Improvements on the density of maximal 1-planar graphs

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Abstract

A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. A graph, together with a 1-planar drawing is called 1-plane. Brandenburg et al. showed that there are maximal 1-planar graphs with only $\frac{45}{17}n + O(1) \approx 2.647n$ edges and maximal 1-plane graphs with only $\frac{7}{3}n + O(1) \approx 2.33n$ edges. On the other hand, they showed that a maximal 1-planar graph has at least $\frac{28}{13}n - O(1) \approx 2.15n - O(1)$ edges, and a maximal 1-plane graph has at least $2.1n - O(1)$ edges.

We improve both lower bounds to $\frac{20n}{9} \approx 2.22n$.

1 Introduction

In a *drawing* of a simple undirected graph G, vertices are represented by distinct points in the plane and edges are represented by simple continuous curves connecting the corresponding points. For simplicity, the points and curves are also called vertices and edges, and if it does not lead to confusion, we do denote them the same way as the original vertices and edges of G, respectively. We assume that edges do not contain vertices in their interior. It follows from Euler's formula that a planar graph of n vertices has at most $3n - 6$ edges. Also if a planar graph G has less edges, then we can add edges to it so that the resulting graph has exactly $3n - 6$ edges and still planar. This holds even if we start with a fixed planar drawing of G.

A drawing of a graph is *1-planar*, if each edge is crossed at most once. A graph is *1-planar*, if it has a 1-planar drawing. It is *maximal* 1-planar, if we cannot add any edge to it so that the resulting graph is still 1-planar. A graph together with a 1-planar drawing is a 1-*plane graph*. It is *maximal* 1-plane, if we cannot add any edge to it so that the resulting drawing is still 1-plane. The

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maximum number of edges of a 1-planar or 1-plane graph is $4n-8$ [4]. Recently, Brandenburg et al. [1, 2] observed a very interesting phenomenon: there are maximal 1-planar and 1-plane graphs with much fewer edges.

Theorem 1 (Brandenburg et al. [2]) *(i) Let $e(n)$ be the minimum number of edges of a maximal 1-planar graph with n vertices. The following holds*

$$\frac{28}{13}n - O(1) \approx 2.15n - O(1) \leq e(n) \leq \frac{45}{17}n + O(1) \approx 2.647n,$$

(ii) Let $e'(n)$ be the minimum number of edges of a maximal 1-plane graph with n vertices. The following holds

$$2.1n - O(1) \leq e(n) \leq \frac{7}{3}n + O(1) \approx 2.33n.$$

In this note, we improve both lower bounds.

Theorem 2 *A maximal 1-planar or 1-plane graph has at least $\frac{20}{9}n - O(1) \approx 2.22n$ edges.*

That is, $e(n), e'(n) \geq \frac{20}{9}n - O(1)$.

2 Preliminaries

Our method is based on the ideas of Brandenburg et al. [2]. We also point out an error in [2], but with our approach their proof goes through as well. The following observations are essentially from their paper. We include the proofs for completeness. Throughout this section, G is a maximal 1-plane graph. The edges of G divide the plane into *faces*. A face is bounded by edges and edge segments. These edges and edge segments end in vertices or crossings.

Lemma 1 (i) *There are at least two vertices on the boundary of each face.*
(ii) *If u and v are two vertices on the boundary of a face, then they are adjacent.*

Proof. (i) Each face is bounded by at least three edges or edge segments, and has at least three vertices or crossings on its boundary. Since there is at most one crossing on each edge, each edge segment contains a vertex as an endpoint. Therefore, there must be at least two vertices on the boundary of the face.
(ii) Suppose that there are two vertices, u and v on the boundary of a face. Now u and v could be connected by a curve in the face without creating any crossing. Therefore, by the maximality of G, u and v are already connected. □

Lemma 2 *There are neither isolated vertices nor vertices of degree 1 in G.*

Proof. Suppose that v is an isolated vertex or a vertex of degree 1 in face F of G. Now $G \setminus \{v\}$ is also maximal 1-planar, since if we can add an edge to $G \setminus \{v\}$, then we could have added it to G. Therefore, by Lemma 1, F has at least two vertices on its boundary, different from v, and v is adjacent to both of them, a contradiction. □

Lemma 3 *If ab and cd are crossing edges in G, then a, b, c, d span a K_4 in G.*

Proof. Let x be the crossing of ab and cd. Since there are no other crossings on ab and cd, there is a face bounded by ax and xc. Now a and c are adjacent by Lemma 1. Similarly a and d, b and c, b and d are also adjacent. □

The smallest degree in G is at least two by Lemma 2. Following [2], we call vertices of degree two *hermits*.

Lemma 4 *If a vertex h has only two neighbors in G, say u and v, then*
(i) *hu and hv are not crossed by any edge,*
(ii) *u and v are adjacent in G.*

Proof. (i) Suppose to the contrary that hu is crossed by an edge. By Lemma 3, vertex h has degree at least 3, a contradiction.
(ii) Since the only neighbors of h are u and v, and edges hu and hv are not crossed, there is a face that has u, x and v on its boundary. Therefore, u and v are adjacent by Lemma 1. □

Lemma 5 *Suppose that h is a hermit, and its neighbors are u and v. Delete h, hu, hv, uv from G, and let G' be the resulting graph with the original embedding. Let F be the face of G' that contains the point corresponding to vertex h. Then F has only two vertices on its boundary, u and v.*

Proof. If there is another vertex on the boundary of F, then we could connect it to h. Either without any crossing or with exactly one crossing with edge uv contradicting the maximality of G. □

We conclude that a hermit is surrounded by two pairs of crossing edges, see Figure 1.

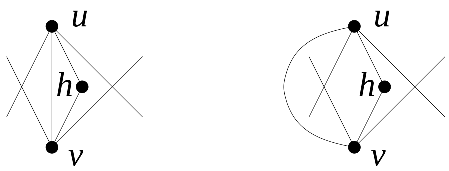

Figure 1: Hermit h, surrounded by two pairs of crossing edges.

Remove all hermits from G. The resulting graph $\hat{G}$ with the inherited drawing is the *skeleton* of G. Notice that $\hat{G}$ is also maximal 1-planar and each vertex of $\hat{G}$ has degree at least 3.

2.1 A Correction

In [2] the lower bound proofs rely on the following statement:

Claim. [2] *Every edge of $\hat{G}$ is covered by a K_4 in $\hat{G}$.*

However, this claim does not hold! See Figure 3 for a counterexample. Call an edge of $\hat{G}$ *exceptional* if it is not part of a K_4 in $\hat{G}$. We have to deal with exceptional edges as well.

Lemma 6 *Suppose that edge ab of $\hat{G}$ is exceptional. That is, ab is not part of a K_4 in $\hat{G}$. Let F_1 and F_2 be the faces bounded by ab. Then*

(i) $F_1 \neq F_2$,
(ii) F_i *has exactly three vertices on its boundary* a, b *and* f_i *for* $i = 1, 2$, *and* $f_1 = f_2$,
(iii) *both* af_i *and* bf_i *are non-exceptional edges of* $\hat{G}$.

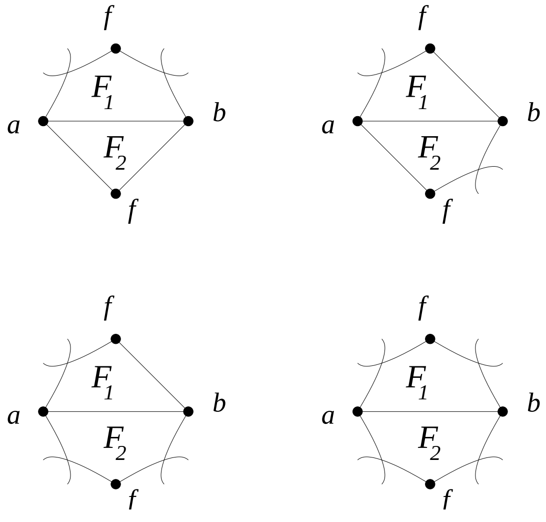

Figure 2: The four possible types of exceptional edges ab

Proof. (i) Suppose that ab is an exceptional edge of $\hat{G}$. If edge ab is crossed by another edge, then it is part of a K_4 by Lemma 3. Therefore, ab does not participate in a crossing. Let F_1 and F_2 be the faces bounded by ab. If $F_1 = F_2$, then ab is a cut edge. In this case, by Lemma 1, both components have at least one other vertex on the boundary of $F_1 = F_2$, and they can be connected. This contradicts the maximality of $\hat{G}$. Consequently, $F_1 \neq F_2$.
(ii) If there is an edge from a and an edge from b which cross, then ab is part of a K_4 by Lemma 3. If there are at least four different vertices on the boundary of F_i, say a, b, x and y, then they form a K_4 by Lemma 1. We conclude that if ab is exceptional, then F_i has exactly three vertices on its boundary, a, b and f_i. If $f_1 \neq f_2$, then we can connect them through F_1 and F_2, a contradiction again. Therefore, $f_1 = f_2$ and we denote it by f for the rest of the proof.
(iii) Vertices a, b and f divide the boundary of F_1 into three parts. Between a and b, we have edge ab by assumption. Between a and f, we either have edge af, or two segments of edges. In the latter case, af is an edge of $\hat{G}$ and part of a K_4 by Lemma 3. We can argue similarly for face F_2. We conclude that af is an edge of $\hat{G}$ and part of a K_4, unless af is on the boundary of both F_1 and F_2. Now the degree of a would be 2 in $\hat{G}$, which is impossible. We can argue the same way for edge bf. $\square$

Remark. Each of the drawings on Figure 2 can be extended to a maximal 1-plane graph so that ab is not part of a K_4. See Figure 3 for an example.

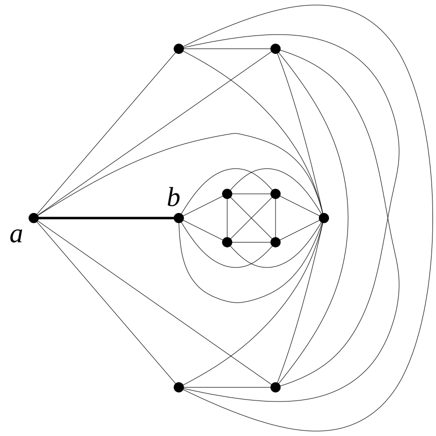

Figure 3: A maximal 1-plane graph, ab is not part of a K_4.

3 Improvement of the lower bounds – Proof of Theorem 2

Let $\hat{G}$ be the skeleton of a maximal 1-plane graph. Recall that the skeleton $\hat{G}$ arises by removing each hermit from G together with its two incident edges. The skeleton inherits its drawing from G, it is maximal 1-plane and each vertex has degree at least three. We distinguish three types of edges in $\hat{G}$: crossing, plain and exceptional. Edges that participate in a crossing are *crossing edges*. A crossing-free edge that is part of a K_4, is a *plain* edge. Any other edge is *exceptional*. Those edges are crossing-free and do not belong to a K_4. Let $n(H)$, $c(H)$, $p(H)$, $e(H)$ denote the number of vertices, crossing edges, plain edges, and exceptional edges of a graph H. In particular, let $n = n(\hat{G})$, $c = c(\hat{G})$, $p = p(\hat{G})$, $e = e(\hat{G})$. We prove the following crucial inequality involving these quantities.

Lemma 7 *If $\hat{G}$ is the skeleton of any drawing of a maximal 1-planar graph G and $n \geq 4$, then*

$$9p + 10e + 7c \geq 20n - 30. \quad (1)$$

Proof. We use induction on the pair (e, n), ordered lexicographically. If there is an exceptional edge, then we use the induction hypothesis on graphs with smaller e. If $e = 0$, then we use induction on n.

Suppose that there is an exceptional edge ab in $\hat{G}$. Let F_1 and F_2 be the two faces bounded by ab. By Lemma 6, $F_1 \neq F_2$, and both F_1 and F_2 have exactly three vertices on their boundaries, a, b, and f, see Figure 2. The closure of $F_1 \cup F_2$ divides the plane into two parts, say S_1, S_2. Now $\overline{S_i}$, the closure of S_i, intersects $\hat{G}$ in G_i for $i = 1, 2$. Remove the edge ab and the interior of F_1, F_2 from $\hat{G}$. Now two almost disjoint subgraphs G_1 and G_2 arise such that they have exactly one vertex f in common, $a \in G_1$ and $b \in G_2$. Both G_1 and G_2 are maximal 1-plane and both have at least four vertices. Therefore, we can use the induction hypothesis on G_1 and G_2. For $i = 1, 2$, let n_i, c_i, p_i, e_i denote the number of vertices, crossing edges, plain edges, and exceptional edges of G_i.

Now $9p_1 + 10e_1 + 7c_1 \geq 20n_1 - 30$ and $9p_2 + 10e_2 + 7c_2 \geq 20n_2 - 30$, where $e_1 + e_2 + 1 = e$, $n_1 + n_2 - 1 = n$ and $p_1 + p_2 = p$, $c_1 + c_2 = c$. Therefore, $9p + 10e - 10 + 7c \geq 20n + 20 - 60$, and the statement follows.

We may now assume $e = 0$, and we should prove $9p + 7c \geq 20n - 30$, where $n \geq 4$. In what follows, we define an increasing sequence of subgraphs $G_0 \subset G_1 \subset \cdots \subset \hat{G}$ recursively and keep track of the number of vertices and edges of G_i. In every step, we maintain the inequality $9p + 7c \geq 20n - 30$.

Since there are no exceptional edges now, we can use the idea of Brandenburg et al. [2]. They defined the K_4-*network* of G, which is an auxiliary graph $\mathcal{K}$. Its vertex set corresponds to the K_4 subgraphs of $\hat{G}$. Two vertices in $\mathcal{K}$ are adjacent if the corresponding subgraphs in $\hat{G}$ share a vertex. Since $\hat{G}$ is connected and every edge is contained in a K_4, the graph $\mathcal{K}$ is connected. Brandenburg et al. proved a lower bound on the number of edges of $\hat{G}$ by finding a certain spanning tree of $\mathcal{K}$ by an algorithm and investigating the number of edges of $\hat{G}$ involved in each step of the algorithm.

We go in their footsteps, but take a closer look. We use a slightly more complex algorithm that sweeps through $\hat{G}$ rather than $\mathcal{K}$.

Let G_0 be a K_4 subgraph of $\hat{G}$. Suppose that we have already defined G_{i-1}, a connected subgraph of $\hat{G}$, and now we construct G_i. Therefore, the vertices, edges, subgraphs of G_{i-1} are *old* and the ones of G_i are *new*. We assume $9p(G_{i-1}) + 7c(G_{i-1}) \geq 20n(G_{i-1}) - 30$. This clearly holds for $i = 1$. To construct G_i from G_{i-1}, we use one of the following operations in this order of preference.

1. Adding an edge between two old vertices.

2. Adding a new vertex x and all K_4's spanned by x and three old vertices.

3. Adding two new vertices, x and y and all K_4's spanned by x, y and two old vertices.

4. Adding two new K_4's such that they share a new vertex and each of them has a vertex in common with the old subgraph.

If none of these operations can be executed, then let $G_{\text{final}} = G_{i-1}$ and the algorithm terminates.

Observe that K_4 has two different 1-planar drawings. Either all edges are crossing-free, or there is exactly one crossing.

We show that the output of the algorithm G_{final} satisfies $G_{\text{final}} = \hat{G}$. We also show by induction, that $9p(G_i) + 7c(G_i) \geq 20n(G_i) - 30$ for every i. This is certainly true for $i = 0$. Suppose $9p(G_{i-1}) + 7c(G_{i-1}) \geq 20n(G_{i-1}) - 30$ for some i, and now we construct G_i. If we apply operation 1, then the left side of inequality (1) increases by at least 5 (it is the case when we add an edge that crosses a previously plain edge), while the right side does not change, so (1) holds for G_i as well.

Suppose that we executed the second operation and we added exactly one K_4 with new vertex x and old vertices a, b, c, see Figure 4. Now either xa, xb and xc are plain edges in G_i, or one of them, say xc, crosses ab. The other two edges, xa and xb do not cross an old edge since in this case there would be more K_4's involving x. The left side of the inequality (1) increased by 27 or 23 while the right side increased by 20. If we added more than one K_4, then we had to

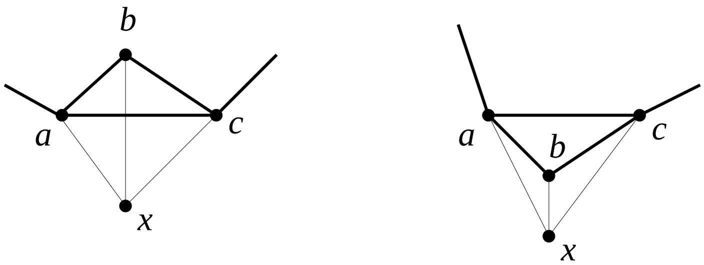

Figure 4: Operation 2.

add at least four edges adjacent to x. The addition of an edge increases the left side of the inequality by at least 5, so it increased by at least 20.

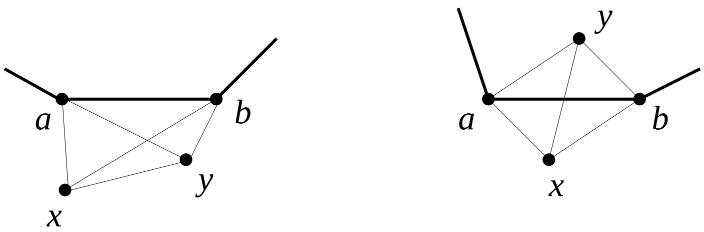

Figure 5: Operation 3.

Suppose that we executed the third operation, see Figure 5. Let x and y be the new vertices, a and b the old vertices of a new K_4. Edges xa, xb, ya, yb cannot cross an old edge, since in that case we find a K_4 with exactly three old vertices contradicting the preference order of the operations. If xy is not crossed by an old edge, then the left side of the inequality (1) increases by 41 or 45, while the right side increases by 40. Suppose that the edge xy crosses an old edge cd. Now x, y, c, d form another K_4, and again none of the other new edges crosses an old edge. We added at least eight new edges, so the left-hand side increased by at least 40 again.

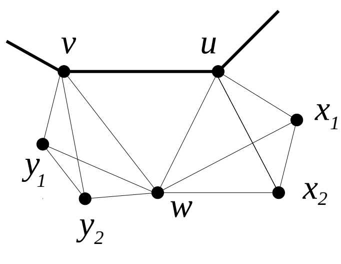

Figure 6: Operation 4.

Suppose now that we arrive to a stage, where we cannot use any of the first three operations. Therefore, there is no K_4 in $\hat{G}$ that has exactly two or three old vertices. Let u be an old vertex that has at least one neighbor not in G_{i-1}. Since $\hat{G}$ is connected, there is such a vertex. The graph G_{i-1} is also connected, so u has a neighbor in G_{i-1} as well. Order all neighbors of u in the circular order the corresponding edges emanate from u.

Recall that edges with a common endpoint do not cross. Let v and w be consecutive neighbors of u such that $v \in G_{i-1}$ and $w \notin G_{i-1}$. We distinguish four cases.

Case 1: Both uv and uw are plain edges in $\hat{G}$. We consider a K_4 that contains the edge uw. By the assumptions, this K_4 has exactly one vertex (u) in G_{i-1} and three vertices, say w, x_1, and x_2 not in G_{i-1}. The vertices v and w can be connected along uv and uw, so by the maximality of $\hat{G}$, they are adjacent in $\hat{G}$. We consider a K_4 that contains the edge vw. By the assumptions, this K_4 has exactly one vertex (v) in G_{i-1} and three vertices not in G_{i-1}: w, y_1, and y_2. By the assumptions, none of the new edges crosses an old edge. If x_1, x_2, y_1, y_2 are all different, then we added five new vertices so the right-hand side of (1) increased by 100, see Figure 6. The left-hand side increased by at least 100 since adding two crossing K_4's means 4 crossing and 8 plain edges and $4 \cdot 7 + 8 \cdot 9 = 100$. If $x_i = y_j$ for some i, j, then the situation is even better, the calculation is very similar.

Case 2: The edge uv is plain in $\hat{G}$ and uw is crossing. In this case, uw is crossed by an edge ab of $\hat{G}$. Now a, b, u, w form a K_4, so a and b are not in G_{i-1}. Vertex v can be connected to a or b along uv, uw, and ab. Suppose that it is a, so by the maximality of $\hat{G}$, v and a are adjacent. Consider a K_4 that contains edge va and the one with vertices a, b, u, w. The calculation is very similar to the previous case.

Case 3: The edge uw is plain in $\hat{G}$ and uv is crossing. Let ab be the edge that crosses uv. Now a, b, u, v form a K_4, and u, v are old vertices, so a and b are also old vertices. Vertex w can be connected to a or b, say, a, along uw, uv, and ab. So again by the maximality of $\hat{G}$, w and a are adjacent. Consider a K_4 that contains wa and a K_4 that contains uw. The calculation is similar to the previous cases.

Case 4: Both uv and uw are crossing edges in $\hat{G}$. This case is the combination of the previous two cases. Edge uw is crossed by ab, uv is crossed by cd. Now one of a and b, say, a, and one of c and d, say, c, can be connected along ab, uv, uw, cd, so they are connected. Take the K_4 formed by u, w, c, d, and a K_4 the contains ac. The calculation is the same again.

In summary, we proved that we can always apply one of the four operations in our algorithm, so the algorithm terminates when $G_{\text{final}} = \hat{G}$. On the other hand, we also proved $9p(G_i) + 7c(G_i) \geq 20n(G_i) - 30$ for every i. Therefore $9p(\hat{G}) + 7c(\hat{G}) \geq 20n(\hat{G}) - 30$. This concludes the proof of Lemma 7. □

3.1 Proof of Theorem 2

Recall that $e(n)$ ($e'(n)$) is the minimum number of edges of a maximal 1-planar (1-plane) graph with n vertices. Since every maximal 1-planar graph with any 1-planar drawing is a maximal 1-plane graph, $e(n) \geq e'(n)$. Therefore, Theorem 2 follows immediately from the next result.

Theorem 3 *Every maximal N-vertex 1-plane graph has at least $\frac{20}{9}N - \frac{10}{3}$ edges, where $N \geq 4$.*

Proof. Let G be a maximal 1-plane graph, N and E denote the number of vertices and edges, and h denotes the number of hermits. Let $\hat{G}$ be the skeleton of G and let $n = n(\hat{G})$, $c = c(\hat{G})$, $p = p(\hat{G})$, $e = e(\hat{G})$ denote the number of vertices, crossing edges, plain edges, and exceptional edges of $\hat{G}$.

Every hermit is surrounded by two pairs of crossing edges. A crossing pair of edges can participate in four such surroundings, on the four sides of the crossing. This gives us $c \geq h$. On the other hand, for each exceptional edge, each of the two neighboring cells has a pair of crossing edges on its boundary, these two crossings cannot participate in a surrounding of a hermit in that direction. This shows $c \geq e$ and $c - e \geq h$. Now $N = n + h$, $E = p + e + c + 2h$.

Let us minimize

$$F(p, e, c, h, n) = E - \frac{20}{9}N$$

under the conditions

$$c - e \geq h, \tag{2}$$

$$9p + 10e + 7c \geq 20n - 30, \tag{3}$$

and

$$p, e, c, h, n \geq 0 \tag{4}$$

$F(p, e, c, h, n) = E - \frac{20}{9}N = p + e + c + 2h - \frac{20}{9}n - \frac{20}{9}h = p + e + c - \frac{20}{9}n - \frac{2}{9}h.$

First we apply the following transformation:
$e' = e - \frac{9}{10}\varepsilon$, $p' = p + \varepsilon$, $h' = h + \frac{9}{10}\varepsilon$, $c' = c$, $n' = n$.

Notice that if conditions (2) and (3) hold for (p, e, c, h, n), then they also hold for (p', e', c', h', n'). On the other hand, $F(p', e', c', h', n') = F(p, e, c, h, n) - \frac{1}{10}\varepsilon$. Therefore, the five-tuple (p, e, c, h, n) that minimizes $F(p, e, c, h, n)$ under conditions (2) and (3) has $e = 0$.

For parameter h, the only condition is that $c \geq h$. If $c > h$ and we increase h, then $F(p, 0, c, h, n)$ decreases, and the conditions still hold. Therefore, we may assume $c = h$. Now we have to minimize $F(p, 0, c, c, n) = p + \frac{7}{9}c - \frac{20}{9}n$ under the condition $9p + 7c \geq 20n - 30$. We get immediately that the minimum of $F(p, 0, c, c, n)$ under the conditions is $-\frac{10}{3}$. Consequently $E - \frac{20}{9}N \geq -\frac{10}{3}$.

Therefore, $E \geq \frac{20}{9}N - \frac{10}{3}$ for any maximal 1-planar drawing with $N \geq 4$ vertices and E edges. □

Remark. We believe that our bound is far from optimal. If our bound was close to optimal, then for some maximal 1-plane graph we would have to use operation 4. in almost every step of the algorithm described in the proof of Lemma 7. However, this seems impossible.

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