Minimum average-case queries of $q+1$-ary search game with small sets

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Abstract

Given a search space $S = \{1, 2, \ldots, n\}$, an unknown element $x^* \in S$ and fixed integers $\ell \geq 1$ and $q \geq 1$, a $q+1$-ary ℓ-restricted query is of the following form: Which one of sets $\{A_0, A_1, \ldots, A_q\}$ is the x^* in? where $(A_0, A_1, \ldots, A_q)$ is a partition of S and $|A_i| \leq \ell$ for $i = 1, 2, \ldots, q$. The problem of finding x^* from S with $q+1$-ary size-restricted queries is called as a $q+1$-ary search game with small sets. In this paper, we consider sequential algorithms for the above problem, and establish the minimum number of average-case sequential queries when x^* satisfies uniform distribution on S.

Keywords: Search game; Sequential algorithm; Average-cost; Huffman tree.

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1. Introduction

The following combinatorial search problem has been extensively researched and many useful results have been obtained (see [1, 6]). Two players, say Paul and Carole, fix a search space $S = \{1, 2, \ldots, n\}$, an integer $q \geq 1$. A $q+1$-ary query is of the form: Which one of sets $\{A_0, A_1, \ldots, A_q\}$ is the x^* in?, where $(A_0, A_1, \ldots, A_q)$ is a partition of S. An answer of the query i indicates that $x^* \in A_i$. Carole chooses $x^* \in S$ as an unknown element (or a target element), and Paul is required to find x^* by asking queries. The main goal of the above problem is to design optimal algorithms that finds x^* with minimum queries. Traditionally, two measures are commonly utilized to estimate the efficiency of an algorithm: the *worst-case* number of queries and the *average-case* number of queries, and two classes of algorithms are usually considered: *sequential algorithms (or adaptive algorithms)* and *predetermined algorithms (or non-adaptive algorithms).*

We consider a natural variant of the above problem. The queries will be restricted with the condition $A_i \leq \ell$ for $i = 1, 2, \ldots q$ and a fixed integer $\ell \geq 1$. Call it a $q+1$-ary search game with small sets and denote it by $(S, q+1, A_{\leq \ell})$. A query that satisfies the above condition is called to be ℓ-admissible. An admissible algorithm of $(S, q+1, A_{\leq \ell})$ is said to be an ℓ-admissible for short.

The problem $(S, q+1, A_{\leq \ell})$ was originally raised by Rényi [3] with stricter conditions on the queries, i.e., $A_i \leq K_i$ for fixed integers $K_i > 0$ and $i = 1, 2, \ldots q$. We denote it as $(S, q+1, A_{\leq})$ throughout the paper.

For $(S, 2, A_{\leq \ell})$, an admissible query is equivalent to the following: Is $x^* \in A$?, where $A \in S$ and $|A| \leq \ell$. Aigner [1] presents a worst-case optimal sequential algorithm, and an average-case optimal

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one when uniform distribution of x^* on S is assumed. The worst-case lower and upper bounds of predetermined algorithms are presented in [7], and improved by many scholars (see [8, 11, 16]).

For $(S, 3, A_{\leq \ell})$, the problem is modeled as a coin-weighting problem, i.e., the queries are restricted with $|A_1| = |A_2| \leq \ell$. Aigner [1] presents the minimum number of worst-case sequential weighings and a lower bound for predetermined weighings. Liu et.al. [10] establish the minimum number of average-case sequential weighings when the uniform distribution is assumed.

For the general game $(S, q+1, A_{\leq \ell})$, based on the model $(S, q+1, A_{\leq})$, Baranyai [3] generalized the results of [7] for the case $q = 1$. He gave lower and upper estimations of the minimum predetermined worst-case number of queries and prove that the estimations are asymptotically optimal.

To the best of our knowledge, how to construct an optimal sequential algorithm for the general game $(S, q+1, A_{\leq \ell})$ is an open problem. In this paper, we focus on the average-case of the problem $(S, q+1, A_{\leq \ell})$ and present an average-case optimal sequential algorithm when uniform distribution on S is assumed. In fact, the algorithm is also worst-case optimal‘and the minimum worst-case sequential queries can be easily obtained from the results given in this paper.

The rest of this paper is structured as follows. In Section 2, we give some notations and main results of this paper. Section 3 proposes some lower bounds of the minimum number of average-case ℓ-admissible sequential queries. Section 4 constructs an average-case optimal ℓ-admissible sequential algorithm when the uniform distribution on S is assumed. In Section 5, we prove the main results given in Section 2 and take an example to verify it. In the end, we conclude this paper and point out future research directions.

2. Notations and main results

A sequential search process can be formalized as a directed tree that satisfies the following conditions (see reference [1]). (1) Each leaf is uniquely assigned with an element of the search space $S = \{1, 2, \ldots, n\}$, each inner node is assigned with a query, and the corresponding edge indicates its answers to its father node. (2)Let $A^j = (A_0^j, A_1^j, \ldots, A_q^j,)$ be queries for $1 \leq j \leq p-1$ and $A^p = \{i\}$, if A^1 is the root and $A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} \ldots \xrightarrow{f_{p-1}} A^p$ is a path to the leaf i, where f_k is the label of the edge (A^k, A^{k+1}) for $1 \leq k \leq p-1$, then $\{i\} = \bigcap_{k=1}^{p-1} A_{f_k}^k$.

A node A is induced with respect to answer i if A has a father node B and the edge (B, A) is labeled with i. Let $h(T, i)$ be the path length from the root to node i in the tree T, $L(T) = \max\{h(T, i) | i \in S$ is a leaf in the tree $T\}$ and $h(T) = \sum_{i \in S} h(T, i)$, which are the number of sequential queries needed to find i, the worst-case length and the *external path length* for algorithm T respectively. Therefore the minimum number of worst-case sequential queries $L(n) = \min\{L(T) | T$ is an admissible tree with n leaves$\}$, and the minimum number of average-case sequential queries $\bar{L}(n) = \frac{H(n)}{n}$, where $H(n) = \min\{h(T) | T$ is an admissible tree with n leaves$\}$ and we call it *Huffman path length.*

For the game $(S, q+1, A_{\leq \ell})$, an ℓ-admissible algorithm (or say an ℓ-*admissible* tree,) is a $q+1$-ary directed tree T with the additional restriction: all its subtrees rooted at any node induced with respect to answer $i = 1, 2, \ldots, q$ have at most ℓ leaves. Obviously, an ℓ_1-admissible tree must be an ℓ_2-admissible tree if $\ell_1 \leq \ell_2$. Analogously, for an ℓ-admissible tree T, let $h_{\leq \ell}(T, i)$ represent the path length from root to leaf i, $h_{\leq \ell}(T) = \sum_{i \in S} h_{\leq \ell}(T, i)$ the external path length of T, and $L_{\leq \ell}(T) = \max\{h_{\leq \ell}(T, i) | i \in S\}$ the worst-case length of the algorithm T. Therefore, for the game $(S, q+1, A_{\leq \ell})$ with $|S| = n$, the minimum number of worst-case sequential queries is $L_{\leq\ell}(n) = \min\{L_{\leq\ell}(T)|T$ is an ℓ-admissible tree with n leaves$\}$ and the minimum number of average-case sequential queries is $\bar{L}_{\leq\ell}(n) = \frac{H_{\leq\ell}(n)}{n}$, where $H_{\leq\ell}(n) = \min\{h_{\leq\ell}(T)|T$ is an ℓ-admissible tree with n leaves$\}$.

In other words, for the game $(S, q+1, A_{\leq\ell})$ with $|S| = n$, given its two ℓ-admissible algorithms T_1 and T_2, we say that T_1 is worst-case (or average-case) better than T_2 if $L_{\leq\ell}(T_1) \leq L_{\leq\ell}(T_2)$ (or $H_{\leq\ell}(T_1) \leq H_{\leq\ell}(T_2)$). An algorithm T^* is worst-case (or average-case) optimal if $L_{\leq\ell}(T^*) = L_{\leq\ell}(n)$ (or $H_{\leq\ell}(T^*) = H_{\leq\ell}(n)$). We say *best* (or optimal) for short if it does not cause confusion hereafter.

Let $\lfloor x \rfloor$ and $\lceil x \rceil$ be the maximal integer that $\leq x$ and the minimal integer that $\geq x$ respectively, and $\mu(j) = 1$ for $j \neq 0$ and $\mu(j) = 0$ for $j = 0$. The following theorem summarizes the main results of this paper.

Theorem 1. *Given integers $n > 0$, $\ell > 0$ and $q > 0$, we have*

(1) *For $\ell = 1$, $H_{\leq\ell}(n) = n(t-1) - \frac{(t-1)(t-2)}{2}q + H(n-(t-1)q)$, where $t = \lceil \frac{n}{q} \rceil$.*

(2) *For $\ell > 1$, let $\ell = (q+1)^\eta + \theta q + \tau$ such that $0 \leq \theta < (q+1)^\eta$ and $0 < \tau \leq q$, and $t = \max\{1, \lceil \frac{n-(q+1)^{\eta+1}}{q\ell} \rceil\}$ and*

$$\begin{aligned} &\Omega = \{(\ell, n)|(q+1)^{\eta+1} + q(\ell-\tau) < n-(t-1)q\ell \leq (q+1)^{\eta+1} + q\ell \text{ and } n > (q+1)^{\eta+1}\}, \\ &\varphi = n(t-1) - \tfrac{(t-1)(t-2)}{2}q\ell + (t-1)qH(\ell) + H(n-(t-1)q\ell), \\ &\delta = \begin{cases} \lceil \frac{n-(t-1)q\ell-(q+1)^{\eta+1}-q(\ell-\tau)}{\tau} \rceil - \lceil \frac{n-(t-1)q\ell-(q+1)^{\eta+1}-q(\ell-\tau)}{q} \rceil, & (\ell, n) \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

then

$$H_{\leq\ell}(n) = \varphi + \delta.$$

3. Lower bounds of $H_{\leq\ell}(n)$

Lemma 2. *Given integers $n > 0$ and $q > 0$, then there exist uniquely non-negative integers α, β and γ such that*

$$n = (q+1)^\alpha + \beta q + \gamma$$

where $0 \leq \beta < (q+1)^\alpha$ and $0 \leq \gamma < q$.

Proof. See reference [1].

Obviously, for $n \geq 2$, if we replace the condition $0 \leq \gamma < q$ with $0 < \gamma \leq q$, Lemma 2 holds too.

Lemma 3. *Given integers $n > 0$, $q > 0$ and $\ell > 0$, let $n = (q+1)^\alpha + \beta q + \gamma$ where $0 \leq \beta < (q+1)^\alpha$ and $0 \leq \gamma < q$, we have (1) $L(n) = \lceil \log_{q+1}^n \rceil$; (2) $L_{\leq\ell}(n) \geq L(n)$; (3) $L_{\leq\ell}(m) \geq L_{\leq\ell}(n)$ if $m \geq n$; (4) if $n \leq (q+1)\ell$, then $L_{\leq\ell}(n) = L(n)$.*

Proof. (1) See [1].
(2) We obtain it by the following fact: an ℓ_1-admissible algorithm is an ℓ_2-admissible one if $\ell_1 \leq \ell_2$.
(3) Suppose T^* is an optimal ℓ-admissible worst-case tree with m leaves such that $h(T^*, 1) \leq h(T^*, 2) \leq \ldots \leq h(T^*, m)$. Delete the leaves $\{n+1, \ldots, m\}$ from T^*, and denote the resulting tree as T'. Therefore T' is also an ℓ-admissible tree with n leaves and $L_{\leq\ell}(m) = L_{\leq\ell}(T^*) \geq L_{\leq\ell}(T') \geq L_{\leq\ell}(n)$.
(4) Let $n = x(q+1) + y$ such that $0 < y \leq q+1$ and $n = (q+1)^\alpha + \beta q + \gamma$ such that $0 \leq \beta < (q+1)^\alpha$ and

$0 \le \xi < q$. If $\beta = (q+1)^\alpha - 1$, then $y = r+1$ and $n+q+1-y \le (q+1)^\alpha + q((q+1)^\alpha - 1) + r + q + 1 - y = (q+1)^{\alpha+1}$. If $\beta < (q+1)^\alpha - 1$, then $n+q+1-y \le (q+1)^\alpha + q((q+1)^\alpha - 2) + r + q < (q+1)^{\alpha+1}$. Since $n+q+1-y \ge n$, therefore $1 + L(x+1) = \lceil \log_{q+1} n+q+1-y \rceil = \lceil \log_{q+1} n \rceil$.

Since $n \le (q+1)\ell$, we have $x < \ell$. Suppose T be tree with n leaves and the first query $A^1 = (A_0^1, A_1^1, \dots, A_q^1)$ such that $|A_i^1| = x+1$ for $i = 0, 1, \dots, y-1$ and $|A_i^1| = x$ for $i = y, \dots, q$. It is clear that T is an ℓ-admissible tree. By (3), we have $L_{\le \ell}(n) \le 1 + \max\{L_{\le \ell}(|A_i^1|) | i = 0, 1, \dots, q\} = 1 + L_{\le \ell}(x+1) = 1 + L(x+1) = \lceil \log_{q+1} n + q + 1 - y \rceil = \lceil \log_{q+1} n \rceil = L(n)$. By (2), we have $L_{\le \ell}(n) = L(n)$. □

Since we need no queries to determine the unknown element for $n = 1$, without lose generality, we let $H_{\le \ell}(0) = H_{\le \ell}(1) = H(0) = H(1) = 0$, then we have the following useful results.

Lemma 4. *Given integers $n > 1$, $q > 0$ and $\ell > 0$, let $n = (q+1)^\alpha + \beta q + \gamma$ where $0 \le \beta < (q+1)^\alpha$ and $0 \le \gamma < q$, we have (1) $H(n) = n\alpha + \lceil (q\beta + \gamma)(q+1)/q \rceil = n\alpha + (q+1)\beta + \gamma + \mu(\gamma)$; (2) $H_{\le \ell}(n) \ge H(n)$;*

Proof. (1) The case is a classical Huffman problem and we call a tree T with n leaves that satisfies $H(T) = H(n)$ a Huffman tree. The detailed proof of (1) can be seen in [1, 6] etc.
(2) Suppose T^* is an optimal ℓ-admissible average-case tree with n leaves, by the definition of $H(n)$, we have $H_{\le \ell}(n) = H_{\le \ell}(T^*) = H(T^*) \ge H(n)$. □

In fact, by Lemma 2, for for $n \ge 2$, $H(n)$ has an alternative form as follows. Let $n = (q+1)^\alpha + \beta q + \gamma$ where $0 \le \beta < (q+1)^\alpha$ and $0 < \gamma \le q$, then $H(n) = n\alpha + \lceil (q\beta + \gamma)(q+1)/q \rceil = n\alpha + (q+1)\beta + \gamma + 1$.

Lemma 5. *Given integers $n > 0$, $q \ge 1$ and $d \ge 0$, let $n = (q+1)^\alpha + \beta q + \gamma$ where $0 \le \beta < (q+1)^\alpha$ and $0 \le \gamma < q$, we have*

(i)

$$H(n+1) - H(n) = \begin{cases} \lfloor \log_{q+1} n \rfloor + 1, & \text{if } \gamma \ne 0, \\ \lfloor \log_{q+1} n \rfloor + 2, & \text{if } \gamma = 0. \end{cases} \tag{1}$$

(ii)

$$d(\lfloor \log_{q+1} n \rfloor + 1) \le H(n+d) - H(n) \le d(\lfloor \log_{q+1} (n+d) \rfloor + 2). \tag{2}$$

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(iii)

$$H(n+q) - H(n) \le q\lceil \log_{q+1}(n+q) \rceil + 1. \tag{3}$$

(iv)

$$H(n) - H(n-i) = i\lceil \log_{q+1} n \rceil, \quad \text{for } \gamma = 0,\ 0 \le i < q \text{ and } i \le n. \tag{4}$$

(v)

$$H(n) - H(m) \le (n-m)\lceil \log_{q+1} n \rceil + \lfloor \frac{n-m}{q} \rfloor, \quad \text{for } \gamma = 0 \text{ and } m \le n. \tag{5}$$

(vi)

$$H_{\le \ell}(n) - H_{\le \ell}(n-1) \ge \lceil \log_{q+1} n \rceil. \tag{6}$$

Proof. By Lemma 2, we know $\alpha = \lfloor \log_{q+1}(n) \rfloor$.
(i) Case 1: $q = 1$. Now γ is an integer satisfying $0 \le \gamma < 1$, i.e., $\gamma = 0$.

Subcase 1. For $\beta < 2^\alpha - 1$, we have $n+1 = 2^\alpha + \beta + 1$ and $\beta + 1 \le 2^\alpha - 1$. By Lemma 4 (1), we have $H(n+1) = (n+1)\alpha + (\beta+1)2$, $H(n) = n\alpha + \beta 2$ and $H(n+1) - H(n) = \alpha + 2 = \lfloor \log_{q+1} n \rfloor + 2$.

Subcase 2. For $\beta = 2^\alpha - 1$, we have $n = 2^\alpha + 2^\alpha - 1$ and $n+1 = (2)^{\alpha+1}$. By Lemma 4 (1), we have $H(n+1) = (n+1)(\alpha+1)$, $H(n) = n\alpha + 2^\alpha - 1$ and $H(n+1) - H(n) = \alpha + 2 = \lfloor \log_{q+1} n \rfloor + 2$.

Case 2. $q \geq 2$.

Subcase 1: For $\gamma = 0$, we have $n+1 = (q+1)^\alpha + \beta q + 1$ and $0 \leq \beta < (q+1)^\alpha$, $1 \leq q-1$. By Lemma 4 (1), we have $H(n+1) = (n+1)\alpha + \beta(q+1) + 2$ and $H(n) = n\alpha + \beta(q+1)$. Therefore $H(n+1) - H(n) = \alpha + 2 = \lfloor \log_{q+1} n \rfloor + 2$.

Subcase 2: For $0 < j < q-1$, we have $n+1 = (q+1)^\alpha + \beta q + \gamma + 1$ and $0 \leq \beta < (q+1)^\alpha$, $\gamma + 1$. By Lemma 4 (1), we have $H(n+1) = (n+1)\alpha + \beta(q+1) + \gamma + 2$ and $H(n) = n\alpha + \beta(q+1) + \gamma + 1$. Therefore $H(n+1) - H(n) = \alpha + 1 = \lfloor \log_{q+1} n \rfloor + 1$.

Subcase 3: For $j = q-1$, We have $n+1 = (q+1)^\alpha + (\beta+1)q$ and $0 \leq \beta + 1 \leq (q+1)^\alpha$. When $0 \leq \beta + 1 \leq (q+1)^\alpha - 1$, by Lemma 4 (1), we have $H(n+1) = (n+1)\alpha + (\beta+1)(q+1)$, $H(n) = n\alpha + \beta(q+1) + q$ and $H(n+1) - H(n) = \alpha + 1 = \lfloor \log_{q+1} n \rfloor + 1$. When $\beta + 1 = (q+1)^\alpha$, we have $n+1 = (q+1)^{\alpha+1}$ and $H(n+1) = (n+1)(\alpha+1)$, $H(n) = n\alpha + ((q+1)^\alpha - 1)(q+1) + q$. Therefore $H(n+1) - H(n) = \alpha + 1 = \lfloor \log_{q+1} n \rfloor + 1$. Summarizing, (1) holds.

(ii) Case 1. For $d = 0$, it is trivial that (2) holds.

Case 2. For $d > 0$, by (1), we have $H(n+i) - H(n+i-1) \geq \lfloor \log_{q+1}(n+i-1) \rfloor + 1 \geq \lfloor \log_{q+1} n \rfloor + 1$ and $H(n+i) - H(n+i-1) \leq \lfloor \log_{q+1}(n+i-1) \rfloor + 2 \leq \lfloor \log_{q+1}(n+d) \rfloor + 2$ for $i \geq 1$. Since $H(n+d) - H(n) = \sum_{i=1}^{d} [H(n+i) - H(n+i-1)]$, we obtain that (2) holds.

(iii) Case 1. $\beta < (q+1)^\alpha - 1$. We have $n+q = (q+1)^\alpha + (\beta+1)q + \gamma$ and $(\beta+1) \leq (q+1)^\alpha - 1$. By Lemma 2 and Lemma 4 (1), we have $\lceil \log_{q+1}(n+q) \rceil = \alpha + 1$, $H(n+q) = (n+q)\alpha + (\beta+1)(q+1) + \gamma + \mu(\gamma)$ and $H(n) = n\alpha + \beta(q+1) + \gamma + \mu(\gamma)$. Therefore $H(n+q) - H(n) = q(\alpha+1) + 1 = q\lceil \log_{q+1}(n+q) \rceil + 1$.

Case 2. $\beta = (q+1)^\alpha - 1$. We have $n+q = (q+1)^\alpha + (\beta+1)q + \gamma = (q+1)^{\alpha+1} + \gamma$. When $\gamma = 0$, by Lemma 2 and Lemma 4 (1), we have $\lceil \log_{q+1}(n+q) \rceil = \alpha + 1$, $H(n+q) = (n+q)(\alpha+1)$, $H(n) = n\alpha + \beta(q+1)$ and $H(n+q) - H(n) = q(\alpha+1) + 1 = q\lceil \log_{q+1}(n+q) \rceil + 1$. When $\gamma > 0$, by Lemma 2 and Lemma 4 (1), we have $\lceil \log_{q+1}(n+q) \rceil = \alpha + 2$, $H(n+q) = (n+q)(\alpha+1) + \gamma + \mu(\gamma)$, $H(n) = n\alpha + \beta(q+1)$ and $H(n+q) - H(n) = q(\alpha+1) + \gamma + 1 \leq q(\alpha+1) + q + 1 = q(\alpha+2) + 1 = q\lceil \log_{q+1}(n+q) \rceil + 1$.

As a result, (3) holds.

(iv) Case 1: $i = 0$. It is trivial that (4) holds.

Case 2. $0 < i < q$.

Subcase 1. For $\beta = 0$, we have $n = (q+1)^\alpha$, $n - i = (q+1)^{\alpha-1} + ((q+1)^{\alpha-1} - 1)q + q - i$. By Lemma 2 and Lemma 4 (1), we have $H(n) = n\alpha$ and $H(n-i) = (n-i)(\alpha-1) + ((q+1)^{\alpha-1} - 1)(q+1) + (q+1) - i = n\alpha - i\alpha$. Therefore, $H(n) - H(n-i) = i\alpha = i\lceil \log_{q+1} n \rceil$.

Subcase 2. For $\beta \geq 1$, we have $n = (q+1)^\alpha + \beta q$, $n - i = (q+1)^\alpha + (\beta-1)q + q - i$. By Lemma 2 and Lemma 4 (1), we have $H(n) = n\alpha + \beta(q+1)$ and $H(n-i) = (n-i)(\alpha-1) + (\beta-1)(q+1) + (q+1) - i = n\alpha + \beta(q+1) - i\alpha$. Therefore, $H(n) - H(n-i) = i\alpha = i\lceil \log_{q+1} n \rceil$.

As a result, (4) holds.

(v) Let $x = \lfloor \frac{n-m}{q} \rfloor$, there exists an integer y such that $n - m = xq + y$, where $0 \leq y < q$. By (iii) and (iv), we have $H(n) - H(n-y) = y\lceil \log_{q+1} n \rceil$ and $H(n-y) - H(m) = H(m+xq) - H(m) = \sum_{k=1}^{x} (H(m+kq) - H(m+(k-1)q)) \leq \sum_{k=1}^{x} (q\lceil \log_{q+1}(m+kq) \rceil + 1) \leq x(q\lceil \log_{q+1}(m+xq) \rceil + 1) \leq x(q\lceil \log_{q+1} n \rceil + 1)$. Therefore $H(n) - H(m) = H(n) - H(n-y) + H(n-y) - H(m) \leq (n-m)\lceil \log_{q+1}(n) \rceil + \lfloor \frac{n-m}{q} \rfloor$.

(vi) Suppose T^* is an optimal ℓ-admissible average-case tree with m leaves, by Lemma 3, there exists a leaf i such that $H_{\leq \ell}(T^*, i) = L_{\leq \ell}(T^*) \geq L(n) = \lceil \log_{q+1} n \rceil$. Let $\widehat{T^*} = T^*/\{i\}$ be an ℓ-admissible average-case tree with $n-1$ leaves, therefore $H_{\leq \ell}(n-1) \leq H_{\leq \ell}(\widehat{T^*}) = H_{\leq \ell}(n) - L_{\leq \ell}(T^*) \leq H_{\leq \ell}(n) - \lceil \log_{q+1} n \rceil$. Thus $H_{\leq \ell}(n) - H_{\leq \ell}(n-1) \geq \lceil \log_{q+1} n \rceil$. $\square$

Corollary 6. *Given integers n_1, n_2 with $\alpha_1 = \lfloor \log_{q+1} n_1 \rfloor$ and $\alpha_2 = \lfloor \log_{q+1} n_2 \rfloor$ respectively. If $\alpha_1 \geq \alpha_2 + 2$, then $H(n_1) + H(n_2) \geq H(n_1 - 1) + H(n_2 + 1)$.*

Proof. Using the Lemma 5 (ii) we have $H(n_1) - H(n_1 - 1) \geq \alpha_1 + 1 \geq \alpha_2 + 3$ and $H(n_2 + 1) - H(n_2) \leq \lfloor \log_{q+1} n_2 + 1 \rfloor + 2 \leq \lfloor \log_{q+1} n_2 \rfloor + 3 = \alpha_2 + 3$. Therefore $H(n_1) + H(n_2) \geq H(n_1 - 1) + H(n_2 + 1)$. $\square$

Lemma 7. *Given integers $\ell > 1$ and $a_i \geq 0$ for $i = 0, 1, \ldots, q$, let $\ell = (q+1)^\eta + \theta q + \tau$ where $0 \leq \theta < (q+1)^\eta$ and $0 < \tau \leq q$. If $a_i \leq \ell$ for $i = 1, \ldots, q$, and $\sum_{i=0}^{q} a_i \geq (q+1)^{\eta+1} + q\ell$, then $\sum_{i=0}^{q} H_{\leq \ell}(a_i) \geq H_{\leq \ell}(n - q\ell) + qH(\ell)$.*

Proof. The result is obtained by using the following property recursively. For $K + 1 > (q+1)^{\eta+1}$ and $M + 1 \leq \ell$, we obtain that

$$H_{\leq \ell}(K+1) + H_{\leq \ell}(M) \geq H_{\leq \ell}(K) + H_{\leq \ell}(M+1). \tag{7}$$

In fact, for $K+1 > (q+1)^{\eta+1}$, by (6), we have $H_{\leq \ell}(K+1) - H_{\leq \ell}(K) \geq \lceil \log_{q+1}^{K+1} \rceil \geq \eta + 2$. For $M + 1 \leq \ell$, by Lemma 4 and (2), we have $H_{\leq \ell}(M+1) - H_{\leq \ell}(M) = H(M+1) - H(M) \leq \lfloor \log_{q+1}^{M+1} \rfloor + 2 \leq \eta + 2$. Therefore we obtain the above result.

Since $\sum_{i=0}^{q} a_i > (q+1)^{\eta+1} + q\ell$, then $n - q\ell + 1 > (q+1)^{\eta+1}$. Recursively applying (7), we have $\sum_{i=0}^{q} H_{\leq \ell}(a_i) \geq H_{\leq \ell}(n - q\ell) + qH(\ell)$. $\square$

Lemma 8. *Given integers $q > 0$, $\ell > 1$, $s > 0$ and $a_i > 0$ for $i = 1, 2, \ldots, s$, let $\ell = (q+1)^\eta + \theta q + \tau$ such that $0 \leq \theta < (q+1)^\eta$ and $0 < \tau \leq q$. If $\ell - \tau < a_i \leq \ell$ for $i = 1, \cdots, s$, then*

$$\sum_{i=1}^{s} H(a_i) \geq (s - x - 1)H(\ell - \tau) + H(\ell - \tau + y) + xH(\ell)$$

where $\sum_{i=1}^{s}(a_i - \ell + \tau) = x\tau + y$ and $0 \leq y < \tau$.

Proof. Since $\ell - \tau < a_i \leq \ell$, we denote $a_i = (q+1)^\eta + \theta q + \gamma_i$ with $q \geq \tau \geq \gamma_i > 0$ for $i = 1, \cdots, s$ and $\sum_{i=1}^{s}(a_i - \ell + \tau) = \sum_{i=1}^{s} \gamma_i = x\tau + y$ with $0 < y \leq \tau$. Therefore $(s - x)\tau = \sum_{i=1}^{s}(\ell - a_i) + y > 0$, i.e., $s \geq x + 1$. Thus we obtain

$$\begin{aligned}
\sum_{i=1}^{s} H(a_i) &= \sum_{i=1}^{s}(a_i \lfloor \log_{q+1} a_i \rfloor + \theta(q+1) + \gamma_i + 1) \\
&= \eta \sum_{i=1}^{s}((q+1)^\eta + \theta q + \gamma_i) + s\theta(q+1) + (x\tau + y) + s \\
&= \eta(s(q+1)^L + s\theta q + x\tau + y) + s\theta(q+1) + (x\tau + y) + s \\
&= xH(\ell) + H(\ell - \tau + y) + (s - x - 1)H(\ell - \tau) + s - x - 1 \\
&\geq xH(\ell) + H(\ell - \tau + y) + (s - x - 1)H(\ell - \tau)
\end{aligned}$$

The following equalities are used to obtain the fourth equality of the ones above.

$$H(\ell) = \eta((q+1)^\eta + \theta q + \tau) + \theta(q+1) + \tau + 1, \tag{8}$$

$$H(\ell-\tau+y)=\eta((q+1)^\eta+\theta q+y)+\theta(q+1)+y+1, \tag{9}$$

$$H(\ell-\tau)=\eta((q+1)^\eta+\theta q)+\theta(q+1). \tag{10}$$

□

Lemma 9. *Given integers $q>0$, $\ell>1$ and $a_i>0$ for $i=0,1,\ldots,q$, let $\ell=(q+1)^\eta+\theta q+\tau$ such that $0\le\theta<(q+1)^\eta$ and $0<\tau\le q$. If $(q+1)^{\eta+1}+q(\ell-\tau)<\sum_{i=0}^q a_i\le(q+1)^{\eta+1}+q\ell$ and $0<a_i\le\ell$ for $i=1,2,\ldots,q$, then*

$$\sum_{i=0}^q H(a_i)\ge H((q+1)^{\eta+1})+xH(\ell)+H(\ell-\tau+y)+(q-x-1)H(\ell-\tau)$$

where $\sum_{i=0}^q a_i-(q+1)^{\eta+1}-q(\ell-\tau)=x\tau+y$ such that integers $x\ge0$ and $0<y\le\tau$.

Proof. Since $\sum_{i=0}^q a_i<(q+1)^{\eta+1}+q\ell$, we know $a_0-(q+1)^{\eta+1}<q\ell-\sum_{i=1}^q a_i=\sum_{i=1}^q(\ell-a_i)$ and that there exist integers b_i such that $a_0-(q+1)^{\eta+1}=\sum_{i=1}^q b_i$ and $0\le b_i\le\ell-a_i$ for $i=1,\ldots,q$. Let $\bar a_i=a_i+b_i$, then $\lfloor\log_{q+1}\bar a_i\rfloor\le\lfloor\log_{q+1}\ell\rfloor\le\eta$. By Lemma 5 (ii), we have

$$H(a_0)\ge H((q+1)^{\eta+1})+(a_0-(q+1)^{\eta+1})(\eta+2), \tag{11}$$

$$H(a_i)\ge H(a_i+b_i)-b_i(\lfloor\log_{q+1}(a_i+b_i)\rfloor+2)\ge H(a_i+b_i)-b_i(\eta+2). \tag{12}$$

Thus $\sum_{i=0}^q H(a_i)\ge H((q+1)^{\eta+1})+\sum_{i=1}^q H(\bar a_i)$ and $\sum_{i=1}^q\bar a_i=\sum_{i=0}^q a_i-(q+1)^{\eta+1}$. Therefore $q(\ell-\tau)<\sum_{i=1}^q\bar a_i\le q\ell$.

Without loss generality, we assume $\bar a_1\ge\bar a_2\ge\ldots\ge\bar a_q$. Since $q(\ell-\tau)<\sum_{i=1}^q\bar a_i$, we know there exists an integer k such that $\bar a_i>\ell-\tau$ for $1\le i\le k$ and $\bar a_i\le\ell-\tau$ for $k<i\le q$. By Lemma 8, we have that there exist integers $m\ge0$, $0<m'\le\tau$ and $\sum_{i=1}^k(\bar a_i-\ell+\tau)=m\tau+m'$ such that $\sum_{i=1}^q H(\bar a_i)\ge\sum_{i=1}^q H(a'_i)$, where $a'_i=\ell$ for $1\le i\le m$, $a'_{m+1}=\ell-\tau+m'$ and $a'_i=\ell-\tau$ for $m+2\le i\le k$.

Let $a'_0=(q+1)^{\eta+1}$ and $a'_i=\bar a_i$ for $k\le i\le q$, we have $\sum_{i=0}^q a'_i=\sum_{i=0}^q a_i$. Let $\sum_{i=0}^q a'_i-(q+1)^{\eta+1}-q(\ell-\tau)=x\tau+y$ such that $x\ge0$ and $0\le y<\tau$. Therefore $(m-x)\tau=(q+1)^{\eta+1}+y-a'_0-a'_{m+1}+(\ell-\tau)+\sum_{i=m+2}^q(\ell-\tau-a_i)\ge(\ell-\tau)-a'_{m+1}\ge-\tau$, i.e., $x\le m+1$.

Case 1. For $x=m+1$, we have $a'_{m+1}=\ell$ and $a'_i=\ell-\tau$ for $i\ge m+2$. Therefore $\sum_{i=0}^q H(a_i)=H((q+1)^{\eta+1})+xH(\ell)+H(\ell-\tau+y)+(q-x-1)H(\ell-\tau)$.

Case 2. For $x\ge m$, since $\lceil\log_{q+1}(\ell-\tau)\rceil\le\lceil\log_{q+1}\ell\rceil=\eta+1$, by Lemma 5 (iv), we obtain, for $m+2\le i\le q$,

$$\begin{aligned}H(a'_i)&\ge H(\ell-\tau)-(\ell-\tau-a'_i)(\eta+1)-\lfloor\frac{\ell-\tau-a'_i}{q}\rfloor\\&\ge H(\ell-\tau)-(\ell-\tau-a'_i)(\eta+1)-\frac{\ell-\tau-a'_i}{q}.\end{aligned}$$

Since $a'_{m+1}=\ell-\tau+m'$ and $m'>0$, we have

$$H(a'_{m+1})\ge H(\ell-\tau)+(a'_{m+1}-\ell+\tau)(\eta+1)+1.$$

Since $\sum_{i=0}^q a'_i-(q+1)^{\eta+1}-q(\ell-\tau)=x\tau+y$ holds with $x\ge0$ and $0\le y<\tau$, and $a'_0=(q+1)^{\eta+1}$ and $a'_i=\ell$ for $1\le i\le m$, we have $\Delta\equiv\sum_{i=m+2}^q(\ell-\tau-a'_i)=s\tau+a_{m+1}-\ell+\tau-x\tau-y\le s\tau+\tau-x\tau-y$.

Therefore

$$\begin{aligned}
\frac{q(s+1)-\Delta}{q} &\geq \frac{q(s+1)-(s\tau+\tau-x\tau-y)}{q} \geq \frac{(q-\tau)(s+1)+x\tau+y}{q} \\
&\geq \frac{q(x+1)-\tau+y}{q} \geq x+1-\frac{\tau-y}{q} \\
&\geq x+1-\frac{q-1}{q}.
\end{aligned}$$

Since $\sum_{i=0}^{q} H(a'_i)$ is an integer, we have

$$\begin{aligned}
\sum_{i=0}^{q} H(a'_i) &= H((q+1)^{\eta+1}) + \sum_{i=m+2}^{q} H(a'_i) + mH(\ell) + H(a'_{m+1}) \\
&\geq H((q+1)^{\eta+1}) + qH(\ell-\tau) + (x\tau+y)(\eta+1) + x + 1 \\
&= H((q+1)^{\eta+1}) + (q-x-1)H(\ell-\tau) + xH(\ell) + H(\ell-\tau+y).
\end{aligned}$$

□

Corollary 10. *Given integers $q > 0$, $\ell > 1$ and $a_i > 0$ for $i = 0, 1, \ldots, q$, let $\ell = (q+1)^\eta + \theta q + \tau$ such that $0 \leq \theta < (q+1)^\eta$ and $0 < \tau \leq q$. If $(q+1)^{\eta+1} + q(\ell-\tau) < \sum_{i=0}^{q} a_i \leq (q+1)^{\eta+1} + q\ell$, then*

$$H_{\leq\ell}(\sum_{i=0}^{q} a_i) \geq H((q+1)^{\eta+1}) + xH(\ell) + H(\ell-\tau+y) + (q-x-1)H(\ell-\tau) + \sum_{i=0}^{q} a_i$$

where $\sum_{i=0}^{q} a_i - (q+1)^{\eta+1} - q(\ell-\tau) = x\tau + y$ with some integers satisfying $x \geq 0$ and $0 < y \leq \tau$.

Proof. Suppose T^* be a optimal ℓ-admissible sequential tree with $\sum_{i=0}^{q} a_i$ leaves, and the first query be $(A_0, A_1, \ldots, A_q)$ with $|A_i| = a_i$ and $0 < a_i \leq \ell$. Then $H_{\leq\ell}(\sum_{i=0}^{q} a_i) = \sum_{i=0}^{q} a_i + \sum_{i=0}^{q} H_{\leq\ell}(a_i) \geq \sum_{i=0}^{q} a_i + \sum_{i=0}^{q} H(a_i)$. By Lemma 9, and the condition $(q+1)^{\eta+1} + q(\ell-\tau) < \sum_{i=0}^{q} a_i \leq (q+1)^{\eta+1} + q\ell$, we can obtain the result directly. □

Theorem 11. *Given integers $n > 0$, $\ell = 1$, we have $H_{\leq 1}(n) \geq H(n)$ if $n \leq q+1$, and $H_{\leq 1}(n) \geq n + H_{\leq 1}(n-q)$ if $n > q+1$.*

Proof. By Lemma 4 (2), we know $H_{\leq\ell}(n) \geq H(n)$ if $n \leq q+1$.

For $n > q+1$, we have $n - q > 1$ and $\lceil \log_{q+1} n - q \rceil \geq 1$. Suppose T^* is an average-case optimal algorithm and the first query is $(A_0, A_1, \ldots, A_q)$ with $|A_i| = a_i$ for $i = 0, 1, \ldots, q$. Therefore $0 < a_i \leq 1$ for $1 \leq i \leq q$ and $H_{\leq\ell}(n) = n + \sum_{i=0}^{q} H_{\leq\ell}(a_i) = n + H_{\leq\ell}(a_0) + \sum_{i=1}^{q} H_{\leq\ell}(a_i)$. By Lemma 5 (vi) and $H_{\leq\ell}(0) = H_{\leq\ell}(1) = 0$, we have $H_{\leq\ell}(n) \geq n + H_{\leq\ell}(n-q)$. □

Theorem 12. *Given integers $n > 0$, $\ell > 1$, let $\ell = (q+1)^\eta + \theta q + \tau$ such that $0 \leq \theta < (q+1)^\eta$ and $0 < \tau \leq q$, we have*

(1) *For $0 < n \leq (q+1)^{\eta+1} + q(\ell-\tau)$, $H_{\leq\ell}(n) \geq H(n)$;*

(2) *For $(q+1)^{\eta+1} + q(\ell-\tau) < n \leq (q+1)^{\eta+1} + q\ell$,*

$$H_{\leq\ell}(n) \geq H((q+1)^{\eta+1}) + xH(\ell) + H(\ell-\tau+y) + (q-x-1)H(\ell-\tau) + n,$$

where $n - (q+1)^{\eta+1} - q(\ell-\tau) = x\tau + y$ with some integers satisfying $x \geq 0$ and $0 < y \leq \tau$.

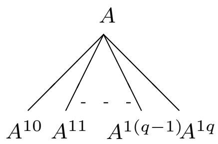

Figure 1: Tree and its subtrees

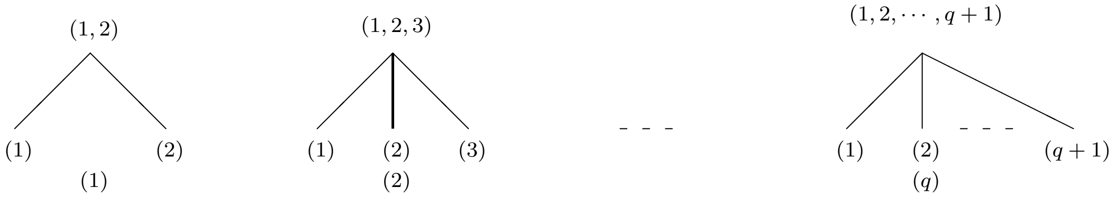

Figure 2: An ℓ-admissible sequential algorithm for $1 \le |A| \le q+1$

(3) *For* $(q+1)^{\eta+1} + q\ell < n$, $H_{\le\ell}(n) \ge n + qH(\ell) + H_{\le\ell}(n - q\ell)$.

Proof. (1) is trivial.
(2) Suppose T^* is an average-case optimal ℓ-admissible sequential tree with n leaves, and the first query is $(A_0, A_1, \dots, A_q)$ with $|A_i| = a_i$ for $i = 0, 1, \dots, q$. Therefore $0 < a_i \le \ell$ for $1 \le i \le q$ and $H_{\le\ell}(n) = n + \sum_{i=0}^{q} H_{\le\ell}(a_i) \ge \sum_{i=0}^{q} a_i + \sum_{i=0}^{q} H(a_i)$. By Lemma 9 and the condition $(q+1)^{\eta+1} + q(\ell - \tau) < \sum_{i=0}^{q} a_i \le (q+1)^{\eta+1} + q\ell$, we obtain the result directly.
(3) Suppose T^* is an average-case optimal ℓ-admissible sequential tree with n leaves, and the first query be $(A_0, A_1, \dots, A_q)$ with $|A_i| = a_i$ for $i = 0, 1, \dots, q$. Therefore $0 < a_i \le \ell$ for $1 \le i \le q$ and $H_{\le\ell}(n) = n + \sum_{i=0}^{q} H_{\le\ell}(a_i)$. By Lemma 7 and the condition $(q+1)^{\eta+1} + q\ell < n$, we obtain the result directly. □

4. An average-case optimal $q+1$-ary ℓ-admissible sequential algorithm

Recall the notations related to the $q+1$-ary ℓ-admissible search process given in Section 2. A $q+1$-ary sequential algorithm is represented by a tree A. In Figure 1 only the root and its children are shown. At the jth child the subtree A^{1j} is attached for $0 \le j \le q$. If A is a tree or subtree, $|A|$ denotes the number of leaves, that is the number of elements of the set (subset) where the algorithm determined by the tree (subtree) finds the unknown element. Let Huff(B) denote a Huffman tree such that the leaf set is the search candidate set B. Obviously, a Huffman tree Huff(B) is an ℓ-admissible if $|B| \le \ell$.

We construct a $q+1$-ary ℓ-admissible tree according to the value of ℓ, and consider two cases $\ell > 1$ and $\ell = 1$ respectively.

For $n = 1$, since the only element is the unknown one, the ℓ-admissible tree is a single node and $H_{\le\ell}(1) = 0$.

For $\ell > 1$ and $n > 1$, let $\ell = (q+1)^{\eta} + \theta q + \tau$ such that $0 \le \beta < (q+1)^{\alpha}$ and $0 < \gamma \le q$; $n = (q+1)^{\alpha} + \beta q + \gamma$ such that $0 \le \beta < (q+1)^{\alpha}$ and $0 < \gamma \le q$. Then we obtain an ℓ-admissible tree as follows in Cases 1-5.

Case 1. For $1 < n \le (q+1)$, we choose the admissible tree as in Fig. 2, i.e., a Huff(S).

Case 2. For $(q+1) < n \le (q+1)^{\eta+1}$, we have $1 \le \alpha \le \eta$. Let $\beta = m(q+1)^{\alpha-1} + t$ where $0 \le t < (q+1)^{\alpha-1}$. Then $(q+1)^{\alpha-1} + tq + \gamma \le (q+1)^{\eta-1} + ((q+1)^{\eta-1} - 1)q + q = (q+1)^{\eta} \le \ell$ and $m(q+1)^{\alpha} + (q-m)(q+1)^{\alpha-1} + (q+1)^{\alpha-1} + tq + \gamma = n$ hold.

We choose the admissible tree as in Fig. 1 in the following way. If $m = 0$, let A^{10} be a Huffman tree with $|A^{10}| = (q+1)^{\alpha-1} + tq + \gamma$, and let A^{1i} be Huffman trees with $|A^{1i}| = (q+1)^{\alpha-1}$ respectively for $1 \le i \le q$.

If $m \ge 1$, let A^{1i} be Huffman trees with $|A^{1i}| = (q+1)^{\alpha}$ respectively for $0 \le i \le m-1$, let A^{1m} be a Huffman tree with $|A^{1m}| = (q+1)^{\alpha-1} + tq + \gamma$, finally let A^{1i} be Huffman trees with $|A^{1i}| = (q+1)^{\alpha-1}$ respectively for $m+1 \le i \le q$.

Case 3. Suppose $(q+1)^{\eta+1} < n \le (q+1)^{\eta+1} + q(\ell-\tau)$. Since $(q+1)^{\eta+1} + q(\ell-\tau) < (q+1)^{\eta+1} + q(q+1)^{\eta+1} = (q+1)^{\eta+2}$, we have $\eta+1 \le \alpha < \eta+2$, i.e., $\alpha = \eta+1$ and $n = (q+1)^{\eta+1} + \beta q + \gamma$. For $(q+1)^{\eta} \le \beta \le \ell-\tau$, let $\beta-(q+1)^{\eta} = b\theta + c$ where $0 \le c < \theta$, then $(q+1)^{\eta+1} + (q+1)\eta + cq + b(\ell-\tau) + \gamma + (q-1)(q+1)^{\eta} = n$.

The inequality $(q+1)^{\eta} + \beta q + \gamma \le (q+1)^{\eta} + ((q+1)^{\eta} - 1)q + q = (q+1)^{\eta+1}$ holds if $\beta < (q+1)^{\eta}$ and $(q+1)^{\eta} + kq + \gamma \le \ell$ holds if $k < \theta$.

Therefore we choose the admissible tree as in Fig. 1 in the following way. If $\beta < (q+1)^{\eta}$, then let A^{10} be an admissible tree given in Cases 1 and 2 with $|A^{10}| = (q+1)^{\eta} + \beta q + \gamma$. Moreover let A^{1i} be admissible trees with $|A^{1i}| = (q+1)^{\eta}$ respectively for $1 \le i \le q$.

If $(q+1)^{\eta} \le \beta \le \ell - \tau$ and $b = 0$, let A^{10} be an admissible tree given in Cases 1 and 2 with $|A^{10}| = (q+1)^{\eta+1}$, A^{11} be an admissible tree with $|A^{11}| = (q+1)^{\eta} + cq + \gamma$, and finally let$A^{1i}$ be admissible trees with $|A^{1i}| = (q+1)^{\eta}$ respectively for $2 \le i \le q$.

If $(q+1)^{\eta} \le \beta \le \ell - \tau$ and $b \ge 1$, let A^{10} be an admissible tree given in Cases 1 and 2 with $|A^{10}| = (q+1)^{\eta+1}$, let A^{1i} be admissible trees with $|A^{1i}| = \ell - \tau$ respectively for $1 \le i \le b$, let $A^{1(b+1)}$ be an admissible tree with $|A^{1(b+1)}| = (q+1)^{\eta} + cq + \gamma$, and let finally A^{1i} be admissible trees with $|A^{1i}| = (q+1)^{\eta}$ respectively for $b+2 \le i \le q$.

Case 4. Suppose $(q+1)^{\eta+1} + q(\ell-\tau) < n \le (q+1)^{\eta+1} + q\ell$. We have $\eta+1 \le \alpha$. By $n \le (q+1)^{\eta+1} + q\ell \le (q+1)^{\eta+1} + q(q+1)^{\eta+1} = (q+1)^{\eta+2}$, we have $\alpha \le \eta+1$. Thus $\alpha = \eta+1$. Let $\beta q - q(\ell-\tau) + \gamma = x\tau + y$ where $0 \le y < \tau$. Since $(q+1)^{\eta+1} + x\ell + \ell - \tau + y + (q-x-1)(\ell-\tau) = n$, we choose the admissible tree as in Fig. 1 in the following way.

If $x = 0$, let A^{10} be an admissible tree given in Cases 1 and 2 with $|A^{10}| = (q+1)^{\eta+1}$, let A^{11} be an admissible tree with $|A^{1(x+1)}| = \ell - \tau + y$, and let finally A^{1i} be admissible trees with $|A^{1i}| = \ell - \tau$ respectively for $2 \le i \le q$.

If $x \ge 1$, let A^{10} be an admissible tree given in Cases 1 and 2 with $|A^{10}| = (q+1)^{\eta+1}$, let A^{1i} be admissible trees with $|A^{1i}| = \ell$ respectively for $1 \le i \le x$, let $A^{1(x+1)}$ be an admissible tree with $|A^{1(x+1)}| = \ell - \tau + y$, and let finally A^{1i} be admissible trees with $|A^{1i}| = \ell - \tau$ respectively for $x+2 \le i \le q$.

Case 5. Suppose $n \ge (q+1)^{\eta+1} + q\ell$. We choose the admissible tree as in Fig. 3 such that A^{i0} is an admissible tree with $|A^{i0}| = n - iq\ell$ for $1 \le i \le t-2$, A^{ij} is an admissible tree with $|A^{ij}| = \ell$ for $1 \le i \le t-1$ and $1 \le j \le q$, and $A^{(t-1)0}$ is an admissible tree given in Cases 1-4 with $|A^{(t-1)0}| = n - (t-1)q\ell$, where $(t-1)q\ell + \Lambda = n - (q+1)^{\eta+1}$ and $0 < \Lambda \le \ell q$.

Case 6. For $\ell = 1$ and $1 < n \le q+1$, we choose the admissible tree as in Fig. 2.

For $\ell = 1$ and $n > q+1$, we choose the admissible tree as in Fig. 3 such that A^{i0} is an admissible tree with $|A^{i0}| = n - iq$ for $1 \le i \le t-2$, A^{ij} is a leaf for $1 \le i \le t-1$ and $1 \le j \le q$, and $A^{(t-1)0}$ is an admissible tree given in Case 1 with $|A^{(t-1)0}| = n - (t-1)q$, where $(t-1)q + \Lambda = n$ and $0 < \Lambda \le q$.

By computing the external length of the admissible tree A, we obtain an upper bound on $H_{\le \ell}(n)$.

Theorem 13. (1) *Given an integer $\ell = 1$, the external length of the proposed ℓ-admissible tree $H_{\le 1}(A) = H(n)$ if $n \le q+1$, and $H_{\le 1}(A) = n + H_{\le \ell}(n-q)$ if $n > q+1$;*

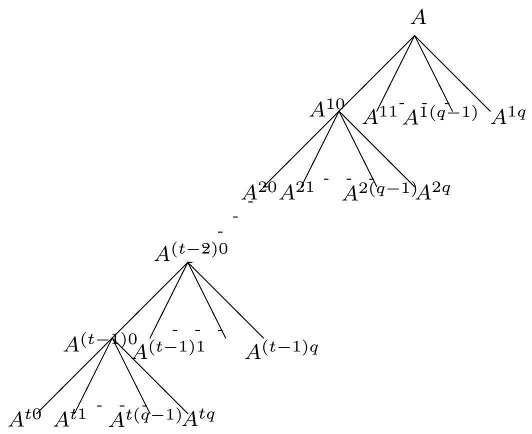

Figure 3: An ℓ-admissible sequential algorithm

(2) *Given an integer $\ell > 1$, the external length of the proposed ℓ-admissible tree $H_{\leq \ell}(A)$ is the lower bound given in Theorem 12.*

Proof. For $n = 1$, the ℓ-admissible tree is a single node and $H_{\leq \ell}(1) = 0$, the results hold. Next we prove the result of the case $n > 1$.

Suppose $n > 1$. (1) By the definition of $H_{\leq \ell}(A)$ and the tree given in Case 6., we obtain $H_{\leq \ell}(A) = n + H_{\leq \ell}(n - q)$.

(2) For Case 1, $H_{\leq \ell}(A) = n = H(n)$.

Consider now Case 2. If $m = 0$, then $H_{\leq \ell}(A) = n + H((q+1)^{\alpha-1} + tq + \gamma) + qH((q+1)^{\alpha-1}) = n\alpha + t(q+1) + \gamma = n\alpha + \beta(q+1) + \gamma = H(n)$.

If $m \geq 1$, then $H_{\leq \ell}(A) = n + mH((q+1)^{\alpha}) + H((q+1)^{\alpha-1} + tq + \gamma) + (q-m)H((q+1)^{\alpha-1}) = n\alpha + (m(q+1)^{\alpha-1} + t)(q+1) + \gamma = n\alpha + \beta(q+1) + \gamma = H(n)$.

Consider now Case 3. If $\beta < (q+1)^{\eta}$, then $H_{\leq \ell}(A) = n + H_{\leq \ell}((q+1)^{\eta} + \beta q + \gamma) + qH_{\leq \ell}((q+1)^{\eta}) = n + H((q+1)^{\eta} + \beta q + \gamma) + qH((q+1)^{\eta}) = ((q+1)^{\eta+1} + \beta q + \gamma)(\alpha+1) + \beta(q+1) + \gamma = H(n)$.

The equation $\beta - (q+1)^{\eta} = b\theta + c$ implies $(q+1)^{\eta+1} + (q+1)\eta + cq + b(\ell - \tau) + \gamma + (q-1)(q+1)^{\eta} = n$.

If $(q+1)^{\eta} \leq \beta \leq \ell - \tau$ and $b = 0$, then $H_{\leq \ell}(A) = n + H_{\leq \ell}((q+1)^{\eta+1}) + H_{\leq \ell}((q+1)\eta + cq + \gamma) + (q-1)H_{\leq \ell}((q+1)^{\eta}) = n + H((q+1)^{\eta+1}) + H((q+1)\eta + cq + \gamma) + (q-1)H((q+1)^{\eta}) = ((q+1)^{\eta+1} + \beta q + \gamma)(\alpha+1) + \beta(q+1) + \gamma = H(n)$.

If $(q+1)^{\eta} \leq \beta \leq \ell - \tau$ and $b \geq 1$, then $H_{\leq \ell}(A) = n + H_{\leq \ell}((q+1)^{\eta+1}) + bH_{\leq \ell}(\ell - \tau) + H_{\leq \ell}((q+1)\eta + cq + \gamma) + (q-b-1)H_{\leq \ell}((q+1)^{\eta}) = n + H((q+1)^{\eta+1}) + bH(\ell-\tau) + H((q+1)\eta + cq + \gamma) + (q-b-1)H((q+1)^{\eta}) = ((q+1)^{\eta+1} + \beta q + \gamma)(\alpha+1) + \beta(q+1) + \gamma = H(n)$.

Consider now Case 4. If $x = 0$, then $H_{\leq \ell}(A) = n + H_{\leq \ell}((q+1)^{\eta+1}) + H_{\leq \ell}(\ell - \tau + y) + (q-1)H_{\leq \ell}(\ell - \tau) = n + H((q+1)^{\eta+1}) + xH(\ell) + H(\ell - \tau + y) + (q - x - 1)H(\ell - \tau)$.

If $x \geq 1$, then $H_{\leq \ell}(A) = n + H_{\leq \ell}((q+1)^{\eta+1}) + xH_{\leq \ell}(\ell) + H_{\leq \ell}(\ell - \tau + y) + (q - x - 1)H_{\leq \ell}(\ell - \tau) = n + H((q+1)^{\eta+1}) + xH(\ell) + H(\ell - \tau + y) + (q - x - 1)H(\ell - \tau)$.

Finally, in Case 5 it is obvious that $H_{\leq \ell}(A) = n + H_{\leq \ell}(n - q\ell) + qH_{\leq \ell}(\ell) = n + H(n - q\ell) + qH(\ell)$.

□

5. Proof of the main result

Proof of Theorem 1

Let $n = (q+1)^\alpha + \beta q + \gamma$ such that $0 \le \beta < (q+1)^\alpha$ and $0 < \gamma \le q$ if $n > 1$, and $\ell = (q+1)^\eta + \theta q + \tau$ such that $0 \le \theta < (q+1)^\alpha$ and $0 < \tau \le q$ if $\ell > 1$.

Using Theorems 11, 12 and 13, we obtain the followings.

In the case $\ell = 1$, we have $H_{\le 1}(n) = H(n)$ if $n \le q+1$, and $H_{\le 1}(n) = n + H(n-q)$ if $n > q+1$.

Suppose now $\ell > 1$. (1) If $0 < n \le (q+1)^{\eta+1} + q(\ell - \tau)$, then $H_{\le \ell}(n) = H(n)$ holds. (2) If $(q+1)^{\eta+1} + q(\ell-\tau) < n \le (q+1)^{\eta+1} + q\ell$, then we have $H_{\le\ell}(n) = H((q+1)^{\eta+1}) + xH(\ell) + H(\ell - \tau + y) + (q-x-1)H(\ell-\tau) + n$, where $n - (q+1)^{\eta+1} - q(\ell-\tau) = x\tau + y$ are integers satisfying $x \ge 0$ and $0 \le y < \tau$. (3) Finally, if $(q+1)^{\eta+1} + q\ell < n$, then $H_{\le\ell}(n) = n + qH(\ell) + H_{\le\ell}(n - q\ell)$.

We use the following statements to complete the proof of Theorem 1.

For $\ell = 1$ and $n > q+1$, the conditions $(t-1)q + \Lambda = n$, $0 < \Lambda \le q$ imply $t = \lceil \frac{n}{q} \rceil$. Therefore we have

$$
\begin{aligned}
H_{\le 1}(n) &= n + H_{\le 1}(n-q) = [(n-q) + H_{\le 1}(n-q)] + q \\
&= [2(n-2q) + H_{\le 1}(n-2q)] + 2q + q \\
&= \cdots \\
&= [(t-1)(n-(t-1)q) + H_{\le 1}(n-(t-1)q)] + \sum_{k=1}^{t-1} kq \\
&= (t-1)n - \frac{(t-1)(t-2)}{2} q + H_{\le 1}(n-(t-1)q) \\
&= (t-1)n - \frac{(t-1)(t-2)}{2} q + H(n-(t-1)q)
\end{aligned}
$$

Suppose $\ell > 1$ and $(q+1)^{\eta+1} + q(\ell-\tau) < n \le (q+1)^{\eta+1} + q\ell$. Since $\ell = (q+1)^\eta + \theta q + \tau$ where $0 \le \theta < (q+1)^\eta$ and $0 < \tau \le q$, we have $\beta = \eta + 1$. Let $n - (q+1)^{\eta+1} - q(\ell-\tau) = \beta q + \gamma - q(\ell - \tau) = (\beta - (q+1)^\eta - \theta q)q + \gamma = x\tau + y$, where $0 < y \le \tau$. We have $x + 1 = \lceil \frac{n-(q+1)^{\eta+1}-q(\ell-\tau)}{\tau} \rceil$ and $\beta - (q+1)^\eta - \theta q + 1 = \lceil \frac{n-(q+1)^{\eta+1}-q(\ell-\tau)}{q} \rceil$. Therefore

$$
\begin{aligned}
H_{\le\ell}(n) &= n + H((q+1)^{\eta+1}) + xH(\ell) + H(\ell-\tau+y) + (q-x-1)H(\ell-\tau) \\
&= n + (q+1)^{\eta+1}(\eta+1) + x(\ell\eta + \theta(q+1) + \tau + 1) + (\ell-\tau+y)\eta + \theta(q+1) + y + 1 \\
&\quad +(q-1-x)[(\ell-\tau)\eta + \theta(q+1)] \\
&= n + [n - x\tau - y - q(\ell-\tau)](q+1)^{\eta+1}(\eta+1) + x(\ell\eta + \theta(q+1) + \tau + 1) \\
&\quad +(\ell-\tau+y)\eta + \theta(q+1) + y + 1 + (q-1-x)[(\ell-\tau)\eta + \theta(q+1)] \\
&= n + n(\eta+1) - q(\ell-\eta) + q\theta(q+1) + x + 1 \\
&= n(\eta+1) + n - q(q+1)^\eta + q\theta + x + 1 \\
&= n(\eta+1) + \beta q + \gamma + (q+1)^\eta + q\theta + x + 1 \\
&= n(\eta+1) + \beta(q+1) + \gamma + 1 - [\beta + 1 - (q+1)^\eta - q\theta] + (x+1) \\
&= H(n) + \lceil \frac{n - (q+1)^{\eta+1} - q(\ell-\tau)}{\tau} \rceil - \lceil \frac{n - (q+1)^{\eta+1} - q(\ell-\tau)}{q} \rceil.
\end{aligned}
$$

Finally, suppose $\ell > 1$ and $n \ge (q+1)^{\eta+1} + q\ell$. Let $(t-1)q\ell + \Lambda = n - (q+1)^{\eta+1}$ with $0 < \Lambda \le \ell q$, then we have $t=\lceil \frac{n-(q+1)^{\eta+1}}{q} \rceil$. Therefore

$$\begin{aligned} H_{\leq \ell}(n) &= n+H_{\leq \ell}(n-q\ell)+qH(\ell)=[(n-q\ell)+H_{\leq \ell}(n-q\ell)]+q(\ell+H(\ell)) \\ &= [2(n-2q\ell)+H_{\leq \ell}(n-2q\ell)]+q(2\ell+H(\ell))+q(\ell+H(\ell)) \\ &= [(t-1)(n-(t-1)q\ell)+H_{\leq \ell}(n-(t-1)q\ell)]+\sum_{k=1}^{t-1} q(k\ell+H(\ell)) \\ &= (t-1)n-\frac{(t-1)(t-2)}{2}q\ell+(t-1)qH(\ell)+H_{\leq \ell}(n-(t-1)q\ell). \end{aligned}$$

□

Example: $(S,2,A_{\leq \ell})$**.** In [1] (Section 1.9, exercise 1.6), Aigner established an average-case optimal algorithm for $(S,2,A_{\leq \ell})$ and obtained the exact value of $H_{\leq \ell}(n)$ as follows. $H_{\leq \ell}(n)=H(n)$ if $n\leq 2\ell$, and $H_{\leq \ell}(n)=nt-\frac{t(t-1)}{2}+tH(\ell)+H(n-t\ell)H(n)$ if $n>2\ell$, where $t=\lceil \frac{n}{\ell} \rceil-2$.

On the other hand, our Theorem 1 gives the following result for $q=1$. $H_{\leq \ell}(n)=H(n)$ for $n\leq 2\ell$, and $H_{\leq \ell}(n)=n(t'-1)-\frac{(t'-1)(t'-2)}{2}+(t'-1)H(\ell)+H(n-(t'-1)\ell)H(n)$ if $n>2\ell$, where $t'=\lceil \frac{n-2^{\eta+1}}{\ell} \rceil$ and $\ell=2^{\eta}+\theta+\tau$ such that $0\leq\theta<2^{\eta}$ and $0<\tau\leq 1$.

In fact, since $\theta+1\leq 2^{\eta}-1+1=2^{\eta}<\ell$, we have $t'=\lceil \frac{n-2^{\eta+1}}{\ell} \rceil=\lceil \frac{n-\ell-(2^{\eta}-\theta-1)}{\ell} \rceil=\lceil \frac{n}{\ell} \rceil-1$. Which implies that the two results are identical in this special case.

6. Conclusion and future reading

For any parameters $n\geq 1$, $q\geq 1$ and $\ell\geq 1$, we study general $q+1$-ary search problem with small sets and obtain an average-case optimal sequential algorithm and the minimum number of average-case sequential queries when the uniform distribution is assumed. In fact, the algorithm proposed in this paper is also worst-case optimal and the minimum number of worst-case sequential queries is implied in the main results.

It would be interesting to extend our results to other problems as follows: Search for more than one element [12, 14, 15, 9] etc., Counterfeit coin weighting with multi-arms balance [5, 4, 10] etc., and Search with lies [2, 8, 13] etc.

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