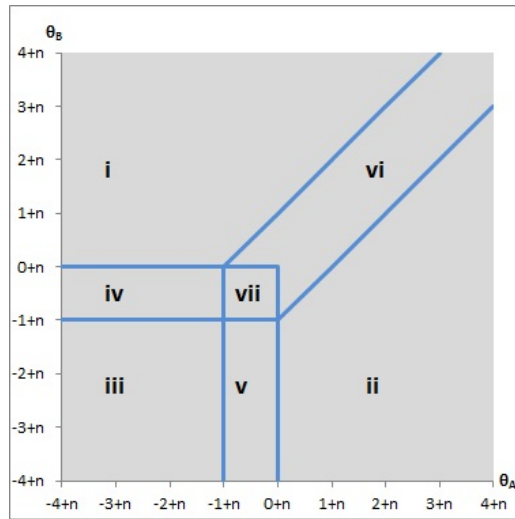


Figure 14: Equilibria in Case of Complete Information with Two Separate Risky Options in the Space of the Fundamental Values

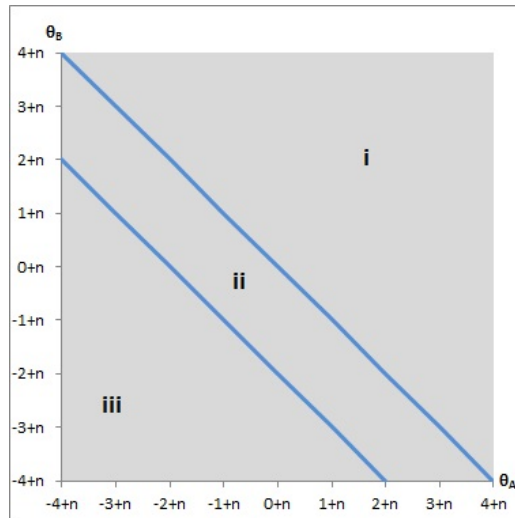


Figure 15: Equilibria in Case of Complete Information with the Unified Risky Option in the Space of the Fundamental Values

A.2 Correlated Fundamentals

The economic fundamentals, θ_A and θ_B , are correlated with correlation coefficient ρ_θ and follow a bivariate normal distribution such that $\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} \sim N\left(\begin{bmatrix} y_A \\ y_B \end{bmatrix}, \begin{bmatrix} \tau_A^2 & \tau_A \tau_B \rho_\theta \\ \tau_A \tau_B \rho_\theta & \tau_B^2 \end{bmatrix}\right)$.¹² However, for simplicity, I assume that the pair of signals received by and agent are not correlated. For this reason, I set the variance of the systematic part of the individual noise term (ε^i) to be 0 (that is $s = 0$), and thus the noise term ε_r^i has mean 0 and standard deviation $s_r = \sigma_r$.

The joint distribution of the fundamentals and the noise terms implies that an agent observing the vector of signals $x^i = (x_A^i, x_B^i)$ considers the fundamental values and the opponents' signal to be distributed normally such as $\theta | x^i \sim N(\bar{\mu}_\theta, \bar{\Sigma}_\theta)$ and $x^j | x^i \sim N(\bar{\mu}_{x^j}, \bar{\Sigma}_{x^j})$ for $\forall j \neq i$, where

$$\bar{\mu}_\theta = \bar{\mu}_{x^j} = \begin{pmatrix} H_{AA}x_A^i + (1 - H_{AA})y_A + H_{AB}x_B^i + (1 - H_{AB})y_B \\ H_{BA}x_A^i + (1 - H_{BA})y_A + H_{BB}x_B^i + (1 - H_{BB})y_B \end{pmatrix}$$

and

$$\bar{\Sigma}_{x^j} = \bar{\Sigma}_\theta + \begin{pmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{pmatrix} = \begin{pmatrix} \sigma_A^2(1 + H_{AA}) & \sigma_A \sigma_B \sqrt{H_{AB}H_{BA}} \\ \sigma_A \sigma_B \sqrt{H_{AB}H_{BA}} & \sigma_B^2(1 + H_{BB}) \end{pmatrix}$$

where the weights are

$$H_{AA} \equiv \frac{(1 - \rho^2)(\tau_A \tau_B)^2 + (\tau_A \sigma_B)^2}{(1 - \rho^2)(\tau_A \tau_B)^2 + (\tau_A \sigma_B)^2 + (\sigma_A \tau_B)^2 + (\sigma_A \sigma_B)^2} \quad (14)$$

$$H_{AB} \equiv \frac{\rho \tau_A \tau_B \sigma_A^2}{(1 - \rho^2)(\tau_A \tau_B)^2 + (\tau_A \sigma_B)^2 + (\sigma_A \tau_B)^2 + (\sigma_A \sigma_B)^2} \quad (15)$$

$$H_{BA} \equiv \frac{\rho \tau_A \tau_B \sigma_B^2}{(1 - \rho^2)(\tau_A \tau_B)^2 + (\tau_A \sigma_B)^2 + (\sigma_A \tau_B)^2 + (\sigma_A \sigma_B)^2} \quad (16)$$

$$H_{BB} \equiv \frac{(1 - \rho^2)(\tau_A \tau_B)^2 + (\sigma_A \tau_B)^2}{(1 - \rho^2)(\tau_A \tau_B)^2 + (\tau_A \sigma_B)^2 + (\sigma_A \tau_B)^2 + (\sigma_A \sigma_B)^2} \quad (17)$$

Similarly as in case of uncorrelated fundamentals, the strategy can be characterized by Proposition 2. However the cutoff function characterizing the equilibrium are differ from the uncorrelated case. Proposition 6 describes them.

¹²An alternative representation is that the fundamentals are independently and randomly drawn from the real line, however each agent observe the pair of noisy public signals $y_A = \theta_A + \varepsilon_A^y$ and $y_B = \theta_B + \varepsilon_B^y$, where $\begin{bmatrix} \varepsilon_A^y \\ \varepsilon_B^y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_A^2 & \tau_A \tau_B \rho_\theta \\ \tau_A \tau_B \rho_\theta & \tau_B^2 \end{bmatrix}\right)$.

Proposition 6 (cutoff functions 2).

The cutoff functions can be characterized by the following equations:

$$k^{A0}(x_B^i) = \frac{1}{H_{AA}} \left\{ n - 1 - (1 - H_{AA})y_A - H_{AB}x_B^i - (1 - H_{AB})y_B + \int_{-\infty}^{\infty} \phi(z) \Phi(D^A(z, k^{A0}(x_B^i), x_B^i)) dz \right\} \quad (18)$$

$$k^{B0}(x_A^i) = \frac{1}{H_{BB}} \left\{ n - 1 - (1 - H_{BB})y_B - H_{BA}x_A^i - (1 - H_{BA})y_A + \int_{-\infty}^{\infty} \phi(z) \Phi(D^B(z, k^{B0}(x_A^i), x_A^i)) dz \right\} \quad (19)$$

$$k^{AB}(x_B^i) = \frac{H_{AB} - H_{BB}}{H_{AA} - H_{BA}}(y_B - x_B^i) + y_A + \frac{1}{H_{AA} - H_{BA}} \cdot \int_{-\infty}^{\infty} \phi(z) [\Phi(D^A(z, k^{AB}(x_B^i), x_B^i)) - \Phi(D^B(z, x_B^i, k^{AB}(x_B^i)))] dz \quad (20)$$

$$k^{BA}(x_A^i) = \frac{H_{BA} - H_{AA}}{H_{BB} - H_{AB}}(y_A - x_A^i) + y_B + \frac{1}{H_{BB} - H_{AB}} \cdot \int_{-\infty}^{\infty} \phi(z) [\Phi(D^B(z, k^{BA}(x_A^i), x_A^i)) - \Phi(D^A(z, x_A^i, k^{BA}(x_A^i)))] dz \quad (21)$$

where $\phi(z)$ and $\Phi(z)$ denote the pdf and the cdf, respectively, of the univariate standard normal distribution, H_{AA} , H_{AB} , H_{BA} and H_{BB} as defined in equation (14), (15), (16) and (17), respectively and

$$D^A(z, x_A^i, x_B^i) \equiv \frac{\sqrt{1 + H_{BB}}}{\sigma_A \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} \cdot \left\{ K^A \left(\sigma_B \sqrt{1 + H_{BB}} z + H_{BA}x_A^i + (1 - H_{BA})y_A + H_{BB}x_B^i + (1 - H_{BB})y_B \right) - H_{AA}x_A^i - (1 - H_{AA})y_A - H_{AB}x_B^i - (1 - H_{AB})y_B - \frac{\sigma_A \sqrt{H_{AB}H_{BA}} z}{\sqrt{1 + H_{BB}}} \right\}$$

$$\begin{aligned}
D^B(z, x_B^i, x_A^i) &\equiv \frac{\sqrt{1+H_{AA}}}{\sigma_B \sqrt{(1+H_{AA})(1+H_{BB}) - H_{AB}H_{BA}}} \cdot \left\{ K^B \left(\sigma_A \sqrt{1+H_{AA}} z \right. \right. \\
&\quad \left. \left. + H_{AB}x_B^i + (1-H_{AB})y_B + H_{AA}x_A^i + (1-H_{AA})y_A \right) - H_{BB}x_B^i \right. \\
&\quad \left. - (1-H_{BB})y_B - H_{BA}x_A^i - (1-H_{BA})y_A - \frac{\sigma_B \sqrt{H_{BA}H_{AB}} z}{\sqrt{1+H_{AA}}} \right\}
\end{aligned}$$

Proposition 7 states that if there is enough noise in the signal generating process, there exists a unique equilibrium in monotone strategies.

Proposition 7 (*existence, uniqueness 2*). *If ... for all $r \in \{A, B\}$, there exists an essentially unique Bayesian equilibrium described by the cutoff functions given in Proposition 6.*

A.3 Proofs

Proof. (of Proposition 1) The joint distribution of the noise terms implies that an agent observing the vector of signals $x^i = (x_A^i, x_B^i)$ considers the fundamental values and the opponents' signal to be distributed normally as follows: $\theta_r | x^i \sim N(x_r^i, \sigma_r)$ and $x_r^j | x^i \sim N(x_r^i, \sqrt{2}\sigma_r)$ for $\forall j \neq i$ and $r \in \{A, B\}$. Thus if x_r^i increases ceteris paribus, $E(\theta_r | x^i)$ increases as well.

Moreover the posterior distribution of opponents' signal is also increasing in x_r^i , thus given that agents follow monotone increasing strategies in related signals, and $E(L_r | x^i) = \Pr(a^j = r | x^i)$, an increase in x_r^i rise $E(L_r | x^i)$ as well. This implies that $E(\theta_r + L_r | x^i)$ increases with x^i .

Given that x_r^i is neutral to both $\theta_{-r} | x^i$ and $x_{-r}^j | x^i$ and opponents follow non-increasing strategy in cross signals, $E(\theta_{-r} + L_{-r} | x^i)$ is non-increasing in x^i .

This makes action r more attractive compared to both action 0 and $-r$. ■

Proof. (of Proposition 3)

Since the noise term is distributed iid., in the symmetric equilibrium, the probability that an agent chooses action r is equal to the aggregate number of agents who chooses that action. Thus given the strategy characterized in Proposition 2 and the conditional distribution $u_r \equiv x_r^j | x^i \sim N(x_r^i, \sqrt{2}\sigma_r)$ for $\forall j \neq i$ and $r \in \{A, B\}$, it can be easily shown that

$$E(L_r | x^i) = P(x_r^j > K^r(x_{-r}^j) | x^i) = \int_{-\infty}^{\infty} \int_{K^r(u_{-r})}^{\infty} f(u_A, u_B) du_r du_{-r}$$

where $f(u_A, u_B)$ is the joint pdf of $x_A^j | x^i$ and $x_B^j | x^i$. Let z_1 and z_2 such that $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim N(\mathbf{0}, \mathbf{I})$, thus by using the Cholesky decomposition $u_A = x_A^i + \sqrt{2}\sigma_A z_A$ and $u_B = x_B^i +$

$\sqrt{2}\sigma_B \left(\rho z_A + \sqrt{1-\rho^2} z_B \right)$ would hold. By introducing $D^r(z) = \frac{K^r(x_{-r}^i + \sqrt{2}\sigma_{-r}z) - x_{-r}^i - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}$, the equations can be rearranged as follows:

$$\begin{aligned} E(L_r | x^i) &= \int_{-\infty}^{\infty} \int_{K^r(u_{-r})}^{\infty} f(u_A, u_B) du_r du_{-r} = \int_{-\infty}^{\infty} \int_{D^r(z_{-r})}^{\infty} \phi(z_A, z_B) dz_r dz_{-r} \\ &= \int_{-\infty}^{\infty} \phi(z) [1 - \Phi(D^r(z))] dz = 1 - \int_{-\infty}^{\infty} \phi(z) \Phi(D^r(z)) dz \end{aligned}$$

where $\phi(z)$ and $\Phi(z)$ denote the pdf and the cdf, respectively, of the univariate standard normal distribution and $\phi(z_A, z_B)$ denotes the pdf of the bivariate standard normal distribution, where z_A and z_B are independent, implying that $\phi(z_A, z_B) = \phi(z_A)\phi(z_B)$. Using that $\theta_r | x^i \sim N(x_r^i, \sigma_r)$ for $r \in \{A, B\}$ results in

$$E(\theta_r + L_r | x^i) = E(\theta_r | x^i) + E(L_r | x^i) = x_r^i + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi(D^r(z)) dz \quad (22)$$

The definition of the cutoff function k^{r0} implies that whenever $x_r^i = k^{r0}(x_{-r}^i)$, $E(\theta_r + L_r | x^i) = n$ has to hold. Substituting this into (22) gives

$$k^{r0}(x_{-r}^i) + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^r(x_{-r}^i + \sqrt{2}\sigma_{-r}z) - k^{r0}(x_{-r}^i) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \right) dz = n$$

implying that

$$k^{r0}(x_{-r}^i) = n - 1 + \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^r(x_{-r}^i + \sqrt{2}\sigma_{-r}z) - k^{r0}(x_{-r}^i) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \right) dz \quad (23)$$

The definition of $k^{r(-r)}$ implies that with $x_r^i = k^{r(-r)}(x_{-r}^i)$, the $E(\theta_r + L_r | x^i) = E(\theta_{-r} + L_{-r} | x^i)$ is satisfied, thus combining this with equation (22) gives

$$\begin{aligned} &k^{r(-r)}(x_{-r}^i) + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^r(x_{-r}^i + \sqrt{2}\sigma_{-r}z) - k^{r(-r)}(x_{-r}^i) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \right) dz \\ &= x_{-r}^i + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^{-r}(k^{r(-r)}(x_{-r}^i) + \sqrt{2}\sigma_r z) - x_{-r}^i - \sqrt{2}\sigma_{-r}\rho z}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}} \right) dz \end{aligned}$$

which can be rearranged into the following form:

$$k^{r(-r)}(x_{-r}^i) = x_{-r}^i + \int_{-\infty}^{\infty} \phi(z) \left[\begin{array}{c} \Phi \left(\frac{K^r(x_{-r}^i + \sqrt{2}\sigma_{-r}z) - k^{r(-r)}(x_{-r}^i) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \right) \\ - \Phi \left(\frac{K^{-r}(k^{r(-r)}(x_{-r}^i) + \sqrt{2}\sigma_r z) - x_{-r}^i - \sqrt{2}\sigma_{-r}\rho z}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}} \right) \end{array} \right] dz \quad (24)$$

■

Proof. (of Proposition 4) By using Banach fixed point theorem, I show that the cutoff function triplet $k = \{k^{A0}, k^{B0}, k^{AB}\}$ given by Proposition 3 indeed exists and is unique. For this in Lemma 8 I show that the set of cutoff function triplets \mathcal{K} with some metric d is a complete metric space. In Lemma 9 I prove that the joint best response mapping $B : \mathcal{K} \rightarrow \mathcal{K}$ is a contraction map.

Lemma 8 (*Non-empty Complete Metric Space*). *The set of cutoff function triplet \mathcal{K} with some metric d is a complete metric space.*

First let me define the metric d for any two sets of functions $F = \{f_1, f_2, \dots, f_N\}$ and $G = \{g_1, g_2, \dots, g_N\}$ where both contain N number of functions, each with domain \mathbb{R} :

$$d(F, G) \equiv \max \left\{ \sup_{y \in \mathbb{R}} |f_1(y) - g_1(y)|, \sup_{y \in \mathbb{R}} |f_2(y) - g_2(y)|, \dots, \sup_{y \in \mathbb{R}} |f_N(y) - g_N(y)| \right\}$$

From (23) we can establish that $k^{r0} : \mathbb{R} \rightarrow [-1 + n, n]$ is bounded. While from (24) follows that $k^{r(-r)}(\mathbf{y}) - y : \mathbb{R} \rightarrow [-1, 1]$ is bounded. Hence (\mathcal{K}, d) is indeed a complete metric space.

Lemma 9 (*Contraction Map*). *The joint best response mapping $B : \mathcal{C} \rightarrow \mathcal{C}$ is a contraction map.*

Given that everybody has cutoffs $k \equiv \{k^{A0}, k^{B0}, k^{AB}\}$, the best response cutoffs of agent i is given by $B(k) = \{b^{A0}(k), b^{B0}(k), b^{AB}(k)\}$. Where from (23) and (24) we have for $r \in \{A, B\}$

$$b^{r0}(y, k) = n - 1 + \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \right) dz \quad (25)$$

$$b^{r(-r)}(y, k) = x_{-r}^i + \int_{-\infty}^{\infty} \phi(z) \left[\begin{array}{c} \Phi \left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r(-r)}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \right) \\ - \Phi \left(\frac{K^{-r}(k^{r(-r)}(y) + \sqrt{2}\sigma_r z) - y - \sqrt{2}\sigma_{-r}\rho z}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}} \right) \end{array} \right] dz \quad (26)$$

with $K^r(y) \equiv \max \{k^{r0}(y), k^{r(-r)}(y)\}$. In order to verify that the map is indeed a contraction map we have to show that $d(k, k') > d(B(k), B(k'))$. For this it is enough to show that both $d(k, k') > \sup_{y \in \mathbb{R}} |b^{r0}(y, k) - b^{r0}(y, k')|$ and $d(k, k') > \sup_{y \in \mathbb{R}} |b^{r(-r)}(y, k) - b^{r(-r)}(y, k')|$ hold if the sufficient conditions are fulfilled.

First let me concentrate on b^{r0}

$$\begin{aligned}
& \sup_{y \in \mathbb{R}} |b^{r0}(y, k) - b^{r0}(y, k')| \\
&= \sup_{y \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right) dz \right. \\
&\quad \left. - \int_{-\infty}^{\infty} \phi(z) \Phi \left(\frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r0'}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right) dz \right| \\
&= \sup_{y \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \phi(z) \left[\begin{array}{c} \Phi \left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right) \\ - \Phi \left(\frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r0'}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right) \end{array} \right] dz \right| \quad (27)
\end{aligned}$$

According to the mean value theorem there exists ξ s.t.:

$$\phi(\xi) = \frac{\Phi \left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right) - \Phi \left(\frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r0'}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right)}{\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} - \frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r0'}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}}}$$

Since $\max_{\xi \in \mathbb{R}} \phi(\xi) = \phi(0) = \frac{1}{\sqrt{2\pi}}$ we have

$$\begin{aligned}
& \phi(0) \frac{K^r(y + \sqrt{2}\sigma_{-r}z) - K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) + k^{r0'}(y)}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \\
&\geq \Phi \left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right) \\
&\quad - \Phi \left(\frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r0'}(y) - \sqrt{2}\sigma_r \rho z}{\sqrt{2}\sigma_r \sqrt{1-\rho^2}} \right)
\end{aligned}$$

Moreover for any y the inequality $-k^{r0}(y) + k^{r0'}(y) \leq \sup_{z \in \mathbb{R}} |k^{r0}(y) - k^{r0'}(y)|$ has to hold. But from the definition of metric d follows that $\sup_{y \in \mathbb{R}} |k^{r0}(y) - k^{r0'}(y)| \leq d(k, k')$, which implies $-k^{r0}(y) + k^{r0'}(y) \leq d(k, k')$. Similarly holds

$$\begin{aligned}
& K^r(y + \sqrt{2}\sigma_{-r}z) - K^{r'}(y + \sqrt{2}\sigma_{-r}z) \\
&\leq \sup_{y \in \mathbb{R}} |K^r(y) - K^{r'}(y)| \\
&\leq \max \left(\sup_{y \in \mathbb{R}} |k^{r0}(y) - k^{r0'}(y)|, \sup_{y \in \mathbb{R}} |k^{r(-r)}(y) - k^{r(-r)'}(y)| \right) \\
&\leq d(k, k')
\end{aligned}$$

for any y . These imply that

$$\begin{aligned} & \phi(0) \frac{2d(k, k')}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \\ & \geq \Phi\left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r0}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) \\ & \quad - \Phi\left(\frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r0'}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) \end{aligned}$$

Combining this with (27) gives:

$$\sup_{y \in \mathbb{R}} |b^{r0}(y, k) - b^{r0}(y, k')| \leq \sup_{y \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \phi(z) \left[\phi(0) \frac{2d(k, k')}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \right] dz \right| = \frac{d(k, k')}{\sqrt{\pi}\sigma_r\sqrt{1-\rho^2}}.$$

Thus $\sup_{z \in \mathbb{R}} |b^{r0}(z, k) - b^{r0}(z, k')| < d(k, k')$ holds if ρ and σ_r are such that $\frac{1}{\sigma_r} < \sqrt{\pi}\sqrt{1-\rho^2}$ (**condition 1**).

Now let me concentrate on $b^{r(-r)}$. Equation (26) gives

$$\begin{aligned} & \sup_{y \in \mathbb{R}} |b^{r(-r)}(y, k) - b^{r(-r)}(y, k')| \\ & = \sup_{y \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \phi(z) \left[\begin{aligned} & \Phi\left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r(-r)}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) \\ & - \Phi\left(\frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r(-r)'}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) \\ & - \Phi\left(\frac{K^{-r}(k^{r(-r)}(y) + \sqrt{2}\sigma_r z) - y - \sqrt{2}\sigma_{-r}\rho z}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}}\right) \\ & + \Phi\left(\frac{K^{-r'}(k^{r(-r)'}(y) + \sqrt{2}\sigma_r z) - y - \sqrt{2}\sigma_{-r}\rho z}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}}\right) \end{aligned} \right] dz \right| \end{aligned}$$

Similarly as above, by using the mean value theorem and the definition of K^r , one can show that

$$\begin{aligned} & \phi(0) \frac{2d(k, k')}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} \\ & \geq \Phi\left(\frac{K^r(y + \sqrt{2}\sigma_{-r}z) - k^{r(-r)}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) \\ & \quad - \Phi\left(\frac{K^{r'}(y + \sqrt{2}\sigma_{-r}z) - k^{r(-r)'}(y) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) \end{aligned}$$

and

$$\begin{aligned}
& \phi(0) \frac{d(k, k')}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}} \\
& \geq -\Phi\left(\frac{K^{-r}(k^{r(-r)}(y) + \sqrt{2}\sigma_r z) - y - \sqrt{2}\sigma_{-r}\rho z}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}}\right) \\
& \quad + \Phi\left(\frac{K^{-r'}(k^{r(-r)'}(y) + \sqrt{2}\sigma_r z) - y - \sqrt{2}\sigma_{-r}\rho z}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}}\right)
\end{aligned}$$

These imply that

$$\begin{aligned}
& \sup_{z \in \mathbb{R}} \left| b^{r(-r)}(y, k) - b^{r(-r)}(y, k') \right| \\
& \leq \int_{-\infty}^{\infty} \phi(z) \left[\phi(0) \frac{2d(k, k')}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} + \phi(0) \frac{d(k, k')}{\sqrt{2}\sigma_{-r}\sqrt{1-\rho^2}} \right] dz \\
& = \frac{d(k, k')}{\sqrt{\pi}\sqrt{1-\rho^2}} \left(\frac{1}{\sigma_r} + \frac{1}{2\sigma_{-r}} \right) \int_{-\infty}^{\infty} \phi(z) dz = \frac{d(k, k')}{\sqrt{\pi}\sqrt{1-\rho^2}} \frac{\sigma_r + 2\sigma_{-r}}{2\sigma_r\sigma_{-r}}
\end{aligned}$$

Thus if ρ , σ_r and σ_{-r} are such that $\frac{2\sigma_A + \sigma_B}{\sigma_A\sigma_B} < 2\sqrt{\pi}\sqrt{1-\rho^2}$ (**condition 2**), then $d(k, k') > \sup_{y \in \mathbb{R}} |b^{r(-r)}(y, k) - b^{r(-r)}(y, k')|$ holds. Since $\frac{2}{\sigma_B} < \frac{2\sigma_A + \sigma_B}{\sigma_A\sigma_B}$ **condition 1** is always satisfied whenever **condition 2** holds. Hence if $\frac{2\sigma_A + \sigma_B}{\sigma_A\sigma_B} < 2\sqrt{\pi}\sqrt{1-\rho^2}$, for all $r \in \{A, B\}$, then

$$\begin{aligned}
& (B(k), B(k')) \\
& = \max \left(\begin{array}{l} \sup_{y \in \mathbb{R}} |b^{A0}(y, k) - b^{A0}(y, k')|, \sup_{y \in \mathbb{R}} |b^{B0}(y, k) - b^{B0}(y, k')|, \\ \sup_{y \in \mathbb{R}} |b^{AB}(y, k) - b^{AB}(y, k')|, \sup_{y \in \mathbb{R}} |b^{BA}(y, k) - b^{BA}(y, k')| \end{array} \right) \\
& < d(k, k')
\end{aligned}$$

is fulfilled. So in this case the mapping is indeed a contraction.

■

Proof. (of Proposition 5) Assume that for some y_0 the inequality $k^{r0}(y_0) > n - 0.5$ does not hold. That is there exists some $\delta_0 \geq 0$ constant such that $k^{r0}(y_0) = n - 0.5 - \delta_0$.

Substituting it into (23) gives

$$n - 0.5 - \delta_0 = n - 1 + \int_{-\infty}^{\infty} \phi(z) \Phi\left(\frac{K^r(y_0 + \sqrt{2}\sigma_{-r}z) - (n - 0.5 - \delta_0) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) dz \quad (28)$$

According to the mean value theorem there exists ξ such that

$$\phi(\xi) = \frac{\Phi\left(\frac{K^r(y_0 + \sqrt{2}\sigma_{-r}z) - (n - 0.5 - \delta_0) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{-\rho z}{\sqrt{1-\rho^2}}\right)}{\frac{K^r(y_0 + \sqrt{2}\sigma_{-r}z) - (n - 0.5 - \delta_0) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} - \frac{-\rho z}{\sqrt{1-\rho^2}}}$$

thus

$$\begin{aligned} & \Phi\left(\frac{K^r(y_0 + \sqrt{2}\sigma_{-r}z) - (n - 0.5 - \delta_0) - \sqrt{2}\sigma_r\rho z}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}\right) \\ &= \phi(\xi) \frac{K^r(y_0 + \sqrt{2}\sigma_{-r}z) - (n - 0.5 - \delta_0)}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} + \Phi\left(\frac{-\rho z}{\sqrt{1-\rho^2}}\right) \end{aligned}$$

Combing it with (28) gives

$$\int_{-\infty}^{\infty} \phi(z) \left[\phi(\xi) \frac{K^r(x_{-r}^i + \sqrt{2}\sigma_{-r}z) - (n - 0.5 - \delta_0)}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} + \Phi\left(\frac{-\rho z}{\sqrt{1-\rho^2}}\right) \right] dz = 0.5 - \delta_0$$

Using that $\phi(z) = \phi(-z)$ and $\Phi(z) = 1 - \Phi(-z)$ implies that with any constant A : $\int_{-\infty}^{\infty} \phi(z) \Phi(Az) dz = \int_{-\infty}^0 \phi(-z) [1 - \Phi(-Az)] + \int_0^{\infty} \phi(z) \Phi(Az) dz = \int_0^{\infty} \phi(z) dz = 0.5$. Hence $\int_{-\infty}^{\infty} \phi(z) \Phi\left(\frac{-\rho z}{\sqrt{1-\rho^2}}\right) dz = 0.5$, and thus $\int_{-\infty}^{\infty} \phi(z) \phi(\xi) \frac{K^r(y_0 + \sqrt{2}\sigma_{-r}z) - (n - 0.5 - \delta_0)}{\sqrt{2}\sigma_r\sqrt{1-\rho^2}} dz = -\delta_0$.

Rearrangement gives $\int_{-\infty}^{\infty} \phi(z) K^r(y_0 + \sqrt{2}\sigma_{-r}z) dz = -\delta_0 \left(\frac{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}{\phi(\xi)} + 1 \right) + n - 0.5$, which implies that for some y_1 the following has to hold

$$K^r(y_1) \leq -\delta_0 \left(1 + \frac{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}{\phi(\xi)} \right) - 0.5 + n$$

Using condition and that $\max_{\xi \in \mathbb{R}} \phi(\xi) = \phi(0) = \frac{1}{\sqrt{2\pi}}$, the inequality $\frac{\sqrt{2}\sigma_r\sqrt{1-\rho^2}}{\phi(\xi)} < 2$ has to hold. Moreover $K^r(y_A) \equiv \max\{k^{r_0}(y_1), k^{r(-r)}(y_1)\}$, thus

$$k^{r_0}(y_1) < -0.5 + n - 3\delta_0$$

That is there exists some constant $\delta_1 > 3\delta_0 \geq 0$ constant such that $k^{r_0}(y_1) = n - 0.5 - \delta_1$. Doing the same transformations as above for y_1 gives that for some y_2 there exists $\delta_2 > 3\delta_1$ such that $k^{r_0}(y_2) = n - 0.5 - \delta_2$. Similarly repeating the same steps m times provides $k^{r_0}(y_m) = n - 0.5 - \delta_m$ with $\delta_m > 3\delta_{m-1} > 3^{m-1}\delta_1 > 0$. This holds for any m and since δ_1 is a positive constant, $\lim_{m \rightarrow \infty} k^{r_0}(y_m) = -\infty$. But given that $k^{r_0}(y_m) \in [-1 + n, n]$ it is a contradiction. Consequently there cannot be y_0 such that the inequality $k^{r_0}(y_0) > n - 0.5$ does not hold. ■

Proof. (of Proposition 6)

Since the noise term is distributed iid., in the symmetric equilibrium, the probability that an agent chooses action r is equal to the aggregate number of agents who chooses that action. Thus given the strategy characterized in Proposition 2 and the conditional distribution $u \equiv x^j | x^i \sim N(\bar{\mu}_{x^j}, \bar{\Sigma}_{x^j})$ for $\forall j \neq i$, it can be easily shown that

$$E(L_r | x^i) = P(x_r^j > K^r(x_{-r}^j) | x^i) = \int_{-\infty}^{\infty} \int_{K^r(u_{-r})}^{\infty} f(u_A, u_B) du_r du_{-r}$$

where $f(u_A, u_B)$ is the joint pdf of $x_A^j | x^i$ and $x_B^j | x^i$. Let z_A and z_B such that $\begin{bmatrix} z_A \\ z_B \end{bmatrix} \sim N(\mathbf{0}, \mathbf{I})$, thus by using the Cholesky decomposition z_A and z_B can be set such that $u_B = H_{BA}x_A^i + (1 - H_{BA})y_A + H_{BB}x_B^i + (1 - H_{BB})y_B + \sigma_B\sqrt{1 + H_{BB}}z_B$ and $u_A = H_{AA}x_A^i + (1 - H_{AA})y_A + H_{AB}x_B^i + (1 - H_{AB})y_B + \frac{\sigma_A}{\sqrt{1 + H_{BB}}}(\sqrt{H_{AB}H_{BA}}z_B + \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}z_A)$ would hold. By introducing

$$D^A(z, x_A^i, x_B^i) = \frac{\sqrt{1 + H_{BB}}}{\sigma_A\sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} \cdot \left\{ K^A \left(\sigma_B\sqrt{1 + H_{BB}}z + H_{BA}x_A^i + (1 - H_{BA})y_A + H_{BB}x_B^i + (1 - H_{BB})y_B \right) - H_{AA}x_A^i - (1 - H_{AA})y_A - H_{AB}x_B^i - (1 - H_{AB})y_B - \frac{\sigma_A\sqrt{H_{AB}H_{BA}}z}{\sqrt{1 + H_{BB}}} \right\}$$

the equation can be rearranged as follows:

$$\begin{aligned} E(L_A | x^i) &= \int_{-\infty}^{\infty} \int_{K^A(u_B)}^{\infty} f(u_A, u_B) du_A du_B = \int_{-\infty}^{\infty} \int_{D^A(z_B)}^{\infty} \phi(z_A, z_B) dz_A dz_B \\ &= \int_{-\infty}^{\infty} \phi(z) [1 - \Phi(D^A(z, x_A^i, x_B^i))] dz = 1 - \int_{-\infty}^{\infty} \phi(z)\Phi(D^A(z, x_A^i, x_B^i)) dz \end{aligned}$$

where $\phi(z)$ and $\Phi(z)$ denote the pdf and the cdf, respectively, of the univariate standard normal distribution and $\phi(z_A, z_B)$ denotes the pdf of the bivariate standard normal distribution, where z_A and z_B are independent, implying that $\phi(z_A, z_B) = \phi(z_A)\phi(z_B)$. Using that $\theta | x^i \sim N(\bar{\mu}_\theta, \bar{\Sigma}_\theta)$ results in

$$\begin{aligned} E(\theta_A + L_A | x^i) &= E(\theta_A | x^i) + E(L_A | x^i) \\ &= H_{AA}x_A^i + (1 - H_{AA})y_A + H_{AB}x_B^i + (1 - H_{AB})y_B \\ &\quad + 1 - \int_{-\infty}^{\infty} \phi(z)\Phi(D^A(z, x_A^i, x_B^i)) dz \end{aligned} \tag{29}$$

The definition of the cutoff function k^{A0} implies that whenever $x_A^i = k^{A0}(x_B^i)$, $E(\theta_A + L_A | x^i) = n$ has to hold. Substituting this into (29) gives

$$H_{AA}k^{A0}(x_B^i) + (1 - H_{AA})y_A + H_{AB}x_B^i + (1 - H_{AB})y_B + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi(D^A(z, k^{A0}(x_B^i), x_B^i)) dz = n$$

implying that

$$k^{A0}(x_B^i) = \frac{1}{H_{AA}} \left\{ n - 1 - (1 - H_{AA})y_A - H_{AB}x_B^i - (1 - H_{AB})y_B + \int_{-\infty}^{\infty} \phi(z) \Phi(D^A(z, k^{A0}(x_B^i), x_B^i)) dz \right\} \quad (30)$$

One can similarly show that

$$E(\theta_B + L_B | x^i) = H_{BB}x_B^i + (1 - H_{BB})y_B + H_{BA}x_A^i + (1 - H_{BA})y_A + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi(D^B(z, x_B^i, x_A^i)) dz \quad (31)$$

and thus

$$k^{B0}(x_A^i) = \frac{1}{H_{BB}} \left\{ n - 1 - (1 - H_{BB})y_B - H_{BA}x_A^i - (1 - H_{BA})y_A + \int_{-\infty}^{\infty} \phi(z) \Phi(D^B(z, k^{B0}(x_A^i), x_A^i)) dz \right\} \quad (32)$$

The definition of k^{AB} implies that with $x_A^i = k^{AB}(x_B^i)$, the equation $E(\theta_A + L_A | x^i) = E(\theta_B + L_B | x^i)$ is satisfied, thus combining this with equations (29) and (31) gives

$$\begin{aligned} & H_{AA}k^{AB}(x_B^i) + (1 - H_{AA})y_A + H_{AB}x_B^i + (1 - H_{AB})y_B \\ & + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi(D^A(z, k^{AB}(x_B^i), x_B^i)) dz \\ = & H_{BB}x_B^i + (1 - H_{BB})y_B + H_{BA}k^{AB}(x_B^i) + (1 - H_{BA})y_A \\ & + 1 - \int_{-\infty}^{\infty} \phi(z) \Phi(D^B(z, x_B^i, k^{AB}(x_B^i))) dz \end{aligned}$$

which can be rearranged into the following form:

$$k^{AB}(x_B^i) = \frac{H_{AB} - H_{BB}}{H_{AA} - H_{BA}}(y_B - x_B^i) + y_A + \frac{1}{H_{AA} - H_{BA}} \cdot \int_{-\infty}^{\infty} \phi(z) [\Phi(D^A(z, k^{AB}(x_B^i), x_B^i)) - \Phi(D^B(z, x_B^i, k^{AB}(x_B^i)))] dz \quad (33)$$

One can similarly show that

$$k^{BA}(x_A^i) = \frac{H_{BA} - H_{AA}}{H_{BB} - H_{AB}}(y_A - x_A^i) + y_B + \frac{1}{H_{BB} - H_{AB}} \cdot \int_{-\infty}^{\infty} \phi(z) [\Phi(D^B(z, k^{BA}(x_A^i), x_A^i)) - \Phi(D^A(z, x_A^i, k^{BA}(x_A^i)))] dz \quad (34)$$

■

Proof. (of Proposition 7) By using Banach fixed point theorem, I show that the cutoff function triplet $k = \{k^{A0}, k^{B0}, k^{AB}\}$ given by Proposition 6 indeed exists and is unique. For this in Lemma 10 I show that the set of cutoff function triplets \mathcal{K} with some metric d is a complete metric space. In Lemma 11 I prove that the joint best response mapping $B : \mathcal{K} \rightarrow \mathcal{K}$ is a contraction map.

Lemma 10 (*Non-empty Complete Metric Space 2*). *The set of cutoff function triplet \mathcal{K} with some metric d is a complete metric space.*

First let me define the metric d for any two sets of functions $F = \{f_1, f_2, \dots, f_N\}$ and $G = \{g_1, g_2, \dots, g_N\}$ where both contain N number of functions, each with domain \mathbb{R} :

$$d(F, G) \equiv \max \left\{ \sup_{y \in \mathbb{R}} |f_1(y) - g_1(y)|, \sup_{y \in \mathbb{R}} |f_2(y) - g_2(y)|, \dots, \sup_{y \in \mathbb{R}} |f_N(y) - g_N(y)| \right\}$$

From (30) we can establish that $k^{\mathbf{r}0} : \mathbb{R} \rightarrow [-1 + n, n]$ is bounded. While from (33) follows that $k^{\mathbf{r}(-\mathbf{r})}(\mathbf{y}) - \mathbf{y} : \mathbb{R} \rightarrow [-1, 1]$ is bounded. Hence (\mathcal{K}, d) is indeed a complete metric space.

Lemma 11 (*Contraction Map 2*). *The joint best response mapping $B : \mathcal{C} \rightarrow \mathcal{C}$ is a contraction map.*

Given that everybody has cutoffs $k \equiv \{k^{A0}, k^{B0}, k^{AB}\}$, the best response cutoffs of agent i is given by $B(k) = \{b^{A0}(k), b^{B0}(k), b^{AB}(k)\}$. Where from (30), (32) and (33) we have

$$b^{A0}(v, k) = \frac{1}{H_{AA}} \left\{ n - 1 - (1 - H_{AA})y_A - H_{AB}v - (1 - H_{AB})y_B + \int_{-\infty}^{\infty} \phi(z) \Phi(D^A(z, k^{A0}(v), v)) dz \right\} \quad (35)$$

$$b^{B0}(v, k) = \frac{1}{H_{BB}} \left\{ n - 1 - (1 - H_{BB})y_B - H_{BA}v - (1 - H_{BA})y_A + \int_{-\infty}^{\infty} \phi(z) \Phi(D^B(z, k^{B0}(v), v)) dz \right\} \quad (36)$$

$$\begin{aligned}
b^{AB}(v, k) &= \frac{H_{AB} - H_{BB}}{H_{AA} - H_{BA}}(y_B - v) + y_A + \frac{1}{H_{AA} - H_{BA}} \\
&\cdot \int_{-\infty}^{\infty} \phi(z) [\Phi(D^A(z, k^{AB}(v), v)) - \Phi(D^B(z, v, k^{AB}(v)))] dz \quad (37)
\end{aligned}$$

with $K^A(v) \equiv \max\{k^{A0}(v), k^{AB}(v)\}$ and $K^B(v) \equiv \max\{k^{B0}(v), k^{BA}(v)\}$. In order to verify that the map is indeed a contraction map we have to show that $d(k, k') > d(B(k), B(k'))$. For this it is enough to show that $d(k, k') > \sup_{v \in \mathbb{R}} |b^{A0}(v, k) - b^{A0}(v, k')|$, $d(k, k') > \sup_{v \in \mathbb{R}} |b^{B0}(v, k) - b^{B0}(v, k')|$ and $d(k, k') > \sup_{v \in \mathbb{R}} |b^{AB}(v, k) - b^{AB}(v, k')|$ hold if the sufficient conditions are fulfilled.

First let me concentrate on b^{A0}

$$\begin{aligned}
&\sup_{v \in \mathbb{R}} |b^{A0}(v, k) - b^{A0}(v, k')| \\
&= \sup_{v \in \mathbb{R}} \left| \frac{\int_{-\infty}^{\infty} \phi(z) \Phi(D^A(z, k^{A0}(v), v)) dz}{H_{AA}} - \frac{\int_{-\infty}^{\infty} \phi(z) \Phi(D^A(z, k^{A0'}(v), v)) dz}{H_{AA}} \right| \\
&= \frac{1}{H_{AA}} \cdot \sup_{v \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \phi(z) [\Phi(D^A(z, k^{A0}(v), v)) - \Phi(D^A(z, k^{A0'}(v), v))] dz \right| \quad (38)
\end{aligned}$$

According to the mean value theorem there exists ξ s.t.:

$$\phi(\xi) = \frac{\Phi(D^A(z, k^{A0}(v), v)) - \Phi(D^A(z, k^{A0'}(v), v))}{D^A(z, k^{A0}(v), v) - D^A(z, k^{A0'}(v), v)}$$

Since $\max_{\xi \in \mathbb{R}} \phi(\xi) = \phi(0) = \frac{1}{\sqrt{2\pi}}$ we have

$$\begin{aligned}
&\Phi(D^A(z, k^{A0}(v), v)) - \Phi(D^A(z, k^{A0'}(v), v)) \\
&\leq \phi(0) [D^A(z, k^{A0}(v), v) - D^A(z, k^{A0'}(v), v)] \\
&= \frac{\phi(0) \sqrt{1 + H_{BB}}}{\sigma_A \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} \cdot \left\{ H_{AA} [-k^{A0}(v) + k^{A0'}(v)] \right. \\
&\quad \left. + K^A \left(\sigma_B \sqrt{1 + H_{BB}} z + H_{BA} k^{A0}(v) + (1 - H_{BA}) y_A + H_{BB} v + (1 - H_{BB}) y_B \right) \right. \\
&\quad \left. - K^{A'} \left(\sigma_B \sqrt{1 + H_{BB}} z + H_{BA} k^{A0'}(v) + (1 - H_{BA}) y_A + H_{BB} v + (1 - H_{BB}) y_B \right) \right\}
\end{aligned}$$

Moreover for any v the inequality $-k^{A0}(v) + k^{A0'}(v) \leq \sup_{y \in \mathbb{R}} |k^{A0}(v) - k^{A0'}(v)|$ has to hold. But from the definition of metric d follows that $\sup_{v \in \mathbb{R}} |k^{A0}(v) - k^{A0'}(v)| \leq d(k, k')$,

which implies $-k^{A0}(v) + k^{A0'}(v) \leq d(k, k')$. Similarly holds

$$\begin{aligned}
& K^A \left(\sigma_B \sqrt{1 + H_{BB}z} + H_{BA} k^{A0}(v) + (1 - H_{BA})y_A + H_{BB}v + (1 - H_{BB})y_B \right) \\
& - K^{A'} \left(\sigma_B \sqrt{1 + H_{BB}z} + H_{BA} k^{A0'}(v) + (1 - H_{BA})y_A + H_{BB}v + (1 - H_{BB})y_B \right) \\
& \leq \sup_{v \in \mathbb{R}} |K^A(v) - K^{A'}(v)| \\
& \leq \max \left(\sup_{v \in \mathbb{R}} |k^{A0}(v) - k^{A0'}(v)|, \sup_{v \in \mathbb{R}} |k^{AB}(v) - k^{AB'}(v)| \right) \\
& \leq d(k, k')
\end{aligned}$$

for any v . These imply that

$$\Phi(D^A(z, k^{A0}(v), v)) - \Phi(D^A(z, k^{A0'}(v), v)) \leq \frac{\phi(0)\sqrt{1 + H_{BB}}(1 + H_{AA})}{\sigma_A \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} d(k, k')$$

Combining this with (38) gives:

$$\begin{aligned}
& \sup_{v \in \mathbb{R}} |b^{A0}(v, k) - b^{A0}(v, k')| \\
& \leq \frac{1}{H_{AA}} \cdot \sup_{v \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \phi(z) \left[\frac{\phi(0)\sqrt{1 + H_{BB}}(1 + H_{AA})}{\sigma_A \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} d(k, k') \right] dz \right| \\
& = \frac{\sqrt{1 + H_{BB}}(1 + H_{AA})}{\sqrt{2\pi}\sigma_A H_{AA} \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} d(k, k').
\end{aligned}$$

Thus $\sup_{z \in \mathbb{R}} |b^{A0}(z, k) - b^{A0}(z, k')| < d(k, k')$ holds if the parameters are such that $\frac{\sqrt{1 + H_{BB}}(1 + H_{AA})}{\sqrt{2\pi}\sigma_A H_{AA} \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} < 1$ (**condition 3A**).

One can similarly show that $\sup_{z \in \mathbb{R}} |b^{B0}(z, k) - b^{B0}(z, k')| < d(k, k')$ holds if the parameters are such that $\frac{\sqrt{1 + H_{AA}}(1 + H_{BB})}{\sqrt{2\pi}\sigma_B H_{BB} \sqrt{(1 + H_{AA})(1 + H_{BB}) - H_{AB}H_{BA}}} < 1$ (**condition 3B**).

Now let me concentrate on b^{AB} . Equation (37) gives

$$\begin{aligned}
& \sup_{v \in \mathbb{R}} |b^{AB}(v, k) - b^{AB}(v, k')| \\
& = \sup_{v \in \mathbb{R}} \left| \frac{\int_{-\infty}^{\infty} \phi(z) [\Phi(D^A(z, k^{AB}(v), v)) - \Phi(D^A(z, k^{AB'}(v), v))] dz}{\int_{-\infty}^{\infty} \phi(z) [\Phi(D^B(z, v, k^{AB}(v))) - \Phi(D^B(z, v, k^{AB'}(v)))] dz} \right| \\
& = \frac{1}{H_{AA} - H_{BA}} \cdot \sup_{v \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \phi(z) \left[\frac{\Phi(D^A(z, k^{AB}(v), v)) - \Phi(D^A(z, k^{AB'}(v), v))}{+\Phi(D^B(z, v, k^{AB}(v))) - \Phi(D^B(z, v, k^{AB'}(v)))} \right] dz \right|
\end{aligned}$$

Similarly as above, by using the mean value theorem, one can show that

$$\Phi(D^A(z, k^{AB}(v), v)) - \Phi(D^A(z, k^{AB'}(v), v)) \leq \frac{\phi(0)\sqrt{1+H_{BB}}(1+H_{AA})}{\sigma_A\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}}d(k, k')$$

and

$$\Phi(D^B(z, v, k^{AB}(v))) - \Phi(D^B(z, v, k^{AB'}(v))) \leq \frac{\phi(0)\sqrt{1+H_{AA}}(1+H_{BA})}{\sigma_B\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}}d(k, k')$$

These imply that

$$\begin{aligned} & \sup_{v \in \mathbb{R}} |b^{AB}(v, k) - b^{AB}(v, k')| \\ & \leq \frac{1}{H_{AA} - H_{BA}} \int_{-\infty}^{\infty} \phi(z) \left[\frac{\phi(0)\sqrt{1+H_{BB}}(1+H_{AA})}{\sigma_A\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}}d(k, k') \right. \\ & \quad \left. + \frac{\phi(0)\sqrt{1+H_{AA}}(1+H_{BA})}{\sigma_B\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}}d(k, k') \right] dz \\ & = \frac{1}{H_{AA} - H_{BA}} \left[\frac{\phi(0)\sqrt{1+H_{BB}}(1+H_{AA})}{\sigma_A\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}}d(k, k') \right. \\ & \quad \left. + \frac{\phi(0)\sqrt{1+H_{AA}}(1+H_{BA})}{\sigma_B\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}}d(k, k') \right] \int_{-\infty}^{\infty} \phi(z) dz \\ & = \frac{\sigma_A\sqrt{1+H_{AA}}(1+H_{BA}) + \sigma_B\sqrt{1+H_{BB}}(1+H_{AA})}{\sqrt{2\pi}\sigma_A\sigma_B(H_{AA} - H_{BA})\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}}d(k, k') \end{aligned}$$

Thus if the parameters are such that $\frac{\sigma_A\sqrt{1+H_{AA}}(1+H_{BA}) + \sigma_B\sqrt{1+H_{BB}}(1+H_{AA})}{\sqrt{2\pi}\sigma_A\sigma_B(H_{AA} - H_{BA})\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}} < 1$ (**condition 4A**), then $d(k, k') > \sup_{y \in \mathbb{R}} |b^{AB}(y, k) - b^{AB}(y, k')|$ holds.

One can similarly show that $\sup_{z \in \mathbb{R}} |b^{BA}(z, k) - b^{BA}(z, k')| < d(k, k')$ holds if the parameters are such that $\frac{\sigma_B\sqrt{1+H_{BB}}(1+H_{AB}) + \sigma_A\sqrt{1+H_{AA}}(1+H_{BB})}{\sqrt{2\pi}\sigma_A\sigma_B(H_{BB} - H_{AB})\sqrt{(1+H_{AA})(1+H_{BB})-H_{AB}H_{BA}}} < 1$ (**condition 4B**).

Note that **condition 3A** and **condition 3B** are always satisfied whenever **condition 4A** and **condition 4B**, respectively, hold. Hence if $\frac{2\sigma_A + \sigma_B}{\sigma_A\sigma_B} < 2\sqrt{\pi}\sqrt{1-\rho^2}$, for all $r \in \{A, B\}$, then

$$\begin{aligned} & (B(k), B(k')) \\ & = \max \left(\begin{array}{l} \sup_{y \in \mathbb{R}} |b^{A0}(y, k) - b^{A0}(y, k')|, \sup_{y \in \mathbb{R}} |b^{B0}(y, k) - b^{B0}(y, k')|, \\ \sup_{y \in \mathbb{R}} |b^{AB}(y, k) - b^{AB}(y, k')|, \sup_{y \in \mathbb{R}} |b^{BA}(y, k) - b^{BA}(y, k')| \end{array} \right) \\ & < d(k, k') \end{aligned}$$

is fulfilled. So in this case the mapping is indeed a contraction.

■