Algorithm for positive realization of transfer functions

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Abstract

The aim of this brief is to present a finite-step algorithm for the positive realization of a rational transfer function $H(z)$. In comparision with previously described algorithms we emphasize that we *do not make an a priori assumption on* (but, instead, include a *finite step procedure for checking*) the nonnegativity of the impulse response sequence of $H(z)$. For primitive transfer functions a new method for reducing the pole order of the dominant pole is also proposed.

Keywords

Positive linear systems, impulse response sequence, positive realization.

I. Introduction

The positive realization problem of a given rational transfer function $H(z)$ of a discrete time-invariant linear system is to find a triple $A \in \mathbb{R}_+^{N\times N}$, $b \in \mathbb{R}_+^N$, $c \in \mathbb{R}_+^N$ (with nonnegative entries) such that $H(z) = c^T(zI - A)^{-1}b$ holds. The nonnegativity restriction on the entries of A, b, c reflect *physical constraints* in applications. Such positive systems appear for example in modelling bio-systems, chemical reaction systems, and socio-economic systems, as described in detail in the monograph by Farina and Rinaldi [5]. A recent application of positive systems in the construction of CRN's (Charge Routing Networks) was given by Benvenuti, Farina and Anderson [2].

In [9] Ohta, Maeda and Kodama reduced the problem of positive realizability of $H(z)$ to finding an appropriate convex polyhedral cone in the room sandwiched by the reachability and observability cones in the state space of an arbitrary minimal realization of $H(z)$. However, the problem of constructing such a polyhedral cone turned out to be highly nontrivial (a characterization of *all* such cones is still lacking). In [1] Anderson, Deistler, Farina and Benvenuti proved that such a cone is always possible to construct if $H(z)$ is a primitive transfer function with nonnegative impulse response. Finally, the case of nonprimitive transfer functions $H(z)$ with nonegative impulse response was settled by Farina in [4] by the method of downsampling the impulse response of $H(z)$ (see also Kitano and Maeda [8], and Förster and Nagy [6]). A common feature of the results mentioned above is that the impulse response of $H(z)$ is *assumed* to be nonnegative. However, it has not been shown so far how one can *check, in finite steps, the nonnegativity of the (clearly infinite) impulse response sequence* corresponding to $H(z)$ (this open problem was raised in [1]).

In the course of this brief we aim to supplement the theory of positive realizability by tackling this problem in Section 2. In Section 3 we propose a new method of constructing a positive realization of a primitive transfer function with multiple dominant pole. In Section 4 we illustrate our results by an example.

II. The nonnegativity of the impulse response sequence

Let

$$H(z) = \frac{p_1 z^{n-1} + ... + p_n}{z^n + q_1 z^{n-1} + ... + q_n} = \sum_{j=1}^{r} \sum_{i=1}^{n_j} \frac{c_j^{(i)}}{(z-\lambda_j)^i} \tag{1}$$

be a strictly proper rational transfer function, where λ_1 denotes *the nonnegative pole of* $H(z)$ *with greatest modulus.* Note that the *coefficients* p_j and q_j are assumed to be *real*, but the *poles* λ_j $(j \neq 1)$ can be *complex.* We will use the notation h_k $(k = 1, 2, 3...)$ for the *impulse response sequence* of $H(z)$, i.e. $H(z) = \sum_{k=1}^{\infty} h_k z^{-k}$. A minimal realization of $H(z)$ will be denoted by (g, F, h). We will describe the structure of our algorithm in several steps. The first two steps are standard in the theory of positive realizations (cf. [1] and [4]), but we include them for completeness and convenience.

Step 1. It is well-known (cf. [1]) that a necessary condition for the existence of a positive realization of $H(z)$ is that λ_1 be a dominant pole of $H(z)$, i.e. the modulus of no pole exceed λ_1. Therefore, if there is no nonnegative pole of $H(z)$, or λ_1 is not dominant, we can conclude that $H(z)$ does not have positive realizations. *Assume that* λ_1 *is dominant.* If $\lambda_1 = 0$, then the realization problem is trivial, and if $\lambda_1 > 0$, then it is also well-known that we may (and will) *assume without loss of generality that* $\lambda_1 = 1$ (cf. [1]).

Step 2. If $H(z)$ is not primitive (i.e. $H(z)$ has dominant poles other than $\lambda_1 = 1$), then a necessary condition for the existence of a positive realization is that *the dominant poles of* $H(z)$ *be cyclic* (see [1]), i.e. there exist $p \in \mathbb{N}$ such that all the dominant poles of $H(z)$ satisfy the equation $z^p = 1$. If the dominant poles are not cyclic, then we conclude that there is no positive realization of $H(z)$. If the dominant poles are cyclic with index p (the smallest of the p's above), then the necessary and sufficient condition for positive realizability of $H(z)$ is that *all the "downsampled" transfer functions* $H_{(j)}(z) := g(zI - F^p)^{-1} F^j h$ (for $j = 0, 1, ..., p-1$) be positively realizable (see [4] and [8]). Notice that $H(z) = \sum_{j=0}^{p-1} z^{p-1-j} H_{(j)}(z^p)$.

If some of the functions $H_{(j)}(z)$ are not primitive, then we apply the downsampling step again to these functions (provided that they are cyclic), and, following the algorithm of [4] (cf. also [8]), we arrive (after a finite number of steps) at a decomposition of the form

$$H(z) = \sum_s z^{\beta_s} H_{(s)}(z^{\alpha_s}) \tag{2}$$

where $0 \leq \beta_s < \alpha_s$, and all the functions $H_{(s)}(z)$ are either primitive or not cyclic. If for any s the function $H_{(s)}(z)$ is not cyclic, then we conclude that $H(z)$ does not have positive realizations (cf. [4]). Assume therefore that all $H_{(s)}(z)$ are primitive. In this case, $H(z)$ is positive realizable if and only if the impulse response of $H(z)$ is nonnegative (cf. [4]).

Step 3. In this step we give an *upper estimate on the finite number of the terms of the impulse response sequence* h_k of $H(z)$ whose nonnegativity we need to check in order to conclude that the *whole impulse response sequence* is nonnegative.

Instead of checking the impulse response of $H(z)$ directly, we take the decomposition $H(z) = \sum_s z^{\beta_s} H_{(s)}(z^{\alpha_s})$ of Step 2, and check the impulse response of each $H_{(s)}(z)$. If the impulse response of each $H_{(s)}(z)$ is non-negative, then clearly so is the impulse response of $H(z)$. The advantage of this method is that all the functions $H_{(s)}(z)$ are primitive.

For the sake of simplicity we will still use the notation $H(z)$ instead of $H_{(s)}(z)$, but we will assume that $H(z)$ is primitive.

We shall use several times the following *observation*: if a transfer function $H(z)$ has the form $H(z) = \sum_{k=1}^{n} \frac{e_k}{(z-s)^k}$, then the impulse response sequence is

$$h_m = \sum_{k=1}^{n} \binom{m-1}{k-1} s^{m-k} e_k \qquad m = 1, 2, 3, \ldots, \tag{3}$$

(with the convention that for $\alpha < \beta$ we define $\binom{\alpha}{\beta} := 0$, and $\binom{0}{0} := 1$ and $0^0 := 1$). This can be proved by using for each $k = 1, 2, \ldots n$ and $|z| > |s|$ the formula $\frac{1}{(z-s)^k} = (\frac{1}{z})^k [1 + \frac{s}{z} + (\frac{s}{z})^2 + (\frac{s}{z})^3 + \ldots]^k$. Since we have $\sum_{k=1}^{n} \frac{e_k}{(z-s)^k} = H(z) = \sum_{m=1}^{\infty} \frac{h_m}{z^m}$, comparing coefficients yields the stated formula.

Recall now that for our transfer function $H(z)$ the only dominant pole is $\lambda_1 = 1$. If $c_1^{(n_1)} < 0$, then it is clear (see the explicit formula for $h_{k,1}$ and the estimates on h_k below) that for large k we have $h_k < 0$, hence there exists no positive realization of $H(z)$. Therefore we can assume that $c_1^{(n_1)} > 0$, and, without loss of generality, that $c_1^{(n_1)} = 1$.

We use the notation $h_{k,j}^{(i)}$ for the impulse response sequence corresponding to the function $\frac{c_j^{(i)}}{(z-\lambda_j)^i}$, and we let $h_{k,j} := \sum_{i=1}^{n_j} h_{k,j}^{(i)}$. Hence we shall have $h_k = \sum_{j=1}^{r} h_{k,j}$. The idea behind the forthcoming calculations is that $h_{k,1}$ turns out to be 'dominant' in the long term behaviour of h_k, and $h_{k,1}^{(n_1)}$ will be 'dominant' in the long term behaviour of $h_{k,1}$.

First we find an index N_0 such that $h_{k,1} \geq 1$ for all $k \geq N_0$. If $n_1 = 1$, then $h_{k,1} = 1$ for all $k \geq 1$, therefore we can take $N_0 = 1$. Assume $n_1 > 1$. Since we have $\lambda_1 = 1$ and $c_1^{(n_1)} = 1$, it follows from (3) that

$$\begin{gathered} h_{k,1} = \binom{k-1}{n_1-1} + c_1^{(n_1-1)}\binom{k-1}{n_1-2} + ... + c_1^{(2)}\binom{k-1}{1} + c_1^{(1)} = \\ 1 + [\binom{k-1}{n_1-1} + c_1^{(n_1-1)}\binom{k-1}{n_1-2} + ... + c_1^{(2)}\binom{k-1}{1} + c_1^{(1)} - 1]. \end{gathered}$$

Let $C := \max\{|c_1^{(n_1-1)}|, \ldots, |c_1^{(2)}|, |c_1^{(1)}|+1\}$, and assume that $k \geq (n_1C+1)(n_1-1) =: N_0$. (Note that $N_0 \geq 3n_1 - 3 \geq 2n_1 - 1$, so the finite sequence of the binomial coefficients $\binom{k-1}{n_1-1}, \binom{k-1}{n_1-2}, ..., \binom{k-1}{0}$ is strictly monotonically decreasing.) For k this large we have $\frac{k-n_1+1}{n_1-1} \geq n_1C$ and this means that $\binom{k-1}{n_1-1}/\binom{k-1}{n_1-2} \geq n_1C$. Hence for any $2 \leq j \leq n_1$ we have $\binom{k-1}{n_1-1}/\binom{k-1}{n_1-j} \geq n_1C$. This means that

$$\begin{gathered} h_{k,1} = 1 + [\binom{k-1}{n_1-1} + c_1^{(n_1-1)}\binom{k-1}{n_1-2} + ... + c_1^{(2)}\binom{k-1}{1} + c_1^{(1)} - 1] \geq \\ 1 + \binom{k-1}{n_1-1} - C\sum_{i=1}^{n_1-1}\binom{k-1}{i-1} \geq 1, \end{gathered}$$

as desired.

Next we find an index M_0 such that $\sum_{j=2}^{r} |h_{k,j}| \leq 1$ for all $k > M_0$. It follows from (3) that $h_{k,j} = \sum_{i=1}^{n_j} h_{k,j}^{(i)} = \sum_{i=1}^{n_j} c_j^{(i)}\lambda_j^{k-i}\binom{k-1}{i-1}$. Therefore

$$|h_{k,j}| \leq \sum_{i=1}^{n_j} |h_{k,j}^{(i)}| \leq \sum_{i=1}^{n_j} |c_j^{(i)}||\lambda_j|^{k-i}\binom{k-1}{i-1}.$$

Now, there are altogether $N_1 := \sum_{j=2}^{r} n_j$ coefficients of type $h_{k,j}^{(i)}$ so it is enough to ensure that the modulus of each of them is not greater than $1/N_1$. That is, we want $|c_j^{(i)}||\lambda_j|^{k-i}\binom{k-1}{i-1} \leq \frac{1}{N_1}$ to hold. If $\lambda_j = 0$, then this is obviously true for $k \geq i+1$. Assume $\lambda_j \neq 0$. To simplify forthcoming calculations we use the notation $\rho := \max\{|\lambda_j| : j = 2, 3, \ldots, r\}$, $\gamma := \max\{|c_j^{(i)}| : j = 2, 3, \ldots, r;\ i = 1, 2, \ldots, n_j\}$ and $\eta := \max\{n_j : j = 2, 3, \ldots, r\}$. The desired inequality $|c_j^{(i)}||\lambda_j|^{k-i}\binom{k-1}{i-1} \leq \frac{1}{N_1}$ is implied by $\gamma\rho^{k-i}\binom{k-1}{i-1} \leq \frac{1}{N_1}$, which is equivalent to $\rho^{k/2}\rho^{k/2}\binom{k-1}{i-1} \leq \frac{\rho^i}{N_1\gamma}$. It is easy to check that for fixed i the value of $\rho^{k/2}\binom{k-1}{i-1}$ is monotonically decreasing (in k) for $k \geq \frac{i-1}{1-\rho^{1/2}} + 1 =: N^{(i)}$.

We use the notation $C^{(i)} := \rho^{(N^{(i)}/2)}\binom{N^{(i)}-1}{i-1}$. If $k \geq N^{(i)}$, then it is sufficient that

$$(\rho^{1/2})^k \leq \tfrac{\rho^i}{N_1\gamma C^{(i)}} =: K^{(i)},$$

therefore we can take

$$M_0 := \max\{\eta, N^{(i)},\ \log_{\rho^{1/2}} K^{(i)},\ i = 1, 2, \ldots \eta\}$$

(Note that we include η because we must not forget about the possible pole at 0.)

This means that if $k > \max\{N_0, M_0\}$, then $h_k \geq h_{k,1} - \sum_{j=2}^{r} |h_{k,j}| \geq 0$, therefore it is definitely enough to check the nonnegativity of the first $K_0 := \max\{N_0, M_0\}$ terms of the impulse response sequence. We remark that the calculations above show that *it is sufficient to know an upper bound on the values of* ρ *and* γ instead of the exact values. This means that it is enough to determine the approximate locations of the poles, and the approximate values of the partial fraction coefficients, and Step 3 can already be applied.

III. A nonnegative realization

Step 4. In order to construct a positive realization of $H(z)$ it is sufficient to find a positive realization for each $H_{(s)}(z)$ in the decomposition (2), and then apply the method of [4] or [8]. As all $H_{(s)}(z)$ are primitive, we could find a positive realization of $H_{(s)}(z)$ by applying the results of [1] directly. However, in this Step we propose a method for reducing the pole order of the dominant pole, and apply the construction of [1] only when the dominant pole is simple. This step seems to simplify the construction of [1] in the case when $H_{(s)}(z)$ has a multiple dominant pole.

Again, for the sake of simplicity we will use the notation $H(z)$ instead of $H_{(s)}(z)$, and we will assume that $H(z)$ is primitive with nonnegative impulse response.

Denote in this Step the transfer function corresponding to the "shifted" impulse response sequence $(h_k, h_{k+1}, \ldots)$ by $H_k(z)$, i.e.

$$H_k(z) := \sum_{j=1}^{\infty} h_{j+k-1} z^{-j}.$$

(Note that $H(z) = H_1(z)$.) First we make the following observation:

If $H_k(z)$ has a nonnegative realization in N dimensions (for some $k > 1$), then so does $H_{k-1}(z)$ in $N+1$ dimensions. (For an easy proof see [7].)

In order to construct a nonnegative realization of $H(z)$ we take the following guidelines:

We find a positive integer k_1 so that we can construct a nonnegative realization of $H_{k_1}(z)$ in some dimensions N, and then use the obsevation above to construct a nonnegative realization of $H(z)$ in dimensions $N+k_1-1$. The index k_1 will be chosen so that

$$H_{k_1}(z) = [\sum_{m=1}^{n_1} \frac{e_m}{(z-1)^m}] + [\frac{d_1}{z-1} + \sum_{j=2}^{r} \sum_{i=1}^{n_j} \frac{e_j^{(i)}}{(z-\lambda_j)^i}] =: f^{(3)}(z) + f^{(4)}(z)$$

holds, where $e_m \geq 0$ for all $1 \leq m \leq n_1$ and *the whole impulse response sequence of* $f^{(4)}(z)$ *is nonnegative*. Here $f^{(3)}(z)$ has a trivial positive realization in n_1 dimensions, and $f^{(4)}(z)$ is primitive with a simple dominant pole and nonnegative impulse response, so that the construction of [1], Theorem 4.1 can be applied. We remark that the index k_1 is not uniquely determined. In order to minimize the dimension of the realization it is important to determine the *optimal* value of k_1. This may be easy to do for a particular transfer function $H(z)$, but a general formula for the optimal value of k_1 does not seem possible to find. The proof below shows only that such an index k_1 always exists.

Assume that $n_1 > 1$. Notice that

$$H_k(z) = \sum_{j=1}^{\infty} h_{k+j-1} z^{-j} = z^{k-1} H(z) - \sum_{r=1}^{k-1} h_r z^{k-r-1}.$$

In particular, the formula $H_2(z) = zH(z) - h_1 = (z-1)H(z) + H(z) - h_1$ shows that in the partial fraction decomposition of $H_2(z)$ the part corresponding to $\lambda_1 = 1$ is given by

$$\frac{1}{(z-1)^{n_1}} + \sum_{j=1}^{n_1-1} \frac{c_1^{(j)} + c_1^{(j+1)}}{(z-1)^j}$$

From this it follows by mathematical induction with respect to k that the partial fraction part corresponding to $\lambda_1 = 1$ in $H_k(z)$ is given by

$$\frac{1}{(z-1)^{n_1}} + \sum_{j=1}^{n_1-1} \frac{\sum_{i=j}^{n_1} \binom{k-1}{i-j} c_1^{(i)}}{(z-1)^j}. \tag{4}$$

If $k \geq N_0$ (as in Step 3), then $\binom{k-1}{j} / \binom{k-1}{j-1} \geq n_1 C$ for all $j = 1, 2, ..., n_1 - 1$. Therefore, we see as in Step 3 that for $k \geq N_0$ all the numerators in (4) are not less than 1. Therefore,

$$H_{N_0}(z) = [\sum_{m=2}^{n_1} \frac{d_m}{(z-1)^m}] + [\frac{d_1}{z-1} + \sum_{j=2}^{r} \sum_{i=1}^{n_j} \frac{d_j^{(i)}}{(z-\lambda_j)^i}] =: f^{(1)}(z) + f^{(2)}(z).$$

where $d_m \geq 1$ for all $1 \leq m \leq n_1$ (but the first few terms of the impulse response sequence of $f^{(2)}(z)$ may be negative!). We can now apply the estimates of Step 3 to the function $f^{(2)}(z)$ and obtain a number $\tilde{K}_0$ such that the impulse response of the function

$f^{(2)}(z)$ becomes nonnegative if $k \geq \tilde{K}_0$. This means that we can take $k_1 = N_0 + \tilde{K}_0$, and the desired decomposition $H_{k_1}(z) = f^{(3)}(z) + f^{(4)}(z)$ holds. Now, the construction of [1] Theorem 4.1 applies to $f^{(4)}(z)$, and we can combine the positive realizations of $f^{(3)}(z)$ and $f^{(4)}(z)$ to get a positive realization of $H_{k_1}(z)$. Then we obtain a positive realization of $H(z)$ by applying the observation above.

We remark that an *upper bound on the dimension* of the positive realization constructed in [1], Theorem 4.1 has not yet been presented. Such dimension estimate is possible to prove, but the proof is fairly long and mathematically involved. It will be presented in a forthcoming publication.

IV. Example

We illustrate the steps of the algorithm by the following example. Let

$$H(z) = \frac{1+z-\frac{1}{4}z^2}{z^3-1} + \frac{1/3}{(z-1)^2} + \frac{2/3}{z-1/2} + \frac{2/3}{z+1/2} - \frac{1}{z+0.9}.$$

Then $p = 3$, and the downsampled functions are

$$\begin{aligned} H_{(0)}(z) &= \frac{-1/4}{z-1} + \frac{1}{(z-1)^2} + \frac{2/3}{z-1/8} + \frac{2/3}{z+1/8} - \frac{1}{z+0.729} \\ H_{(1)}(z) &= \frac{4/3}{z-1} + \frac{1}{(z-1)^2} + \frac{1/3}{z-1/8} - \frac{1/3}{z+1/8} + \frac{0.9}{z+0.729} \\ H_{(2)}(z) &= \frac{5/3}{z-1} + \frac{1}{(z-1)^2} + \frac{1/6}{z-1/8} + \frac{1/6}{z+1/8} - \frac{0.81}{z+0.729}. \end{aligned}$$

Next we check the nonnegativity of the impulse response of $H(z)$ by checking each $H_{(s)}(z)$ $(s = 0, 1, 2)$. For $s = 0$ we have $n_1 = 2$, $C = 5/4$ and $N_0 = 4$. Further, following the definitions, we have $N_1 = 3$, $\rho = 0.729$, $\gamma = 1$, $\eta = 1$, $N^{(1)} = 1$, $C^{(1)} = \sqrt{0.729}$, $K^{(1)} = \frac{\sqrt{0.729}}{3}$, and $\log_{\rho^{1/2}} K^{(1)} = 7.95$. Therefore it is sufficient to check the first $K_0 = 8$ terms of the impulse response sequence of $H_{(0)}(z)$. Following similar calculations we deduce that it is enough to check the nonnegativity of the first 8 terms of the impulse response sequence of $H_{(1)}(z)$, and the first 7 terms in the case of $H_{(2)}(z)$.

Next, we construct a positive realization for each $H_{(s)}(z)$ $(s = 0, 1, 2)$. In the case of $H_{(0)}(z)$ we apply Step 4 with $k_1 = 1$. The 'shifted' transfer function is given by $H_2(z) = [\frac{1}{(z-1)^2}] + [\frac{3/4}{z-1} + \frac{1/12}{z-1/8} - \frac{1/12}{z+1/8} + \frac{0.729}{z+0.729}] = f^{(3)}(z) + f^{(4)}(z)$

Now, a positive realization for $f^{(3)}(z)$ and $f^{(4)}(z)$ is given by the triplets (c_3, A_3, b_3) and (c_4, A_4, b_4), respectively, where

$$c_3 = \begin{pmatrix} 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$c_4 = \begin{pmatrix} 1 & 0 & 1.972 & 0.291412 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.729 \\ 0 & 0 & 1 & 0.271 \end{pmatrix}, b_4 = \begin{pmatrix} 0 \\ 1/48 \\ 0.75 \\ 0 \end{pmatrix}.$$

Now we apply the observation of Step 4 (cf. [7]), and give a positive realization of $H_{(0)}(z)$ by

$$c_0 = \begin{pmatrix} 1/12, & c_3, & c_4 \end{pmatrix}, A_0 = \begin{pmatrix} 0 & 0 & 0 \\ b_3 & A_3 & 0 \\ b_4 & 0 & A_4 \end{pmatrix}, b_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The cases of $H_{(1)}(z)$ and $H_{(2)}(z)$ are 'trivial' because we can take $k_1 = 0$ in Step 4, and write

$$\begin{aligned} H_{(1)}(z) &= [\tfrac{1}{(z-1)^2}] + [\tfrac{4/3}{z-1} + \tfrac{1/3}{z-1/8} - \tfrac{1/3}{z+1/8} + \tfrac{0.9}{z+0.729}] \\ H_{(2)}(z) &= [\tfrac{1}{(z-1)^2}] + [\tfrac{5/3}{z-1} + \tfrac{1/6}{z-1/8} + \tfrac{1/6}{z+1/8} - \tfrac{0.81}{z+0.729}]. \end{aligned}$$

In both cases positive realizations of dimension 6 can be given as follows:

$$c_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1.675 & 0.507925 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.729 \\ 0 & 0 & 0 & 0 & 1 & 0.271 \end{pmatrix}, b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1/12 \\ 4/3 \\ 0 \end{pmatrix}$$

$$c_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0.514 & 1.354294 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.729 \\ 0 & 0 & 0 & 0 & 1 & 0.271 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 1 \\ 1/3 \\ 0 \\ 5/3 \\ 0 \end{pmatrix}$$

A positive realization of the original $H(z)$ is then possible to construct as in [4] or [8].

V. Conclusions

In this brief we provided a general *finite step procedure* for checking the nonnegativity of the impulse response sequence of $H(z)$, which answers an open problem raised in [1]. For primitive transfer functions a new method of positive realization was proposed by reducing the pole order of the dominant pole.

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