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ON $\mathcal{M}_p$-EMBEDDED PRIMARY SUBGROUPS OF FINITE GROUPS*

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Abstract

A subgroup H of G is called $\mathcal{M}_p$-embedded in G, if there exists a p-nilpotent subgroup B of G such that $H_p \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. In this paper, we use $\mathcal{M}_p$-embedded subgroups to study the structure of finite groups.

1. Introduction

All groups considered in this paper will be finite. We shall adhere to the notation employed in [3, 5]. In particular, let $|G|$ denote the order of a group G, and let $\pi(G)$ denote the set of all prime divisors of $|G|$. In addition, $\mathcal{U}$ denotes the class of all supersoluble groups.

The topic of the embedding properties of subgroups is one of the most fruitful fields in Finite Group Theory. This idea provides a new approach to characterize the structure of a group by borrowing some local properties of subgroups. By using the embedding properties of subgroups, many scholars have deeply studied the structure of a group. For instance, in 1998, Ballester-Bolinches and Pedraza–Aguilera [2] introduced the following definition: A subgroup H of G is S-quasinormally embedded (normally embedded) in G if, for every Sylow subgroup P of H, there is an S-quasinormal

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(normal) subgroup K in G such that P is also a Sylow subgroup of K. Furthermore, they proved that if all maximal subgroups of all Sylow subgroups of G are S-quasinormally embedded in G, then G is supersoluble. Further, in 2001, Asaad and Heliel [1] obtained an interesting result about p-nilpotency of a group. Moreover, in 2009, Shemetkov and Skiba [10] introduced $\mathcal{F}\Phi$-hypercentral subgroup of a group. In 2012, Guo and Skiba [7] stated $\mathcal{F}\Phi^*$-hypercentral subgroups of a group. They obtained some new results about the structure of finite groups by using these embedding properties of subgroups.

It is well known that the generalized supplementation of subgroups is significant to determine the structure of a group. In 2007, Skiba [12] defined weakly s-permutable subgroups and studied the structure of a group. In 2008, Guo [6] gave the concept of $\mathcal{F}$-supplemented subgroups and investigated solubility and supersolubility of a group. In 2009, Miao and Lempken [8] considered $\mathcal{M}$-supplemented subgroups of a group and obtained some characterizations of saturated formations containing all supersoluble groups. In 2009, Monakhov and Shnyparkov [11] introduced the definition of $\mathcal{M}_p$-supplemented subgroups and obtained some interesting results about p-supersoluble groups.

As a continuation of above work, naturally, we should consider a question that we construct new embedding properties of subgroups by using the generalized supplementation of subgroups to study the structure of a group. Based on this point, we will introduce the following concept of $\mathcal{M}_p$-embedded subgroups which is closely related to $\mathcal{M}_p$-supplementation. Our motivation is to investigate the construction of a group extensively and obtain some new characterizations about p-supersolubility and supersolubility with the embedding property of subgroups.

Definition 1.1. Let π be a set of primes. A subgroup H of a group G is called $\mathcal{M}_\pi$-supplemented in G, if there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for every maximal subgroup H_1 of H with $\pi\big(|H : H_1|\big) \subseteqq \pi$. In particular, if $\pi=\{p\}$, then H is called $\mathcal{M}_p$-supplemented in G.

Definition 1.2. A subgroup H of G is called $\mathcal{M}_p$-embedded in G, if there exists a p-nilpotent subgroup B of G such that $H_p \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G.

It is not necessary for an $\mathcal{M}_p$-embedded subgruop to be $\mathcal{M}_p$-supplemented.

Example 1.3. Consider the group $G = A_5$. Let $H = \big\langle (12345), (15)(24) \big\rangle$ with $|H| = 10$ and $B = \big\langle (12345) \big\rangle$ a Sylow 5-subgroup of H.

Clearly, B is $\mathcal{M}_5$-supplemented in G and hence H is $\mathcal{M}_5$-embedded in G.

Furthermore, it is easy to verify that H is not $\mathcal{M}_5$-supplemented in G.

2. Preliminaries

For the sake of convenience, we first list here some known results which will be useful in the sequel.

LEMMA 2.1. *Let G be a group. Then*

(1) *Let $N \trianglelefteq G$ and $N \leqq H$. If H is $\mathcal{M}_p$-embedded in G, then H/N is $\mathcal{M}_p$-embedded in G/N.*

(2) *Let π be a set of primes. Let N be a normal π'-subgroup and H be a π-subgroup of G. If H is $\mathcal{M}_p$-embedded in G, then HN/N is $\mathcal{M}_p$-embedded in G/N.*

PROOF. The claims are easy exercises left to the reader. □

LEMMA 2.2 (see [9, Lemma 2.6]). *If H is a subgroup of a group G with $|G : H| = p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.*

LEMMA 2.3 (see [5, Gaschütz]). *Let G be a group. Suppose that N and D are normal subgroups of G, and also $D \leqq N$, $D \leqq \Phi(G)$. Then N/D is nilpotent if and only if N is nilpotent.*

LEMMA 2.4 (see [5, Theorem 1.8.17]). *Let N be a nontrivial soluble normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

LEMMA 2.5. *Let R be a soluble minimal normal subgroup of a group G. If there exists a maximal subgroup R_1 of R such that R_1 is $\mathcal{M}_p$-embedded in G, then R is a cyclic group of prime order.*

PROOF. Since R is a soluble minimal normal subgroup of G, R is an elementary abelian p-group for some prime $p \in \pi(G)$. By hypothesis, R_1 is $\mathcal{M}_p$-embedded in G, then there exists a p-nilpotent subgroup B of G such that $R_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G such that $G = BK = R_1 B_{p'} K$ where $B_{p'}$ is the normal p-complement of B and $T B_{p'} K < G$ for every maximal subgroup T of R_1. It follows from R is a minimal normal subgroup of G that $R \leqq T B_{p'} K$ or $R \cap T B_{p'} K = 1$. If $R \leqq T B_{p'} K$, then $T B_{p'} K = R T B_{p'} K = BK = G$, a contradiction. If $R \cap T B_{p'} K = 1$, then we have $|G : T B_{p'} K| = p$ and hence $|R| = p$. □

LEMMA 2.6 (see [11, Lemma 4]). *Let H be a $\mathcal{M}_\pi$-supplemented subgroup in a group G and B be a $\mathcal{M}_\pi$-supplement to H. If H_1 is a maximal subgroup in H and $\pi(H : H_1) \subseteqq \pi$, then $|G : H_1 B| = |H : H_1|$.*

LEMMA 2.7 (see [13, Theorem 3.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If for every maximal subgroup M of G, either $F(H) \leqq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

LEMMA 2.8 (see [11, Theorem 1]). *Let p be the smallest prime divisor of $|G|$ and H be a p-nilpotent subgroup containing a Sylow p-subgroup of G. If H is $\mathcal{M}_p$-supplemented in G, then G is p-nilpotent.*

LEMMA 2.9 (see [11, Theorem 2]). *Let G be a group, $\pi(G) = \{p_1, p_2 = p, \dots, p_n\}$, $p_1 < p_2 = p < \dots < p_n$ and H be a p-nilpotent subgroup containing a Sylow p-subgroup of G. If H is $\mathcal{M}_p$-supplemented in G, then G is p-supersoluble.*

LEMMA 2.10 (see [4]). *Let G be a group and N a subgroup of G. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Then*

(1) *If N is normal in G, then $F^*(N) \leqq F^*(G)$.*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact,*

$$F^*(G)/F(G) = \mathrm{Soc}\left(F(G)C_G\left(F(G)\right)/F(G)\right).$$

(3) *$F^*\left(F^*(G)\right) = F^*(G) \geqq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *$C_G\left(F^*(G)\right) \leqq F(G)$.*

(5) *If $P \trianglelefteq G$ with $P \leqq O_p(G)$, then $F^*\left(G/\Phi(P)\right) = F^*(G)/\Phi(P)$.*

(6) *If $K \leqq Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

3. Main results

THEOREM 3.1. *Let G be a p-soluble group and P be a Sylow p-subgroup of G. If every maximal subgroup of P is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

PROOF. Assume that the assertion is false and choose G to be a counterexample of minimal order. Furthermore, we have

(1) $O_{p'}(G) = 1$.

If $L = O_{p'}(G) \neq 1$, we consider the factor group G/L. By Lemma 2.1(2), G/L satisfies the condition of the theorem, the minimal choice of G implies that G/L is p-supersoluble, and hence G is p-supersoluble, a contradiction.

(2) $O_p(G) \neq 1$.

Since G is p-soluble and $O_{p'}(G)$=1, we have that the minimal normal subgroup of G is abelian p-group and hence $O_p(G) \neq 1$.

(3) *Final contradiction.*

By (2), we pick a minimal normal subgroup N of G contained in $O_p(G)$. By Lemma 2.1(1), we know that G/N satisfies the hypothesis of the theorem, the minimal choice of G implies that G/N is p-supersoluble. On the other hand, since the class of all p-supersoluble groups is a saturated formation, we have N is the unique minimal normal subgroup of G contained in $O_p(G)$ and $N = O_p(G) = F(G) \nleqq \Phi(G) = 1$ by Lemma 2.4. Clearly, by [3, Theorem A.9.2], there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$, $P = NM_p$ where M_p is a Sylow p-subgroup of M. We may choose a maximal subgroup P_1 with $N \leqq P_1$, then $M_p \nleqq P_1$. By hypothesis, P_1 is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $P_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G, $G = BK = P_1B_{p'}K$ where $B_{p'}$ is the normal p-complement of B and $TB_{p'}K < G$ for every maximal subgroup T of P_1. If $N \leqq \Phi(P_1)$, then $N \leqq \Phi(G)$ and hence G is p-supersoluble, a contradiction. Then there exists a maximal subgroup T^* of P_1 such that $N \nleqq T^*$. Since $T^*B_{p'}K$ is a maximal subgroup of G, $N \leqq T^*B_{p'}K$ or $N \cap T^*B_{p'}K = 1$. If $N \leqq T^*B_{p'}K$, then $T^*B_{p'}K = NT^*B_{p'}K \leqq P_1B_{p'}K = G$, a contradiction. On the other hand, if $N \cap T^*B_{p'}K = 1$, then $|N| = p$ and hence G is p-supersoluble.

The final contradiction completes the proof. □

Corollary 3.2. *Let G be a group and P a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is $\mathcal{M}_p$-embedded in G, then G is p-nilpotent.*

Proof. Let P_1 be a maximal subgroup of P. Then there exists a p-nilpotent subgroup B of G such that $P_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G, $G = BK = P_1B_{p'}K$ where $B_{p'}$ is the normal p-complement of B and $|G : TB_{p'}K| = p$ for every maximal subgroup T of P_1. Then $TB_{p'}K \trianglelefteq G$ by Lemma 2.2. Let $L = (TB_{p'}K)_p$. Obviously, L is a maximal subgroup of P. By hypothesis, L is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B_1 of G such that $L \in \mathrm{Syl}_p(B_1)$ and B_1 is $\mathcal{M}_p$-supplemented in G. If $B_1 \nleqq TB_{p'}K$, then we have $G = B_1TB_{p'}K$ and $p = |G : TB_{p'}K| = |B_1 : B_1 \cap TB_{p'}K|$, but $|B_1 : B_1 \cap TB_{p'}K|$ is a p'-number, a contradiction. Thus we have $B_1 \leqq TB_{p'}K$. It is easy to see that L is $\mathcal{M}_p$-embedded in $TB_{p'}K$, and hence $TB_{p'}K$ is p-nilpotent by Lemma 2.8 and G is p-soluble. By Theorem 3.1, G is p-nilpotent. □

Corollary 3.3. *Suppose that G is a group. $\pi(G) = \{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 < \cdots < p_n$. If every maximal subgroup of P is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

Proof. Let P^* be a maximal subgroup of P. By hypothesis, there exists a p-nilpotent subgroup B of G such that $P^* \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G, $G = BK = P^*B_{p'}K$ where $B_{p'}$ is the normal p-complement of B and $B_iK = P_iB_{p'}K < G$ with $|P^* : P_i| = p$. By Lemma 2.6, $|G : B_iK| = p$. Now, $G/(B_iK)_G$ is a subgroup of the symmetric group S_p and hence $\left|G : (B_iK)_G\right| = p_1^{\alpha}p$. Then

$G/(B_iK)_G$ is p-supersoluble. Let $A = (B_iK)_G$ and $A_p = \big((B_iK)_G\big)_p$. Since A_p is a maximal subgroup of P, A_p is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B^* of G such that $A_p \in \mathrm{Syl}_p(B^*)$ and B^* is $\mathcal{M}_p$-supplemented in G. We consider the group AB^*. Obviously, A_p is a sylow p-subgroup of AB^*. Thus we have AB^* is p-supersoluble by Lemma 2.9 and hence $(B_iK)_G$ is p-supersoluble. Then G is p-soluble and hence G is p-supersoluble by Theorem 3.1. □

THEOREM 3.4. *Let G be a group. If for each $p \in \pi(G)$ every maximal subgroup of a Sylow p-subgroup of G is $\mathcal{M}_p$-embedded in G, then G is supersoluble.*

PROOF. Assume that the assertion is false and choose G to be a counterexample of minimal order. Furthermore, we have

(1) *G is soluble and there exists $p \in \pi(G)$ such that $O_p(G) \neq 1$.*

By Corollary 3.2, G is q-nilpotent where q is the smallest prime divisor of $|G|$. Hence G is soluble and there exists $p \in \pi(G)$ such that $N \leqq O_p(G) \neq 1$.

(2) *G has the unique minimal normal subgroup N with $G = N \rtimes M$. M is supersoluble and $C_G(N) = N = F(G)$.*

Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then N is an elementary abelian p-group. We claim that G/N is supersoluble. Suppose that $P \in \mathrm{Syl}_p(G)$ with $N \leqq P$. If $N = P$, then $|N| = p$ by Lemma 2.5, a contradiction. Assume that $N < P$. Let $P_1/N < \cdot P/N$, $P_1 < \cdot P$. By Lemma 2.1(1), P_1/N is $\mathcal{M}_p$-embedded in G/N. Now consider $p \neq q$. Let $QN/N \in \mathrm{Syl}_q(G/N)$ and we can assume that $Q_1N/N < \cdot QN/N$ with $Q_1 < \cdot Q$. By hypothesis, Q_1 is $\mathcal{M}_q$-embedded in G, then Q_1N/N is $\mathcal{M}_q$-embedded in G/N by Lemma 2.1(2). Hence G/N satisfies the hypothesis of G. The minimal choice of G implies that G/N is supersoluble. Since the class of supersoluble groups is a saturated formation, we have N is a unique minimal normal subgroup of G contained in $O_p(G)$ and $N \nleqq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$ by [3, Theorem A.9.2]. The supersolubility of G/N implies that M is supersoluble.

(3) *Final contradiction.*

By (2), $G = NM$ and $P = NM_p$. By hypothesis, P_1 is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $P_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G, $G = BK = P_1B_{p'}K$ where $B_{p'}$ is the normal p-complement of B and $T_iB_{p'}K < G$ for every maximal subgroup T_i of P_1. We choose a maximal subgroup T_j of P_1 such that $M_p \leqq T_j$ and $|G : T_jB_{p'}K| = p$. Then $N \leqq T_jB_{p'}K$ or $N \cap T_jB_{p'}K = 1$. If $N \leqq T_jB_{p'}K$, then $T_jB_{p'}K = NT_jB_{p'}K = PB_{p'}K = G$, a contradiction. If $N \cap T_jB_{p'}K = 1$, then $|N| = |G : T_jB_{p'}K| = p$ and hence G is supersoluble, a contradiction.

The final contradiction completes the proof. $\square$

THEOREM 3.5. *Let G be a p-soluble group and p a prime divisor of $|G|$. If every maximal subgroup of each noncyclic Sylow p-subgroup of $F_p(G)$ is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

PROOF. Assume that the assertion is false and choose G to be a counterexample of minimal order. Let P be a Sylow p-subgroup of $F_p(G)$. Furthermore, we have

(1) $O_{p'}(G)$=1.

In fact, if $O_{p'}(G) = L \neq 1$, we consider G/L. Since $F_p(G/L) = F_p(G)/L$ and $F_p(G) = O_{p'p}(G)$, we have $F_p(G)/L = O_p(G/L) = PL/L$ and hence $F_p(G)/L$ is a p-group. Clearly, there exists a maximal subgroup P_1 of P such that $P_1L/L = H/L$ for every maximal subgroup H/L of $F_p(G)/L$. By hypothesis, every maximal subgroup of P is $\mathcal{M}_p$-embedded in G, $P_1L/L = H/L$ is also $\mathcal{M}_p$-embedded in G/L by Lemma 2.1(2). Hence G/L satisfies the condition of the theorem. The minimal choice of G implies that G/L is p-supersoluble, a contradiction.

(2) $\Phi(G)$=1.

Assume that $\Phi(G) \neq 1$. The p-solubility of $G/\Phi(G)$ implies that

$$F_p\big(G/\Phi(G)\big) \neq 1.$$

By (1), $F_p(G) = F(G) = P$. Since $F_p\big(G/\Phi(G)\big) = F_p(G)/\Phi(G)$, we see that $P_1/\Phi(G)$ is $\mathcal{M}_p$-embedded in $G/\Phi(G)$ for every maximal subgroup $P_1/\Phi(G)$ of $P/\Phi(G)$. The minimal choice of G implies that $G/\Phi(G)$ is p-supersoluble. Since the class of all p-supersoluble groups is a saturated formation, G is p-supersoluble, a contradiction.

(3) *Every minimal normal subgroup of G contained in $F(G)$ is cyclic of order p.*

By Lemma 2.4, $F(G)$ is the direct product of minimal normal subgroups of G contained in $F(G)$. Since G is p-soluble and $O_{p'}(G)$=1, we have $C_G\big(F(G)\big) \leqq F(G)$. Now $\Phi(G) = 1$ implies that $F(G)$ is nontrivial elementary abelian p-group and $C_G\big(F(G)\big) = F(G)$. Thus we may assume that $P = F(G) = R_1 \times \cdots \times R_t$ where R_i $(i = 1, 2, \ldots, t)$ is a minimal normal subgroup of G contained in $F(G)$ and we will show that all are cyclic of order p. Let R_1^* be a maximal subgroup of R_1, then $P_1 = R_1^* \times R_2 \times \cdots \times R_t$ is a maximal subgroup of P. Set $M = R_2 \times \cdots \times R_t$. Since P_1 is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $P_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G such that $G = BK = P_1B_{p'}K$ where $B_{p'}$ is the normal p-complement of B and $TB_{p'}K < G$ for every maximal subgroup T of P_1. Let $B_1 = R_1^*B_{p'}$ is p-nilpotent and $R_1^* \in \mathrm{Syl}_p(B_1)$, there exists a subgroup $K_1 = MK$ such that $G = B_1K_1$ and $T^*B_{p'}K_1 < G$ for every maximal subgroup T^* of R_1^*. It follows that R_1^* is $\mathcal{M}_p$-embedded in G, then $|R_1| = p$ by Lemma 2.5. Similarly, R_i $(i = 2, \dots, t)$ are also cyclic of order p.

(4) *Final contradiction.*

Since $P = F(G) = R_1 \times \cdots \times R_t$ where R_i is the minimal normal subgroup of G of order p, $G/C_G(R_i)$ is isomorphic to a subgroup of $\mathrm{Aut}(R_i)$, $G/C_G(R_i)$ is cyclic and it is p-supersoluble for each i. This implies that $G/\bigcap_{i=1}^t C_G(R_i)$ is p-supersoluble. Again, since $C_G\big(F(G)\big) = \bigcap_{i=1}^t C_G(R_i) = F(G)$, we have $G/F(G)$ is p-supersoluble for each i. But all chief factors of G below $F(G)$ are cyclic groups of order p and hence G is p-supersoluble.

The final contradiction completes our proof. $\square$

THEOREM 3.6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, suppose that G has a soluble normal subgroup N with $G/N \in \mathcal{F}$. If every maximal subgroup of each noncyclic Sylow p-subgroup P of $F(N)$ is $\mathcal{M}_p$-embedded in G for all $p \in \pi\big(F(N)\big)$, then $G \in \mathcal{F}$.*

PROOF. Assume that the theorem is false and let G be a counterexample of minimal order. Furthermore, we have

(1) $N \cap \Phi(G) = 1$.

If $N \cap \Phi(G) \neq 1$, then there exists a minimal normal subgroup R of G such that $R \leqq N \cap \Phi(G)$. Since N is soluble, we know that R is an elementary abelian p-group. We will show that G/R satisfies the hypothesis of the theorem. Clearly, $(G/R)/(N/R) \cong G/N \in \mathcal{F}$. By Lemma 2.3, $F(N/R) = F(N)/R$. Let P_1/R be a maximal subgroup of P/R. Then P_1 is a maximal subgroup of P. By the hypothesis, P_1 is $\mathcal{M}_p$-embedded in G, hence P_1/R is $\mathcal{M}_p$-embedded in G/R by Lemma 2.1(1). Now, let Q/R be a maximal subgroup of the Sylow q-subgroup of $F(N)/R$ where $q \neq p$. Then $Q = Q_1R$, where Q_1 is a maximal subgroup of the Sylow q-subgroup of $F(N)$. By hypothesis, Q_1 is $\mathcal{M}_q$-embedded in G and hence $Q/R = Q_1R/R$ is $\mathcal{M}_q$-embedded in G/R by Lemma 2.1(2). By the minimality of G, $G/R \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, it follows that $G \in \mathcal{F}$, a contradiction.

(2) *Every minimal normal subgroup of G contained in $O_p(N)$ is cyclic of order p where p is a prime divisor of $|N|$.*

If $N = 1$, the assertion is trivially true. Thus we may assume that $N \neq 1$, the solubility of N implies that $F(N) \neq 1$. By Lemma 2.4, $F(N)$ is the direct product of minimal normal subgroups of G contained in N. There at least exists a maximal subgroup W of G not containing $F(N)$ and hence there at least exists a prime p of $\pi(N)$ with $O_p(N) \nleqq W$ by Lemma 2.7. Applying Lemma 2.7 again, we have $|G : W|$ is not of prime order.

Denote $P = O_p(N)$. Then P is the direct product of some minimal normal subgroups of G. We assume that $P = R_1 \times R_2 \times \cdots \times R_t$ where R_i is a minimal normal subgroup of G, $i = 1, 2, \dots, t$ and we will show that all are cyclic of order p. Let R_1^* be a maximal subgroup of R_1, then $P_1 = R_1^* \times R_2 \times \cdots \times R_t$ is a maximal subgroup of P. Set $M = R_2 \times \cdots \times R_t$. Since P_1 is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G

such that $P_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G such that $G = BK = P_1B_{p'}K$ where $B_{p'}$ is the normal p-complement of B and $TB_{p'}K < G$ for every maximal subgroup T of P_1. Let $B_1 = R_1^*B_{p'}$ is p-nilpotent and $R_1^* \in \mathrm{Syl}_p(B_1)$, there exists a subgroup $K_1 = MK$ such that $G = B_1K_1$ and $T^*B_{p'}K_1 < G$ for every maximal subgroup T^* of R_1^*. It follows that R_1^* is $\mathcal{M}_p$-embedded in G, then $|R_1| = p$ by Lemma 2.5. Similarly, R_i $(i = 2, \ldots, t)$ are also cyclic of order p.

(3) *Final contradiction.*

Since $N \cap \Phi(G) = 1$, for every minimal normal subgroup R of G contained in P, there exists a maximal subgroup M of G such that $G = RM$ and $R \cap M{=}1$ by [3, Theorem A.9.2]. It is clear that $|G : M| = p$ and hence $G \in \mathcal{F}$ by Lemma 2.7, a contradiction.

The final contradiction completes the proof. □

COROLLARY 3.7. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a soluble group. If every maximal subgroup of each noncyclic Sylow p-subgroup of $F(G)$ is $\mathcal{M}_p$-embedded in G for all $p \in \pi\big(F(G)\big)$, then $G \in \mathcal{F}$.*

THEOREM 3.8. *Let G be a group. If $F^*(G)$ is soluble and every maximal subgroup of every noncyclic Sylow p-subgroup of $F^*(G)$ is $\mathcal{M}_p$-embedded in G for all $p \in \pi\big(F^*(G)\big)$, then G is supersoluble.*

PROOF. Assume that the assertion is false and let G be a counterexample of minimal order. Furthermore, we have

(1) $\Phi\big(O_p(G)\big) = 1$ *for any $p \in \pi\big(F^*(G)\big)$ and $F^*(G) = F(G)$ is abelian.*

If $\Phi\big(O_p(G)\big) \neq 1$, we consider the quotient group $G/\Phi\big(O_p(G)\big)$. By Lemma 2.10(5), we have $F^*\big(G/\Phi\big(O_p(G)\big)\big) = F^*(G)/\Phi\big(O_p(G)\big)$. Clearly, $G/\Phi\big(O_p(G)\big)$ satisfies the condition of the theorem, the minimal choice of G implies that $G/\Phi\big(O_p(G)\big)$ is supersoluble, and hence G is supersoluble, a contradiction. Furthermore, we know that $O_p(G)$ is an elementary abelian group and $F^*(G) = F(G)$ is abelian because $F^*(G)$ is soluble group.

(2) *There exists a noncyclic Sylow p-subgroup P of $F^*(G)$ where p is a prime divisor of $\big|F^*(G)\big|$.*

If every Sylow subgroup of $F^*(G)$ is cyclic, then we denote that $F^*(G) = T_1 \times \cdots \times T_r$ where T_i $(i = 1, 2, \ldots, r)$ is the cyclic Sylow subgroup of $F^*(G)$ and hence $G/C_G(T_i)$ is abelian for each i. This implies that $G/\bigcap_{i=1}^r C_G(T_i)$ is supersoluble. Morever, we have $G/F(G)$ is supersoluble since $C_G\big(F^*(G)\big) = \bigcap_{i=1}^r C_G(T_i) = F^*(G) = F(G)$. Therefore G is supersoluble, a contradiction. Then we may assume that P is a noncyclic Sylow p-subgroup of $F^*(G)$.

(3) $P \cap \Phi(G) = 1$.

If $P \cap \Phi(G) \neq 1$, then there exists a minimal normal subgroup L of G contained in $P \cap \Phi(G)$. It is easy to see that there exists a maximal subgroup P_1 of P such that $L \leqq P_1$. Clearly, $L \nleqq \Phi(P_1)$ by (1). By hypothesis, P_1 is

$\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $P_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G, $G = BK = P_1 B_{p'} K$ where $B_{p'}$ is the normal p-complement of B and $\left|G : P_1^* B_{p'} K\right| = p$ for every maximal subgroup P_1^* of P_1. Since $L \nleqq \Phi(P_1)$, there exists a maximal subgroup T of P_1 such that $P_1 = LT$. Obviously, we have $TB_{p'}K = LTB_{p'}K = P_1 B_{p'} K = G$, a contradiction.

(4) *Every minimal normal subgroup of G contained in the noncyclic Sylow subgroup of $F^*(G)$ is cyclic of prime order.*

By (2) and Lemma 2.4, P is the direct product of minimal normal subgroups of G contained in P. Hence we assume that $P = R_1 \times R_2 \times \cdots \times R_t$ where R_i is a minimal normal subgroup of G contained in P, $i = 1, 2, \ldots, t$ and we will show that all are cyclic of order p. Let R_1^* be a maximal subgroup of R_1, then $P_1 = R_1^* \times R_2 \times \cdots \times R_t$ is a maximal subgroup of P. Set $M = R_2 \times \cdots \times R_t$. Since P_1 is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $P_1 \in \mathrm{Syl}_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. That is, there exists a subgroup K of G such that $G = BK = P_1 B_{p'} K$ where $B_{p'}$ is the normal p-complement of B and $TB_{p'}K < G$ for every maximal subgroup T of P_1. Let $B_1 = R_1^* B_{p'}$ is p-nilpotent and $R_1^* \in \mathrm{Syl}_p(B_1)$, there exists a subgroup $K_1 = MK$ such that $G = B_1 K_1$ and $T^* B_{p'} K_1 < G$ for every maximal subgroup T^* of R_1^*. It follows that R_1^* is $\mathcal{M}_p$-embedded in G, then $|R_1| = p$ by Lemma 2.5. Similarly, R_i $(i = 2, \ldots, t)$ are also cyclic of order p. On the other hand, we see that $F^*(G)$ is the direct product of Sylow subgroups of $F^*(G)$. Thus we may assume that every Sylow subgroup of $F^*(G)$ is the direct product of minimal normal subgroups of prime order.

(5) *Final contradiction.*

By (3), we have $F^*(G) = H_1 \times H_2 \times \cdots \times H_n$ where H_i is a minimal normal subgroup of G of prime order of $\left|F^*(G)\right|$. Therefore, $F(G) \leqq Z_{\mathcal{U}}(G)$. Since $C_G\big(F^*(G)\big) = C_G\big(F(G)\big) = \bigcap_{i=1}^n C_G(H_i) = F^*(G) = F(G)$, we have $G/F(G)$ is supersoluble and hence G is supersoluble, a contradiction.

The final contradiction completes the proof. □

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REFERENCES

[1] Asaad, M. and Heliel, A. A., On S-quasinormality embedded subgroups of finite groups, *J. Pure Appl. Algebra,* **165** (2001), 129–135.

[2] Ballester-Bolinches, A. and Pedraza Aguilera, M. C., Sufficient conditions for supersolubility of finite groups, *J. Pure Appl. Algebra,* **127** (1998), 113–223.

[3] Doerk, K. and Hawkes, T., *Finite Soluble Groups,* de Gruyter, Berlin/New York, 1992.

[4] Huppert, B. and Blackburn, N., *Finite Groups III,* Springer-Verlag, Berlin-New York, 1982.

[5] Guo, W., The Theory of Classes of Groups, Science Press-Kluwer Academic Publishers, Beijing–New York–Dordrecht–Boston–London, 2000.

[6] Guo, W., On $\mathcal{F}$-supplemented subgroups of finite groups, *Manuscripta Math.,* **127** (2008), 139–150.

[7] Guo, W. and Skiba, A. N., On $\mathcal{F}\Phi^*$-hypercentre of finite groups, *J. Algebra,* **372** (2012), 275–292.

[8] Miao, L. and Lempken, W., On $\mathcal{M}$-supplemented subgroups of finite groups, *J. Group Theory,* **12** (2009), 271–289.

[9] Miao, L., On p-nilpotency of finite groups, *Bull. Braz. Math. Soc.,* **38**(4) (2007), 585–594.

[10] Shemetkov, L. A. and Skiba, A. N., On the $\mathcal{X}\Phi$-hypercentre of finite groups, *J. Algebra,* **322** (2009), 2106–2117.

[11] Monakhov, V. S. and Shnyparkov, A. V., On the p-supersolubility of a finite group with a $\mathcal{M}$-supplemented Sylow p-subgroup, *Siberian Math. J.,* **50**(4) (2009), 681–686.

[12] Skiba, A. N., On weakly s-permutable subgroups of finite groups, *J. Algebra,* **315** (2007), 192–209.

[13] Wang, Y., Wei, H. and Li, Y., A generalization of Kramer's theorem and its applications, *Bull. Austral. Math. Soc.,* **65** (2002), 467–475.