ON \( \mathcal{M}_p \)-EMBEDDED PRIMARY SUBGROUPS OF FINITE GROUPS∗

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Abstract

A subgroup \( H \) of \( G \) is called \( \mathcal{M}_p \)-embedded in \( G \), if there exists a \( p \)-nilpotent subgroup \( B \) of \( G \) such that \( H_p \in \text{Syl}_p(B) \) and \( B \) is \( \mathcal{M}_p \)-supplemented in \( G \). In this paper, we use \( \mathcal{M}_p \)-embedded subgroups to study the structure of finite groups.

1. Introduction

All groups considered in this paper will be finite. We shall adhere to the notation employed in [3, 5]. In particular, let \(|G|\) denote the order of a group \( G \), and let \( \pi(G) \) denote the set of all prime divisors of \(|G|\). In addition, \( \mathcal{U} \) denotes the class of all supersoluble groups.

The topic of the embedding properties of subgroups is one of the most fruitful fields in Finite Group Theory. This idea provides a new approach to characterize the structure of a group by borrowing some local properties of subgroups. By using the embedding properties of subgroups, many scholars have deeply studied the structure of a group. For instance, in 1998, Ballester-Bolinches and Pedraza–Aguilera [2] introduced the following definition: A subgroup \( H \) of \( G \) is \( S \)-quasinormally embedded (normally embedded) in \( G \) if, for every Sylow subgroup \( P \) of \( H \), there is an \( S \)-quasinormal

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(normal) subgroup $K$ in $G$ such that $P$ is also a Sylow subgroup of $K$. Furthermore, they proved that if all maximal subgroups of all Sylow subgroups of $G$ are $S$-quasinormally embedded in $G$, then $G$ is supersoluble. Further, in 2001, Asaad and Heliel [1] obtained an interesting result about $p$-nilpotency of a group. Moreover, in 2009, Shemetkov and Skiba [10] introduced $\mathcal{F}\Phi$-hypercentral subgroup of a group. In 2012, Guo and Skiba [7] stated $\mathcal{F}\Phi^*$-hypercentral subgroups of a group. They obtained some new results about the structure of finite groups by using these embedding properties of subgroups.

It is well known that the generalized supplementation of subgroups is significant to determine the structure of a group. In 2007, Skiba [12] defined weakly $s$-permutable subgroups and studied the structure of a group. In 2008, Guo [6] gave the concept of $\mathcal{F}$-supplemented subgroups and investigated solubility and supersolubility of a group. In 2009, Miao and Lempken [8] considered $\mathcal{M}$-supplemented subgroups of a group and obtained some characterizations of saturated formations containing all supersoluble groups. In 2009, Monakhov and Shnyparkov [11] introduced the definition of $\mathcal{M}_p$-supplemented subgroups and obtained some interesting results about $p$-supersoluble groups.

As a continuation of above work, naturally, we should consider a question that we construct new embedding properties of subgroups by using the generalized supplementation of subgroups to study the structure of a group. Based on this point, we will introduce the following concept of $\mathcal{M}_p$-embedded subgroups which is closely related to $\mathcal{M}_p$-supplementation. Our motivation is to investigate the construction of a group extensively and obtain some new characterizations about $p$-supersolubility and supersolubility with the embedding property of subgroups.

**Definition 1.1.** Let $\pi$ be a set of primes. A subgroup $H$ of a group $G$ is called $\mathcal{M}_\pi$-supplemented in $G$, if there exists a subgroup $B$ of $G$ such that $G = HB$ and $H_1B < G$ for every maximal subgroup $H_1$ of $H$ with $\pi(|H : H_1|) \subseteq \pi$. In particular, if $\pi = \{p\}$, then $H$ is called $\mathcal{M}_p$-supplemented in $G$.

**Definition 1.2.** A subgroup $H$ of $G$ is called $\mathcal{M}_p$-embedded in $G$, if there exists a $p$-nilpotent subgroup $B$ of $G$ such that $H_p \in \text{Syl}_p(B)$ and $B$ is $\mathcal{M}_p$-supplemented in $G$.

It is not necessary for an $\mathcal{M}_p$-embedded subgroup to be $\mathcal{M}_p$-supplemented.

**Example 1.3.** Consider the group $G = A_5$. Let $H = \langle (12345), (15)(24) \rangle$ with $|H| = 10$ and $B = \langle (12345) \rangle$ a Sylow 5-subgroup of $H$.

Clearly, $B$ is $\mathcal{M}_5$-supplemented in $G$ and hence $H$ is $\mathcal{M}_5$-embedded in $G$. Furthermore, it is easy to verify that $H$ is not $\mathcal{M}_5$-supplemented in $G$.  

2. Preliminaries

For the sake of convenience, we first list here some known results which will be useful in the sequel.

**Lemma 2.1.** Let $G$ be a group. Then

1. Let $N \leq G$ and $N \leq H$. If $H$ is $\mathcal{M}_p$-embedded in $G$, then $H/N$ is $\mathcal{M}_p$-embedded in $G/N$.

2. Let $\pi$ be a set of primes. Let $N$ be a normal $\pi'$-subgroup and $H$ be a $\pi$-subgroup of $G$. If $H$ is $\mathcal{M}_p$-embedded in $G$, then $HN/N$ is $\mathcal{M}_p$-embedded in $G/N$.

**Proof.** The claims are easy exercises left to the reader. \(\square\)

**Lemma 2.2** (see [9, Lemma 2.6]). If $H$ is a subgroup of a group $G$ with $|G : H| = p$, where $p$ is the smallest prime divisor of $|G|$, then $H \leq G$.

**Lemma 2.3** (see [5, Gaschütz]). Let $G$ be a group. Suppose that $N$ and $D$ are normal subgroups of $G$, and also $D \leq N$, $D \leq \Phi(G)$. Then $N/D$ is nilpotent if and only if $N$ is nilpotent.

**Lemma 2.4** (see [5, Theorem 1.8.17]). Let $N$ be a nontrivial soluble normal subgroup of a group $G$. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of $N$ is the direct product of minimal normal subgroups of $G$ which are contained in $N$.

**Lemma 2.5.** Let $R$ be a soluble minimal normal subgroup of a group $G$. If there exists a maximal subgroup $R_1$ of $R$ such that $R_1$ is $\mathcal{M}_p$-embedded in $G$, then $R$ is a cyclic group of prime order.

**Proof.** Since $R$ is a soluble minimal normal subgroup of $G$, $R$ is an elementary abelian $p$-group for some prime $p \in \pi(G)$. By hypothesis, $R_1$ is $\mathcal{M}_p$-embedded in $G$, then there exists a $p$-nilpotent subgroup $B$ of $G$ such that $R_1 \in \text{Syl}_p(B)$ and $B$ is $\mathcal{M}_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$ such that $G = BK = R_1B_{p'}K$ where $B_{p'}$ is the normal $p$-complement of $B$ and $TB_{p'}K < G$ for every maximal subgroup $T$ of $R_1$. It follows from $R$ is a minimal normal subgroup of $G$ that $R \leq TB_{p'}K$ or $R \cap TB_{p'}K = 1$. If $R \leq TB_{p'}K$, then $TB_{p'}K = RTB_{p'}K = BK = G$, a contradiction. If $R \cap TB_{p'}K = 1$, then we have $|G : TB_{p'}K| = p$ and hence $|R| = p$. \(\square\)

**Lemma 2.6** (see [11, Lemma 4]). Let $H$ be a $\mathcal{M}_\pi$-supplemented subgroup in a group $G$ and $B$ be a $\mathcal{M}_\pi$-supplement to $H$. If $H_1$ is a maximal subgroup in $H$ and $\pi(H : H_1) \subseteq \pi$, then $|G : H_1B| = |H : H_1|$.

**Lemma 2.7** (see [13, Theorem 3.1]). Let $F$ be a saturated formation containing $\mathcal{U}$, $G$ a group with a soluble normal subgroup $H$ such that $G/H \in F$. If for every maximal subgroup $M$ of $G$, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in F$. The converse also holds, in the case where $F = \mathcal{U}$.
Lemma 2.8 (see [11, Theorem 1]). Let $p$ be the smallest prime divisor of $|G|$ and $H$ be a $p$-nilpotent subgroup containing a Sylow $p$-subgroup of $G$. If $H$ is $\mathcal{M}_p$-supplemented in $G$, then $G$ is $p$-nilpotent.

Lemma 2.9 (see [11, Theorem 2]). Let $G$ be a group, $\pi(G) = \{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 = p < \cdots < p_n$ and $H$ be a $p$-nilpotent subgroup containing a Sylow $p$-subgroup of $G$. If $H$ is $\mathcal{M}_p$-supplemented in $G$, then $G$ is $p$-supersolvable.

Lemma 2.10 (see [4]). Let $G$ be a group and $N$ a subgroup of $G$. The generalized Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasinilpotent subgroup of $G$. Then

1. If $N$ is normal in $G$, then $F^*(N) \leq F^*(G)$.
2. $F^*(G) \neq 1$ if $G \neq 1$; in fact,

$$F^*(G)/F(G) = \text{Soc} \left( F(G)C_G(F(G))/F(G) \right).$$

3. $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.
4. $C_G(F^*(G)) \leq F(G)$.
5. If $P \trianglelefteq G$ with $P \leq O_p(G)$, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.
6. If $K \leq Z(G)$, then $F^*(G/K) = F^*(G)/K$.

3. Main results

Theorem 3.1. Let $G$ be a $p$-soluble group and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is $\mathcal{M}_p$-embedded in $G$, then $G$ is $p$-supersolvable.

Proof. Assume that the assertion is false and choose $G$ to be a counterexample of minimal order. Furthermore, we have

1. $O_p'(G) = 1$.
   If $L = O_p'(G) \neq 1$, we consider the factor group $G/L$. By Lemma 2.1(2), $G/L$ satisfies the condition of the theorem, the minimal choice of $G$ implies that $G/L$ is $p$-supersolvable, and hence $G$ is $p$-supersolvable, a contradiction.
2. $O_p(G) \neq 1$.
   Since $G$ is $p$-soluble and $O_p(G) = 1$, we have that the minimal normal subgroup of $G$ is abelian $p$-group and hence $O_p(G) \neq 1$.
3. Final contradiction.
   By (2), we pick a minimal normal subgroup $N$ of $G$ contained in $O_p(G)$. By Lemma 2.1(1), we know that $G/N$ satisfies the hypothesis of the theorem, the minimal choice of $G$ implies that $G/N$ is $p$-supersolvable. On the other
hand, since the class of all $p$-supersoluble groups is a saturated formation, we have $N$ is the unique minimal normal subgroup of $G$ contained in $O_p(G)$ and $N = O_p(G) = F(G) \nleq \Phi(G) = 1$ by Lemma 2.4. Clearly, by [3, Theorem A.9.2], there exists a maximal subgroup $M$ of $G$ such that $G = NM$ and $N \cap M = 1$, $P = NM_p$ where $M_p$ is a Sylow $p$-subgroup of $M$. We may choose a maximal subgroup $P_1$ with $N \leq P_1$, then $M_p \nleq P_1$. By hypothesis, $P_1$ is $M_p$-embedded in $G$, there exists a $p$-nilpotent subgroup $B$ of $G$ such that $P_1 \in \text{Syl}_p(B)$ and $B$ is $M_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$, $G = BK = P_1B_p'K$ where $B_p'$ is the normal $p$-complement of $B$ and $TB_p'K < G$ for every maximal subgroup $T$ of $P_1$. If $N \leq \Phi(P_1)$, then $N \leq \Phi(G)$ and hence $G$ is $p$-supersoluble, a contradiction. Then there exists a maximal subgroup $T'$ of $P_1$ such that $N \nleq T'$. Since $T^*B_p'K$ is a maximal subgroup of $G$, $N \leq T^*B_p'K$ or $N \cap T^*B_p'K = 1$. If $N \leq T^*B_p'K$, then $T^*B_p'K = NT^*B_p'K \leq P_1B_p'K = G$, a contradiction. On the other hand, if $N \cap T^*B_p'K = 1$, then $|N| = p$ and hence $G$ is $p$-supersoluble.

The final contradiction completes the proof. □

COROLLARY 3.2. Let $G$ be a group and $P$ a Sylow $p$-subgroup of $G$ where $p$ is the smallest prime divisor of $|G|$. If every maximal subgroup of $P$ is $M_p$-embedded in $G$, then $G$ is $p$-nilpotent.

PROOF. Let $P_1$ be a maximal subgroup of $P$. Then there exists a $p$-nilpotent subgroup $B$ of $G$ such that $P_1 \in \text{Syl}_p(B)$ and $B$ is $M_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$, $G = BK = P_1B_p'K$ where $B_p'$ is the normal $p$-complement of $B$ and $|G : TB_p'K| = p$ for every maximal subgroup $T$ of $P_1$. Then $TB_p'K \leq G$ by Lemma 2.2. Let $L = (TB_p'K)_p$. Obviously, $L$ is a maximal subgroup of $P$. By hypothesis, $L$ is $M_p$-embedded in $G$, there exists a $p$-nilpotent subgroup $B_1$ of $G$ such that $L \in \text{Syl}_p(B_1)$ and $B_1$ is $M_p$-supplemented in $G$. If $B_1 \nleq TB_p'K$, then we have $G = B_1TB_p'K$ and $p = |G : TB_p'K| = |B_1 : B_1 \cap TB_p'K|$, but $|B_1 : B_1 \cap TB_p'K|$ is a $p'$-number, a contradiction. Thus we have $B_1 \leq TB_p'K$. It is easy to see that $L$ is $M_p$-embedded in $TB_p'K$, and hence $TB_p'K$ is $p$-nilpotent by Lemma 2.8 and $G$ is $p$-soluble. By Theorem 3.1, $G$ is $p$-nilpotent. □

COROLLARY 3.3. Suppose that $G$ is a group. $\pi(G) = \{p_1, p_2 = p_1, \ldots, p_n\}$, $p_1 < p_2 < \cdots < p_n$. If every maximal subgroup of $P$ is $M_p$-embedded in $G$, then $G$ is $p$-supersoluble.

PROOF. Let $P^*$ be a maximal subgroup of $P$. By hypothesis, there exists a $p$-nilpotent subgroup $B$ of $G$ such that $P^* \in \text{Syl}_p(B)$ and $B$ is $M_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$, $G = BK = P^*B_p'K$ where $B_p'$ is the normal $p$-complement of $B$ and $B_pK = P_pB_p'K < G$ with $|P^* : P_p| = p$. By Lemma 2.6, $|G : B_pK| = p$. Now, $G/(B_pK)_G$ is a subgroup of the symmetric group $S_p$ and hence $|G : (B_pK)_G| = p_1/p$. Then
$G/(B_iK)_G$ is $p$-supersoluble. Let $A = (B_iK)_G$ and $A_p = ((B_iK)_G)_p$. Since $A_p$ is a maximal subgroup of $P$, $A_p$ is $\mathcal{M}_p$-embedded in $G$, there exists a $p$-nilpotent subgroup $B^*$ of $G$ such that $A_p \in \text{Syl}_p(B^*)$ and $B^*$ is $\mathcal{M}_p$-supplemented in $G$. We consider the group $AB^*$. Obviously, $A_p$ is a sylow $p$-subgroup of $AB^*$. Thus we have $AB^*$ is $p$-supersoluble by Lemma 2.9 and hence $(B_iK)_G$ is $p$-supersoluble. Then $G$ is $p$-soluble and hence $G$ is $p$-supersoluble by Theorem 3.1.

**Theorem 3.4.** Let $G$ be a group. If for each $p \in \pi(G)$ every maximal subgroup of a Sylow $p$-subgroup of $G$ is $\mathcal{M}_p$-embedded in $G$, then $G$ is supersoluble.

**Proof.** Assume that the assertion is false and choose $G$ to be a counterexample of minimal order. Furthermore, we have

1. $G$ is soluble and there exists $p \in \pi(G)$ such that $O_p(G) \neq 1$.

By Corollary 3.2, $G$ is $q$-nilpotent where $q$ is the smallest prime divisor of $|G|$. Hence $G$ is soluble and there exists $p \in \pi(G)$ such that $N \leq O_p(G) \neq 1$.

2. $G$ has the unique minimal normal subgroup $N$ with $G = N \rtimes M$. $M$ is supersoluble and $C_G(N) = N = F(G)$.

Let $N$ be a minimal normal subgroup of $G$ contained in $O_p(G)$. Then $N$ is an elementary abelian $p$-group. We claim that $G/N$ is supersoluble. Suppose that $P \in \text{Syl}_p(G)$ with $N \leq P$. If $N = P$, then $|N| = p$ by Lemma 2.5, a contradiction. Assume that $N < P$. Let $P_1/N < \cdot P/N$, $P_1 < \cdot P$. By Lemma 2.1(1), $P_1/N$ is $\mathcal{M}_p$-embedded in $G/N$. Now consider $p \neq q$. Let $QN/N \in \text{Syl}_q(G/N)$ and we can assume that $Q_1N/N < \cdot QN/N$ with $Q_1 < \cdot Q$. By hypothesis, $Q_1$ is $\mathcal{M}_q$-embedded in $G$, then $Q_1N/N$ is $\mathcal{M}_q$-embedded in $G/N$ by Lemma 2.1(2). Hence $G/N$ satisfies the hypothesis of $G$. The minimal choice of $G$ implies that $G/N$ is supersoluble. Since the class of supersoluble groups is a saturated formation, we have $N$ is a unique minimal normal subgroup of $G$ contained in $O_p(G)$ and $N \not\leq \Phi(G)$. Hence there exists a maximal subgroup $M$ of $G$ such that $G = NM$ and $N \cap M = 1$ by [3, Theorem A.9.2]. The supersolubility of $G/N$ implies that $M$ is supersoluble.

3. Final contradiction.

By (2), $G = NM$ and $P = NM_p$. By hypothesis, $P_1$ is $\mathcal{M}_p$-embedded in $G$, there exists a $p$-nilpotent subgroup $B$ of $G$ such that $P_1 \in \text{Syl}_p(B)$ and $B$ is $\mathcal{M}_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$, $G = BK = P_1B_pK$ where $B_p$ is the normal $p$-complement of $B$ and $T_iB_pK < G$ for every maximal subgroup $T_i$ of $P_1$. We choose a maximal subgroup $T_j$ of $P_1$ such that $M_p \leq T_j$ and $|G : T_jB_pK| = p$. Then $N \leq T_jB_pK$ or $N \cap T_jB_pK = 1$. If $N \leq T_jB_pK$, then $T_jB_pK = NT_jB_pK = PB_pK = G$, a contradiction. If $N \cap T_jB_pK = 1$, then $|N| = |G : T_jB_pK| = p$ and hence $G$ is supersoluble, a contradiction.
The final contradiction completes the proof.

\[ \square \]

**Theorem 3.5.** Let \( G \) be a \( p \)-soluble group and \( p \) a prime divisor of |\( G \)|. If every maximal subgroup of each noncyclic Sylow \( p \)-subgroup of \( F_p(G) \) is \( \mathcal{M}_p \)-embedded in \( G \), then \( G \) is \( p \)-supersoluble.

**Proof.** Assume that the assertion is false and choose \( G \) to be a counterexample of minimal order. Let \( P \) be a Sylow \( p \)-subgroup of \( F_p(G) \). Furthermore, we have

1. \( O_{p'}(G) = 1 \).
2. \( \Phi(G) = 1 \). The \( p \)-solubility of \( G/\Phi(G) \) implies that

\[
F_p\left( G/\Phi(G) \right) \neq 1
\]

By (1), \( F_p(G) = F(G) = P \). Since \( F_p\left( G/\Phi(G) \right) = F_p(G)/\Phi(G) \), we see that \( P/\Phi(G) \) is \( \mathcal{M}_p \)-embedded in \( G/\Phi(G) \) for every maximal subgroup \( P/\Phi(G) \) of \( P/\Phi(G) \). The minimal choice of \( G \) implies that \( G/\Phi(G) \) is \( p \)-supersoluble. Since the class of all \( p \)-supersoluble groups is a saturated formation, \( G \) is \( p \)-supersoluble, a contradiction.

3. Every minimal normal subgroup of \( G \) contained in \( F(G) \) is cyclic of order \( p \).

By Lemma 2.4, \( F(G) \) is the direct product of minimal normal subgroups of \( G \) contained in \( F(G) \). Since \( G \) is \( p \)-soluble and \( O_{p'}(G) = 1 \), we have \( C_G(F(G)) \leq F(G) \). Now \( \Phi(G) = 1 \) implies that \( F(G) \) is nontrivial elementary abelian \( p \)-group and \( C_G(F(G)) = F(G) \). Thus we may assume that \( G = F(G) = R_1 \times \cdots \times R_t \) where \( R_i \) \((i = 1, 2, \ldots, t)\) is a minimal normal subgroup of \( G \) contained in \( F(G) \) and we will show that all are cyclic of order \( p \). Let \( R_i^* \) be a maximal subgroup of \( R_i \), then \( P_i = R_i^* \times R_2 \times \cdots \times R_t \) is a maximal subgroup of \( P \). Set \( M = R_2 \times \cdots \times R_t \). Since \( P_1 \) is \( \mathcal{M}_p \)-embedded in \( G \), there exists a \( p \)-nilpotent subgroup \( B \) of \( G \) such that \( P_1 \in \text{Syl}_p(B) \) and \( B \) is \( \mathcal{M}_p \)-supplemented in \( G \). That is, there exists a subgroup \( K \) of \( G \) such that \( G = BM \) where \( B \) is the normal \( p \)-complement of \( B \) and \( TB/K < G \) for every maximal subgroup \( T \) of \( P_1 \). Let \( B_1 = R_i^* \in \text{Syl}_p(B_1) \), there exists a subgroup \( K_1 = MK \) such that \( G = B_1K_1 \) and \( T^*B/K_1 < G \) for every maximal subgroup \( T^* \) of \( R_i^* \). It fol-
allows that \( R_i^* \) is \( \mathcal{M}_p \)-embedded in \( G \), then \( |R_1| = p \) by Lemma 2.5. Similarly, \( R_i \) (\( i = 2, \ldots, t \)) are also cyclic of order \( p \).

(4) Final contradiction.

Since \( P = F(G) = R_1 \times \cdots \times R_t \) where \( R_i \) is the minimal normal subgroup of \( G \) of order \( p \), \( G/C_G(R_i) \) is isomorphic to a subgroup of \( \text{Aut}(R_i) \), \( G/C_G(R_i) \) is cyclic and it is \( p \)-supersoluble for each \( i \). This implies that \( G/\bigcap_{i=1}^t C_G(R_i) \) is \( p \)-supersoluble. Again, since \( C_G(F(G)) = \bigcap_{i=1}^t C_G(R_i) = F(G) \), we have \( G/F(G) \) is \( p \)-supersoluble for each \( i \). But all chief factors of \( G \) below \( F(G) \) are cyclic groups of order \( p \) and hence \( G \) is \( p \)-supersoluble.

The final contradiction completes our proof. \( \Box \)

**Theorem 3.6.** Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{U} \), suppose that \( G \) has a soluble normal subgroup \( N \) with \( G/N \in \mathcal{F} \). If every maximal subgroup of each noncyclic Sylow \( p \)-subgroup \( P \) of \( F(N) \) is \( \mathcal{M}_p \)-embedded in \( G \) for all \( p \in \pi(F(N)) \), then \( G \in \mathcal{F} \).

**Proof.** Assume that the theorem is false and let \( G \) be a counterexample of minimal order. Furthermore, we have

1. \( N \cap \Phi(G) = 1 \).

   If \( N \cap \Phi(G) \neq 1 \), then there exists a minimal normal subgroup \( R \) of \( G \) such that \( R \subseteq N \cap \Phi(G) \). Since \( N \) is soluble, we know that \( R \) is an elementary abelian \( p \)-group. We will show that \( G/R \) satisfies the hypothesis of the theorem. Clearly, \( (G/R)/(N/R) \cong G/N \in \mathcal{F} \). By Lemma 2.3, \( F(N/R) = F(N)/R \). Let \( P_1/R \) be a maximal subgroup of \( P/R \). Then \( P_1 \) is a maximal subgroup of \( P \). By the hypothesis, \( P_1 \) is \( \mathcal{M}_p \)-embedded in \( G \), hence \( P_1/R \) is \( \mathcal{M}_p \)-embedded in \( G/R \) by Lemma 2.1(1). Now, let \( Q/R \) be a maximal subgroup of the Sylow \( q \)-subgroup of \( F(N)/R \) where \( q \neq p \). Then \( Q = Q_1R \), where \( Q_1 \) is a maximal subgroup of the Sylow \( q \)-subgroup of \( F(N) \). By hypothesis, \( Q_1 \) is \( \mathcal{M}_q \)-embedded in \( G \) and hence \( Q/R = Q_1/R \) is \( \mathcal{M}_q \)-embedded in \( G/R \) by Lemma 2.1(2). By the minimality of \( G \), \( G/R \in \mathcal{F} \). Since \( \mathcal{F} \) is a saturated formation, it follows that \( G \in \mathcal{F} \), a contradiction.

2. Every minimal normal subgroup of \( G \) contained in \( O_p(N) \) is cyclic of order \( p \) where \( p \) is a prime divisor of \( |N| \).

   If \( N = 1 \), the assertion is trivially true. Thus we may assume that \( N \neq 1 \), the solubility of \( N \) implies that \( F(N) \neq 1 \). By Lemma 2.4, \( F(N) \) is the direct product of minimal normal subgroups of \( G \) contained in \( N \). There at least exists a maximal subgroup \( W \) of \( G \) not containing \( F(N) \) and hence there at least exists a prime \( p \) of \( \pi(N) \) with \( O_p(N) \not\subseteq W \) by Lemma 2.7. Applying Lemma 2.7 again, we have \( |G:W| \) is not of prime order.

   Denote \( P = O_p(N) \). Then \( P \) is the direct product of some minimal normal subgroups of \( G \). We assume that \( P = R_1 \times R_2 \times \cdots \times R_t \) where \( R_i \) is a minimal normal subgroup of \( G \), \( i = 1, 2, \ldots, t \) and we will show that all are cyclic of order \( p \). Let \( R_1^* \) be a maximal subgroup of \( R_1 \), then \( P_1 = R_1^* \times R_2 \times \cdots \times R_t \) is a maximal subgroup of \( P \). Set \( M = R_2 \times \cdots \times R_t \). Since \( P_1 \) is \( \mathcal{M}_p \)-embedded in \( G \), there exists a \( p \)-nilpotent subgroup \( B \) of \( G \).
such that $P_1 \in \text{Syl}_p(B)$ and $B$ is $\mathcal{M}_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$ such that $G = BK = P_1 B_{p'} K$ where $B_{p'}$ is the normal $p$-complement of $B$ and $TB_{p'} K < G$ for every maximal subgroup $T$ of $P_1$. Let $B_1 = R_1^* B_{p'}$ is $p$-nilpotent and $R_1^* \in \text{Syl}_p(B_1)$, there exists a subgroup $K_1 = MK$ such that $G = B_1 K_1$ and $T^* B_{p'} K_1 < G$ for every maximal subgroup $T^*$ of $R_1^*$. It follows that $R_1^*$ is $\mathcal{M}_p$-embedded in $G$, then $|R_1| = p$ by Lemma 2.5. Similarly, $R_i$ ($i = 2, \ldots, t$) are also cyclic of order $p$. Hence $G$, $G/F$ and $G/O$ are also cyclic of order $p$. 

(3) Final contradiction.

Since $N \cap \Phi(G) = 1$, for every minimal normal subgroup $R$ of $G$ contained in $P$, there exists a maximal subgroup $M$ of $G$ such that $G = RM$ and $R \cap M = 1$ by [3, Theorem A.9.2]. It is clear that $|G : M| = p$ and hence $G \in \mathcal{F}$ by Lemma 2.7, a contradiction.

The final contradiction completes the proof. ☐

**Corollary 3.7.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ be a soluble group. If every maximal subgroup of each noncyclic Sylow subgroup of $F(G)$ is $\mathcal{M}_p$-embedded in $G$ for all $p \in \pi(F(G))$, then $G \in \mathcal{F}$.

**Theorem 3.8.** Let $G$ be a group. If $F^*(G)$ is soluble and every maximal subgroup of every noncyclic Sylow $p$-subgroup of $F^*(G)$ is $\mathcal{M}_p$-embedded in $G$ for all $p \in \pi(F^*(G))$, then $G$ is supersoluble.

**Proof.** Assume that the assertion is false and let $G$ be a counterexample of minimal order. Furthermore, we have

(1) $\Phi(O_p(G)) = 1$ for any $p \in \pi(F^*(G))$ and $F^*(G) = F(G)$ is abelian.

If $\Phi(O_p(G)) \neq 1$, we consider the quotient group $G/\Phi(O_p(G))$. By Lemma 2.10(5), we have $F^*(G/\Phi(O_p(G))) = F^*(G)/\Phi(O_p(G))$. Clearly, $G/\Phi(O_p(G))$ satisfies the condition of the theorem, the minimal choice of $G$ implies that $G/\Phi(O_p(G))$ is supersoluble, and hence $G$ is supersoluble, a contradiction. Furthermore, we know that $O_p(G)$ is an elementary abelian group and $F^*(G) = F(G)$ is abelian because $F^*(G)$ is soluble group.

(2) There exists a noncyclic Sylow $p$-subgroup $P$ of $F^*(G)$ where $p$ is a prime divisor of $|F^*(G)|$.

If every Sylow subgroup of $F^*(G)$ is cyclic, then we denote that $F^*(G) = T_1 \times \cdots \times T_r$ where $T_i$ ($i = 1, 2, \ldots, r$) is the cyclic Sylow subgroup of $F^*(G)$ and hence $G/C_G(T_i)$ is abelian for each $i$. This implies that $G/\bigcap_{i=1}^r C_G(T_i)$ is supersoluble. Moreover, we have $G/F(G)$ is supersoluble since $C_G(F^*(G)) = \bigcap_{i=1}^r C_G(T_i) = F^*(G) = F(G)$. Therefore $G$ is supersoluble, a contradiction. Then we may assume that $P$ is a noncyclic Sylow $p$-subgroup of $F^*(G)$.

(3) $P \cap \Phi(G) = 1$.

If $P \cap \Phi(G) \neq 1$, then there exists a minimal normal subgroup $L$ of $G$ contained in $P \cap \Phi(G)$. It is easy to see that there exists a maximal subgroup $P_1$ of $P$ such that $L \leq P_1$. Clearly, $L \not\leq \Phi(P_1)$ by (1). By hypothesis, $P_1$ is
$\mathcal{M}_p$-embedded in $G$, there exists a $p$-nilpotent subgroup $B$ of $G$ such that $P_1 \in \text{Syl}_p(B)$ and $B$ is $\mathcal{M}_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$, $G = BK = P_1B_{p'}K$ where $B_{p'}$ is the normal $p$-complement of $B$ and $|G : P_1^*B_{p'}K| = p$ for every maximal subgroup $P_1^*$ of $P_1$. Since $L \not\subseteq \Phi(P_1)$, there exists a maximal subgroup $T$ of $P_1$ such that $P_1 = LT$. Obviously, we have $TB_{p'}K = LTB_{p'}K = P_1B_{p'}K = G$, a contradiction.

(4) Every minimal normal subgroup of $G$ contained in the noncyclic Sylow subgroup of $F^*(G)$ is cyclic of prime order.

By (2) and Lemma 2.4, $P$ is the direct product of minimal normal subgroups of $G$ contained in $P$. Hence we assume that $P = R_1 \times R_2 \times \cdots \times R_t$ where $R_i$ is a minimal normal subgroup of $G$ contained in $P$, $i = 1, 2, \ldots, t$ and we will show that all are cyclic of order $p$. Let $R_i^*$ be a maximal subgroup of $R_i$, then $P_1 = R_1^* \times R_2 \times \cdots \times R_t$ is a maximal subgroup of $P$. Set $M = R_2 \times \cdots \times R_t$. Since $P_1$ is $\mathcal{M}_p$-embedded in $G$, there exists a $p$-nilpotent subgroup $B$ of $G$ such that $P_1 \in \text{Syl}_p(B)$ and $B$ is $\mathcal{M}_p$-supplemented in $G$. That is, there exists a subgroup $K$ of $G$ such that $G = BK = P_1B_{p'}K$ where $B_{p'}$ is the normal $p$-complement of $B$ and $TB_{p'}K < G$ for every maximal subgroup $T$ of $P_1$. Let $B_1 = R_1^*B_{p'}$ be a maximal subgroup of prime order. It follows that $R_1^*$ is $\mathcal{M}_p$-embedded in $G$, then $|R_1| = p$ by Lemma 2.5. Similarly, $R_i$ ($i = 2, \ldots, t$) are also cyclic of order $p$. On the other hand, we see that $F^*(G)$ is the direct product of Sylow subgroups of $F^*(G)$. Thus we may assume that every Sylow subgroup of $F^*(G)$ is the direct product of minimal normal subgroups of prime order.

(5) Final contradiction.

By (3), we have $F^*(G) = H_1 \times H_2 \times \cdots \times H_n$ where $H_i$ is a minimal normal subgroup of $G$ of prime order of $|F^*(G)|$. Therefore, $F(G) \leq Z_\ell(G)$. Since $C_G(F^*(G)) = C_G(F(G)) = \bigcap_{i=1}^{n} C_G(H_i) = F^*(G) = F(G)$, we have $G/F(G)$ is supersoluble and hence $G$ is supersoluble, a contradiction.

The final contradiction completes the proof.

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