

ANALYTIC SINGULARITIES SUPPORTED BY A SPECIFIC INTEGRAL HOMOLOGY SPHERE LINK

ANDRÁS NÉMETHI AND TOMOHIRO OKUMA

ABSTRACT. The main question we target is the following: If one fixes a topological type (of a complex normal surface singularity) then what are the possible analytic types supported by it, and/or, what are the possible values of the geometric genus? We answer the question for a specific (in some sense pathological) topological type, which supports rather different analytic structures. These structures are listed together with some of their key analytic invariants.

Dedicated to Henry Laufer on the occasion of his 70th birthday

1. INTRODUCTION

The topological type of a normal complex surface singularity (X, o) is determined by its link (an oriented smooth connected 3-manifold), or, by the dual graph of any good resolution (a connected graph with a negative definite intersection form [5, 15], which serves also as plumbing graphs of the link [21]).

The main question we target is the following:

Question 1.1. If one fixes a topological type (say, a minimal good resolution graph) and varies the possible analytic types supported on this fixed topological type, then what are the possible values of the geometric genus p_g ?

Slightly more concrete version is formulated as follows:

Problem 1.2. Associate combinatorially an integer $\text{MAX}(\Gamma)$ to any (resolution) graph Γ , such that for any analytic type supported by Γ one has $p_g \leq \text{MAX}(\Gamma)$, and furthermore, for certain analytic structure one has equality.

Moreover, define by symmetric properties $\text{MIN}(\Gamma)$ as well.

A possible topological lower bound for p_g can be constructed as follows. Fix a resolution $\tilde{X} \rightarrow X$ and for any divisor l supported by the exceptional divisor set $\chi(l) := -(l, l - Z_K)/2$, where Z_K is the anti-canonical cycle (see below) and $(,)$ denotes the intersection form. Set also $\min\chi$ as $\min_l \chi(l)$. Then $\min\chi$ is a topological invariant computable from Γ ; Wagreich considered the expression $p_a(X, o) = 1 - \min\chi$, and called it the ‘arithmetical genus’ [33]. Moreover, for any analytic structure, whenever $p_g > 0$, one also has (see e.g. [33, p. 425])

$$(1.3) \quad 1 - \min\chi \leq p_g.$$

Indeed, one verifies that $\min\chi$ can be realized by an effective cycle $l_0 > 0$ (see e.g. [19]). Then from the cohomological long exact sequence associated with $0 \rightarrow \mathcal{O}_{\tilde{X}}(-l_0) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{l_0} \rightarrow 0$ one has

$$p_g + \chi(l_0) = \dim H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-l_0)) + h^1(\mathcal{O}_{\tilde{X}}(-l_0)) \geq 1$$

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(since $H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-l_0))$ contains the class of constants). (1.3) sometimes is sharp: e.g. for elliptic singularities (when $\min\chi = 0$) Laufer proved that for the generic analytic structure one has indeed $p_g = 1 - \min\chi = 1$ [12].

For different generalizations of (1.3) (inequalities, which involve besides $\min\chi$ and p_g some other analytic invariants as well) see e.g. [32, (2.6)] or [7, Prop. 8].

However, the authors do not know if the above bound (1.3) is always optimal:

Question 1.4. Does there exist for any Γ an analytic structure with $p_g = 1 - \min\chi$?

A possible upper bound for p_g is constructed as follows [17].

Let $\{E_i\}_{i \in \mathcal{I}}$ denote the set of irreducible exceptional curves, and for simplicity we will assume that each E_i is rational. For any effective cycle $Z > 0$ let $\mathcal{P}(Z)$ be the set of monotone computation sequences $\gamma = \{l_k\}_{k=0}^t$ of cycles supported on the exceptional curve with the following properties: $l_0 = 0$, $l_t = Z$, and $l_{k+1} = l_k + E_{i(k)}$ for some $i(k) \in \mathcal{I}$. Associated with such γ we define

$$S(\gamma) := \sum_{k=0}^{t-1} \max\{0, (E_{i(k)}, l_k) - 1\}.$$

Set also $\text{Path}(Z) := \min_{\gamma \in \mathcal{P}(Z)} S(\gamma)$. Then for any analytic structure supported on Γ one has

$$(1.5) \quad h^1(\mathcal{O}_Z) \leq \text{Path}(Z).$$

Indeed, from the exact sequence $0 \rightarrow \mathcal{O}_{E_{i(k)}}(-l_k) \rightarrow \mathcal{O}_{l_{k+1}} \rightarrow \mathcal{O}_{l_k} \rightarrow 0$ we get

$$h^1(\mathcal{O}_{l_{k+1}}) - h^1(\mathcal{O}_{l_k}) \leq h^1(\mathcal{O}_{E_{i(k)}}(-l_k)) = \max\{0, (E_{i(k)}, l_k) - 1\} \quad (0 \leq k < t),$$

hence the inequality follows by summation. Since $p_g = h^1(\mathcal{O}_{\lfloor Z_K \rfloor}) = h^1(\mathcal{O}_Z)$ for any $Z \geq \lfloor Z_K \rfloor$ when $Z_K \geq 0$, it is natural to define $\text{Path}(\Gamma) := \min_{Z \geq \lfloor Z_K \rfloor} \text{Path}(Z)$. It satisfies

$$(1.6) \quad p_g \leq \text{Path}(\Gamma).$$

The computation of $\text{Path}(\Gamma)$ is rather hard. In [17] (see also [20]) is related with the Euler characteristic of the ‘path lattice cohomology’ of Γ . In the next statement we collect some families of singularities when (1.6) is sharp.

Theorem 1.7. *In the next statement we consider singularities with rational homology sphere link. In the following cases $p_g = \text{Path}(\Gamma)$ (hence these analytic families realize the maximal p_g on their topological type):*

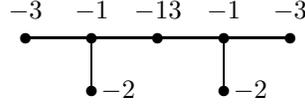
- *weighted homogeneous normal surface singularities [19] (in fact, for star shaped graphs with all E_i rational, $\text{Path}(\Gamma)$ equals the topological expression of Pinkham valid for p_g [29]),*
- *superisolated hypersurface singularities [20],*
- *isolated hypersurface Newton–nondegenerate singularities [20],*
- *rational singularities [19],*
- *Gorenstein elliptic singularities [19].*

One can expect that the realization $p_g = \text{Path}(\Gamma)$ is even more general.

However, the main aim of the present article is to show that the upper bound (1.6) in general is not sharp: for certain graph Γ the bound $\text{Path}(\Gamma)$ cannot be realized. Surprisingly, the very same example shows some additional statements as well: (the third part is motivated by the ‘conviction’ that usually ‘large’ p_g is realized simultaneously with ‘small’ maximal ideal cycle):

Theorem 1.8. *There exists a numerically Gorenstein topological type for which*

- *$p_g < \text{Path}(\Gamma)$ for any analytic type supported on Γ ;*
- *even if an analytic type realizes the maximal p_g (among all analytic types supported on the topological type under discussion) it is not necessarily Gorenstein;*
- *even if an analytic type realizes the maximal p_g , the maximal ideal cycle is not necessarily the Artin cycle.*

FIGURE 1. The graph Γ

Our fixed topological type, which has the above properties, is given by the minimal good graph from Figure 1.

In the next statements we assume that (X, o) has the resolution graph Γ from Figure 1 and \tilde{X} is its minimal good resolution. Let Z_{\min} be the Artin cycle, while Z_{\max} the maximal ideal cycle introduced by S. S.-T. Yau [35] (see Definition 2.1). For this graph one has $\min\chi = -1$ and $\text{Path}(\Gamma) = 4$. The first equality follows from [17, Example 4.4.1], or by using (1.3), $\chi(Z_{\min}) = -1$ and the existence of an analytic structure with $p_g = 2$. The second equality follows again from [17] (see also the description of the χ -function for graphs with two nodes in [10]). Nevertheless, we will verify it below as well.

With these notations we prove the following.

Theorem 1.9 (Cf. Section 2, Section 3). *For any analytic structure one has $\text{coeff}_{E_0}(Z_{\max}) \leq 2$ (where E_0 is the (-13) -curve), and $p_g(X, o) \leq 3$. If (X, o) is Gorenstein, then $p_g(X, o) = 3$.*

Theorem 1.10. *Any analytic structure satisfies one of the following properties:*

- (1) $Z_{\max} = Z_{\min}$, $p_g(X, o) = 3$, and (X, o) is a non-Gorenstein Kodaira singularity (cf. Theorem 4.1).
- (2) $Z_{\max} = 2Z_{\min}$ and (X, o) is of splice type (hence Gorenstein with $p_g(X, o) = 3$, cf. Theorem 5.2).
- (3) $2Z_{\min} \leq Z_{\max} < 3Z_{\min}$ (there are three cases, see below), $p_g(X, o) = 2$ and (X, o) is not Gorenstein (cf. Theorem 5.10 and Section 6).

Corollary 1.11. *The following are equivalent:*

- (1) $Z_{\max} = Z_{\min}$;
- (2) (X, o) is a Kodaira singularity.

Corollary 1.12. *The following are equivalent:*

- (1) $Z_{\max} = 2Z_{\min}$, $p_g(X, o) = 3$;
- (2) (X, o) is of splice type (complete intersection);
- (3) (X, o) is Gorenstein.

For Kodaira (or Kulikov) singularities see [6, 31], for splice singularities see [23].

Remark 1.13. (1) In general, a Gorenstein singularity with integral homology sphere link and with $Z_{\min}^2 = -1$ is not necessarily of splice type. An example can be found in [13, 4.6] (where the minimal good graph is even star-shaped).

(2) For the two cases with $p_g = 3$ (non-Gorenstein Kodaira and splice complete intersection) we provide precise realizations; however for the $p_g = 2$ cases we will not give the realizations (e.g. equations) in this article.

(3) The next table lists all the possible analytic structures supported by Γ with some of their key properties. E is the exceptional curve of the minimal resolution. For the notation E_i^* see Section 5.

Z_{\max}	Gorenstein	p_g	$h^1(\mathcal{O}_E(-E))$	$h^1(\mathcal{O}_E(-2E))$	mult	embdim
Z_{\min}	No (Kodaira)	3	1	0	3	4
$2Z_{\min}$	Yes (splice)	3	0	1	4	4
$2Z_{\min}$ or E_1^* or E_4^*	No	2	0	0	6	7

(4) In most of the proofs we use ‘computation sequences’. Computation sequences were introduced and deeply exploited by Laufer, they constitute a powerful machinery in the theory of surface singularities. The present manuscript supports this fact as well.

Remark 1.14. After we finished our manuscript the referee drew our attention to the excellent article [9] of K. Konno, which we were not aware of. We thank the referee for this information. Indeed, our proofs and arguments and some of the statements have overlaps with the results of this article, which contains several important results regarding the key cycles of a resolution of a normal surface singularity.

After this information, however, we decided not to change the structure (and the proofs) of our statements, in this way the present manuscript still remains (more or less) self-contained and more readable. In this Remark we wish to list some of the overlaps and give the credits to [9]. (Definitely, this list covers only the overlaps, and not the huge amount of results of [9].)

In [9] the author studies singularities with $Z_{min}^2 = -1$. Our main example belongs to this family too, in fact, it even belongs to the simplest class of ‘essentially irreducible Z_{min} ’ of Konno. For example, in ‘essentially irreducible Z_{min} ’ case, the fact that $p_g \leq 4$ when $Z_{min}^2 = -1$ and $\chi(Z_{min}) = \min \chi = -1$ is shown in Theorem 3.9 of [9]. Furthermore, in [9, Th. 3.9] is also stated that the singularity must be a doublepoint whenever $p_g = 4$. (This overlaps with the first part of our Theorem 3.1.) Also, the calculations of the present note in the Gorenstein case (§5,I) is much similar to [9, Th. 3.11], which might even shorten slightly the proof of our Theorem 5.2. A related statement can be found also in [9, Lemma 3.4].

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2. PRELIMINARY

Let (X, o) be a normal complex surface singularity and $\pi: \tilde{X} \rightarrow X$ a resolution with exceptional set E . Let $\{E_i\}_{i \in \mathcal{I}}$ denote the set of irreducible components of E . We denote by Γ the resolution graph of (X, o) . The group of cycles is defined by $L := \sum_{i \in \mathcal{I}} \mathbb{Z}E_i$. Let us simplify into DD' the intersection number (D, D') . For any function $f \in H^0(\mathcal{O}_{\tilde{X}})$, $f \neq 0$, let $(f)_E$ denote the exceptional part of $\text{div}(f)$, namely, $(f)_E = \sum_{i \in \mathcal{I}} \text{ord}_{E_i}(f \circ \pi)E_i \in L$. A divisor D on \tilde{X} is said to be nef (resp. anti-nef) if $DE_i \geq 0$ (resp. $DE_i \leq 0$) for all $i \in \mathcal{I}$.

We write $h^i(*) = \dim_{\mathbb{C}} H^i(*)$. Moreover, for an effective cycle $l \in L$ we write $H^i(l) := H^i(\mathcal{O}_l)$, and $\chi(l)$ denotes the Euler characteristic $\chi(\mathcal{O}_l) = h^0(l) - h^1(l)$. By Riemann–Roch formula, for a divisor D on \tilde{X} ,

$$\chi(\mathcal{O}_l(D)) = h^0(\mathcal{O}_l(D)) - h^1(\mathcal{O}_l(D)) = \chi(l) + Dl = -(l^2 - Z_K l)/2 + Dl,$$

where Z_K denotes the canonical cycle (see Definition 2.1). The expression $\chi(l) = -(l^2 - Z_K l)/2$ is extended for any $l \in L$.

Definition 2.1. We define the (minimal) Artin cycle Z_{\min} , the maximal ideal cycle Z_{\max} , and the cohomological cycle $Z_{\text{coh}} \in L$ as follows:

- (1) $Z_{\min} = \min \{Z > 0 \mid Z \text{ is anti-nef}\}$.
- (2) $Z_{\max} = \min \{(f)_E \mid f \in \mathfrak{m}_{X,o}\}$, where $\mathfrak{m}_{X,o}$ is the maximal ideal of $\mathcal{O}_{X,o}$.
- (3) $Z_{\text{coh}} = \min \{Z > 0 \mid h^1(\mathcal{O}_Z) = p_g(X, o)\}$ if $p_g(X, o) > 0$. $Z_{\text{coh}} = 0$ if $p_g(X, o) = 0$.
- (4) The canonical cycle $Z_K \in L \otimes \mathbb{Q}$ is defined by $K_{\tilde{X}} E_i = -Z_K E_i$ for all $i \in \mathcal{I}$. If $Z_K \in L$, then (X, o) or Γ is said to be numerically Gorenstein.

For the existence of the unique cohomological cycle on any resolution (with the property $h^1(Z) < p_g$ for any $Z \not\geq Z_{\text{coh}}$) see Reid [30, §4.8]. One has $Z_{\text{coh}} \leq \lfloor Z_K \rfloor$.

Recall that (X, o) is Gorenstein if and only if $-Z_K \sim K_{\tilde{X}}$ (linear equivalence on \tilde{X}).

Remark 2.2. Let k be a positive integer.

(1) If $Z_{\max} = kZ_{\min}$, \tilde{X} is the minimal resolution, and $\mathcal{O}_{\tilde{X}}(-Z_{\max})$ has no base point, then the same equality holds on any resolution.

(2) If $Z_{\max} = kZ_{\min}$ on a resolution, then the same equality holds on the minimal resolution.

Theorem 2.3 (Konno [8, §3]). (1) If (X, o) is Gorenstein and $p_g(X, o) \geq 2$, then $p_g(X, o) > p_a(Z_{\min}) = 1 - \chi(Z_{\min})$.

(2) Assume that (X, o) is numerically Gorenstein and $Z_K \geq 0$. Then (X, o) is Gorenstein if and only if $Z_K = Z_{\text{coh}}$.

Next, assume that the link of (X, o) is a \mathbb{Q} -homology sphere and the graph Γ is numerically Gorenstein. It is not hard to verify that in the numerically Gorenstein case $\text{Path}(\Gamma) = \text{Path}(Z_K)$ (a detailed proof can be found in [19]). The next results analyse certain cases when the inequality $p_g(X, o) \leq \text{Path}(\Gamma)$ from (1.6) is strict.

Theorem 2.4. Assume that Γ is numerically Gorenstein and $Z_K > Z_{\text{coh}}$ for some analytic structure (X, o) (that is, (X, o) is not Gorenstein). Then, if one of the following properties hold:

(1) either $\{\gamma \in \mathcal{P}(Z_K) : S(\gamma) = \text{Path}(\Gamma)\} \rightarrow \{E_i\}_{i \in \mathcal{I}}, \gamma \mapsto E_{i(t-1)}$, is surjective, or

(2) the support $|Z_K - Z_{\text{coh}}|$ is E ,
then $p_g(X, o) < \text{Path}(\Gamma)$.

Proof. We prove that if $p_g = \text{Path}(\Gamma)$ and the surjectivity (1) holds then $Z_{\text{coh}} = Z_K$. Indeed, the assumption $p_g = \text{Path}(\Gamma)$ implies that along a path (any path) γ with $p_g = \text{Path}(\Gamma) = S(\gamma)$, whenever p_g can grow with $E_{i(k)}l_k - 1 > 0$, it necessarily grows with this amount. On the other hand, for any choice of γ , l_{t-1} has the form $Z_K - E_{i(t-1)}$. Since $l_{t-1}E_{i(t-1)} - 1 = 2\chi(E_{i(t-1)}) - 1 = 1$, the assumption $p_g = \text{Path}(\Gamma)$ implies that necessarily $h^1(Z_K - E_{i(t-1)}) < h^1(Z_K) = p_g$. By the surjectivity (1) we get that this must be the case for any E_i , that is, $h^1(Z_K - E_i) < h^1(Z_K) = p_g$ for any $i \in \mathcal{I}$. This shows that $Z_{\text{coh}} = Z_K$.

Suppose that the condition (2) holds. Fix $\gamma \in \mathcal{P}(Z_K)$, $\gamma = \{l_k\}_{k=0}^t$, with $S(\gamma) = \text{Path}(\Gamma)$. Let γ' be the shorter path $\gamma' = \{l_k\}_{k=0}^{t-1}$. Then by similar computation as above $S(\gamma') = S(\gamma) - 1$. Hence, by (1.5), $p_g = h^1(Z_{\text{coh}}) \leq h^1(\mathcal{O}_{Z_K - E_{i(t-1)}}) \leq S(\gamma') < S(\gamma) = \text{Path}(\Gamma)$. \square

Assumption 2.5. From now on, we assume that the minimal good resolution graph Γ of (X, o) is as in Figure 1.

The cycles Z_{\min} and Z_K are shown in the next picture:



One easily verifies that $\chi(Z_{\min}) = -1$, hence $h^1(Z_{\min}) = 2$, which implies $p_g \geq 2$. (In fact, $\min \chi$ is also -1 , cf. [17, 4.4.1].)

For any path $\gamma = \{l_k\}_k$ we say that γ has a simple jump at k if $E_{i(k)}l_k = 2$.

Let us prove first that for the above graph one has $\text{Path}(\Gamma) \leq 4$. For this we have to construct a path with (at most) four simple jumps.

We start with $l_0 = 0$, then we add a base-element, say the (-13) -vertex E_0 . Then there exists a ‘Laufer computation sequence’ starting from E_0 and ending with Z_{\min} , determined by Laufer’s algorithm (for the Artin cycle) [11], which has exactly two simple jumps, and at all the other steps $E_{i(k)}l_k = 1$. Next, we add a base-element

(say E_5 , one of the (-1) -base cycles) to Z_{\min} . Then, again, there is a computation sequence starting with $Z_{\min} + E_5$ and ending with $2Z_{\min}$ with exactly one simple jump and at all the other steps $E_{i(k)}l_k = 1$. Finally, constructed in similar way, there is a increasing sequence starting with $2Z_{\min}$ and ending with Z_K such that there are two steps with $E_{i(k)}l_k = 0$ (including the very first one), one simple jump, and at all the other steps $E_{i(k)}l_k = 1$. (Since $\chi(Z_K - E_i) = 1 > \chi(Z_K) = 0$, a jump necessarily must appear.)

This shows that $\text{Path}(\Gamma) \leq 4$, hence for any analytic structure $p_g \leq 4$.

In Section 4 we show (using also from Section 3 that $p_g < 4$) that the Kodaira analytic structure satisfies $p_g = 3$ and $Z_{\text{coh}} \leq 2Z_{\min} \leq Z_K - E$ (cf. (4.2)). Hence, by Theorem 2.4, $\text{Path}(\Gamma) = 4$.

Moreover, analysing the long exact cohomological sequences at each step along the pathes considered above, we obtain that

$$(2.6) \quad \begin{cases} h^1(Z_{\min}) = 2, \\ h^1(2Z_{\min}) \leq h^1(Z_{\min}) + 1, \\ h^1(Z_K) = p_g \leq h^1(2Z_{\min}) + 1. \end{cases}$$

Furthermore, the reader is invited to verify (by constructing the corresponding pathes) that the above sequence-construction procedure has the following additional property as well. For any $i \in \mathcal{I}$, there is a sequence starting with $2Z_{\min}$ and ending with Z_K , with all the properties listed above, and which ends with E_i (that is, at the very last step we have to add E_i). Therefore, Theorem 2.4 and (2.6) read as follows.

Corollary 2.7. *If there exists a singularity (X, o) with graph Γ (as in Figure 1) and $p_g = 4$ then (X, o) should be Gorenstein and necessarily $h^1(mZ_{\min}) = m + 1$ for $m = 1, 2, 3$. (Note that $3Z_{\min} \geq Z_K$.)*

This will be an important ingredient in proving that $p_g = 4$ is not realized.

In the rest of this section, we assume that $\pi: \tilde{X} \rightarrow X$ is the *minimal* resolution. Then E is an irreducible curve with $E^2 = -1$ and it has two ordinary cusps; it corresponds to the (-13) -curve in Figure 1. One verifies the following facts.

$$(2.8) \quad h^1(E) = 2, \quad \chi(\mathcal{O}_E(-nE)) = n - 1, \quad \chi(nE) = (n^2 - 3n)/2 \text{ for } n \geq 0.$$

From the exact sequence

$$(2.9) \quad 0 \rightarrow \mathcal{O}_{\tilde{X}}(-E) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_E \rightarrow 0,$$

we have

$$(2.10) \quad h^1(\mathcal{O}_{\tilde{X}}(-E)) = p_g(X, o) - 2.$$

By adjunction formula, we obtain that $Z_K = 3E$.

By the Grauert-Riemenschneider vanishing theorem, $H^1(\mathcal{O}_{\tilde{X}}(-3E)) = 0$. Therefore, the exact sequence $0 \rightarrow \mathcal{O}_{\tilde{X}}(-3E) \rightarrow \mathcal{O}_{\tilde{X}}(-2E) \rightarrow \mathcal{O}_E(-2E) \rightarrow 0$, implies

$$(2.11) \quad \begin{cases} (a) \quad \dim \frac{H^0(\mathcal{O}_{\tilde{X}}(-2E))}{H^0(\mathcal{O}_{\tilde{X}}(-3E))} = \dim H^0(\mathcal{O}_E(-2E)) \geq \chi(\mathcal{O}_E(-2E)) = 1, \\ (b) \quad h^1(\mathcal{O}_{\tilde{X}}(-2E)) = h^1(\mathcal{O}_E(-2E)). \end{cases}$$

Hence, the definition of Z_{\max} and (2.11)(a) imply the following.

Proposition 2.12. *$Z_{\max} \leq 2E$ on the minimal resolution.*

3. A SINGULARITY WITH $p_g \geq 4$ DOES NOT EXIST

The aim of this section is to prove the following.

Theorem 3.1. *For all analytic structures (X, o) supported on Γ one has $2 \leq p_g(X, o) \leq 3$. If (X, o) is Gorenstein, then $p_g(X, o) = 3$.*

The proof consists of several step. Notice that the second part follows from (1.3) and Theorem 2.3, since $1 - \chi(Z_{\min}) = 2$ (provided that we verify that $p_g \leq 3$).

Hence we need to prove that $p_g = 4$ cannot occur. To do this, we assume that $p_g(X, o) = 4$ for certain (X, o) and we will deduce a contradiction.

By Corollary 2.7 (X, o) is necessarily Gorenstein.

Let \tilde{X} be the minimal resolution. Then $K_{\tilde{X}} = -Z_K = -3E$.

Moreover, by Corollary 2.7 again, in the minimal good resolution $h^1(mZ_{\min}) = m + 1$ for $m = 1, 2, 3$. Hence in the minimal resolution (e.g. by Leray spectral sequence argument)

$$(3.2) \quad h^1(mE) = m + 1 \quad (m = 1, 2, 3).$$

From (2.10) $h^1(\mathcal{O}_{\tilde{X}}(-E)) = 2$, and from (2.11) we also have $h^1(\mathcal{O}_{\tilde{X}}(-2E)) = 1$, since $h^1(\mathcal{O}_E(-2E)) = h^0(\mathcal{O}_E) = 1$ by duality. From the exact sequence

$$(3.3) \quad 0 \rightarrow \mathcal{O}_E(-E) \rightarrow \mathcal{O}_{2E} \rightarrow \mathcal{O}_E \rightarrow 0,$$

we also obtain $h^1(\mathcal{O}_E(-E)) = 1$. So $h^0(\mathcal{O}_E(-E)) = 1$ since $\chi(\mathcal{O}_E(-E)) = 0$.

Since $h^1(\mathcal{O}_{\tilde{X}}(-2E)) - h^1(\mathcal{O}_{\tilde{X}}(-E)) + h^1(\mathcal{O}_E(-E)) = 0$, from the exact sequence

$$(3.4) \quad 0 \rightarrow \mathcal{O}_{\tilde{X}}(-2E) \rightarrow \mathcal{O}_{\tilde{X}}(-E) \rightarrow \mathcal{O}_E(-E) \rightarrow 0,$$

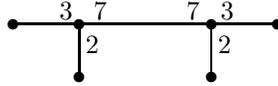
$H^0(\mathcal{O}_{\tilde{X}}(-E)) \rightarrow H^0(\mathcal{O}_E(-E)) \cong \mathbb{C}$ is surjective. Therefore, $Z_{\max} = E$. Let $s \in H^0(\mathcal{O}_E(-E))$ be the image of a general function $f \in H^0(\mathcal{O}_{\tilde{X}}(-E))$. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_E(-E) \xrightarrow{\times s} \mathcal{O}_E(-2E) \rightarrow \mathcal{O}_P(-2E) \rightarrow 0,$$

where $P \in E$ is the zero of s . Since $\deg \mathcal{O}_E(-E) = 1$, P is a nonsingular point of E . Since $h^1(\mathcal{O}_E(-2E)) = 1 = h^1(\mathcal{O}_E(-E))$ we get that $H^0(\mathcal{O}_E(-2E)) \rightarrow H^0(\mathcal{O}_P(-2E))$ is surjective, hence P is not a base point of $H^0(\mathcal{O}_E(-2E))$. Furthermore, since $H^0(\mathcal{O}_{\tilde{X}}(-2E)) \xrightarrow{r} H^0(\mathcal{O}_E(-2E))$ is surjective, there exists a function $g \in H^0(\mathcal{O}_{\tilde{X}}(-2E))$ such that $r(g)(P) \neq 0$ and $(g)_E = 2E$. We can choose local coordinates x, y at P such that $E = \{x = 0\}$, $f = xy$, $g = x^2$. Then $\mathfrak{m}_{X,o}\mathcal{O}_{\tilde{X}} = (x, y)\mathcal{O}_{\tilde{X}}(-E)$ at P , or, $\mathfrak{m}_{X,o}\mathcal{O}_{\tilde{X}} = \mathfrak{m}_P\mathcal{O}_{\tilde{X}}(-E)$. Hence $\text{mult}(X, o) = -E^2 + 1 = 2$.

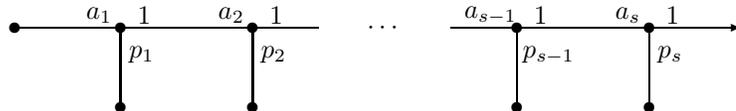
Now, it is well-known that a normal surface singularity with multiplicity two is necessarily a hypersurface of suspension type: $(X, o) = (\{z^2 + h(x, y) = 0\}, o)$ in suitable local coordinates.

However, this is impossible by the following proposition and by the fact that the splice diagram of Γ is



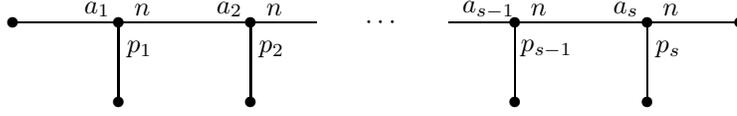
Proposition 3.5. [22] *Assume that the link of $\{z^n + h(x, y) = 0\}$ is an integral homology sphere. Then the following facts hold.*

- (1) h is irreducible;
- (2) Assume that the splice diagram of h is the following (for details see [3]):



Then $(a_i p_i, n) = 1$ for all i .

- (3) The splice diagram of $\{z^n + h(x, y) = 0\}$ is



4. THE CASE $Z_{\max} = Z_{\min}$

Proposition 2.7 and Lemma 2.9.1 of [6, §2] guarantee the existence of a normal complex surface singularity (X, o) with minimal good resolution graph Γ on which $Z_{\max} = Z_{\min}$. Indeed, let us construct an ‘extended’ graph Γ^e by gluing a (-1) -vertex to the (-13) -vertex of Γ by a new edge. In this way we get a negative semi-definite graph. By a theorem of Winters [34] there exists a family of projective curves $h_W : W \rightarrow (\mathbb{C}, 0)$ such that W is smooth, the central fiber is encoded by Γ^e , and the nearby fibers are smooth. Let \tilde{X} be a convenient small neighbourhood of the union of central curves indexed by Γ . Then this union of curves can be contracted by Grauert theorem [5] to get a singularity (X, o) and \tilde{X} serves as its minimal good resolution, on which the restriction h of h_W is a function with $(h)_E = Z_{\min}$.

An analytic type constructed in this way is called Kodaira [6] (or Kulikov [31]). We shall prove the following.

Theorem 4.1. *If $Z_{\max} = Z_{\min}$ on the minimal good resolution, then (X, o) necessarily is a non-Gorenstein Kodaira singularity with $p_g(X, o) = 3$, $\text{embdim}(X, o) = 4$ and $\text{mult}(X, o) = 3$. Furthermore such (X, o) is the total space of a one-parameter family of the curve singularity defined by $\text{rank} \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1^2 \end{pmatrix} < 2$ in $(\mathbb{C}^3, 0)$.*

Proof. We note that $Z_{\max} = E$ on the minimal resolution if and only if $Z_{\max} = Z_{\min}$ on the minimal good resolution, because if $\text{div}(f) = E + H$ on the minimal resolution, then H intersects E transversally.

Assume that \tilde{X} is the minimal resolution and that $Z_{\max} = E$. Note that $H^1(\mathcal{O}_{\tilde{X}}(-nE)) = 0$ for $n \geq 3$ by the vanishing theorem (cf. [4]). Then (X, o) is a Kodaira singularity by [6, 2.9.1] and $\mathcal{O}_{\tilde{X}}(-E)$ has no fixed component. Hence $\dim H^0(\mathcal{O}_{\tilde{X}}(-E))/H^0(\mathcal{O}_{\tilde{X}}(-2E)) \neq 0$. From the exact sequence (3.4), we have

$$\begin{aligned} h^1(\mathcal{O}_{\tilde{X}}(-E)) &\geq h^1(\mathcal{O}_E(-E)) = h^0(\mathcal{O}_E(-E)) \\ &\geq \dim_{\mathbb{C}} H^0(\mathcal{O}_{\tilde{X}}(-E))/H^0(\mathcal{O}_{\tilde{X}}(-2E)) \geq 1. \end{aligned}$$

Since by Theorem 3.1 $p_g(X, o) \leq 3$, in fact we have $p_g(X, o) = 3$ by (2.10), and all the inequalities above are equalities. Hence, via (3.3),

$$(4.2) \quad Z_{\text{coh}} = 2E.$$

By Theorem 2.3, (X, o) is not Gorenstein. Since $H^1(\mathcal{O}_{\tilde{X}}(-3E)) = 0$, it follows from [26, 3.1] (cf. also with the exact sequence from (3.4)) that $1 = h^1(\mathcal{O}_{\tilde{X}}(-E)) > h^1(\mathcal{O}_{\tilde{X}}(-nE))$ for $n \geq 2$. In particular, $h^1(\mathcal{O}_{\tilde{X}}(-nE)) = 0$ for $n \geq 2$. Thus we obtain that $H^0(\mathcal{O}_{\tilde{X}}(-nE)) \rightarrow H^0(\mathcal{O}_E(-nE))$ is surjective for $n \geq 0$ and $h^0(\mathcal{O}_E(-nE)) = n - 1$ for $n \geq 2$ by (2.8).

Let us compute the multiplicity of (X, o) . Since $h^0(\mathcal{O}_E(-E)) = h^0(\mathcal{O}_E(-2E)) = 1$, $\mathcal{O}_{\tilde{X}}(-E)$ and $\mathcal{O}_{\tilde{X}}(-2E)$ have a base point P . Take a general section $s \in H^0(\mathcal{O}_E(-E))$, and consider the exact sequence

$$0 \rightarrow \mathcal{O}_E(-2E) \xrightarrow{\times s} \mathcal{O}_E(-3E) \rightarrow \mathcal{O}_P(-3E) \rightarrow 0.$$

Then $H^0(\mathcal{O}_E(-3E)) \rightarrow H^0(\mathcal{O}_P(-3E))$ is surjective since $h^0(\mathcal{O}_E(-2E)) = 1$ and $h^0(\mathcal{O}_E(-3E)) = 2$. Since $H^0(\mathcal{O}_{\tilde{X}}(-3E)) \xrightarrow{\tau} H^0(\mathcal{O}_E(-3E))$ is surjective, $\mathcal{O}_{\tilde{X}}(-3E)$ has no base point. Hence a general function $g \in H^0(\mathcal{O}_{\tilde{X}}(-3E))$ satisfies $r(g)(P) \neq 0$ and $(g)_E = 3E$. As in Section 3, for suitable coordinates x, y at P , $\mathfrak{m}_{X, o} \mathcal{O}_{\tilde{X}} =$

$(y, x^2)\mathcal{O}_{\tilde{X}}(-E)$, where $E = \{x = 0\}$. Taking the blowing up $\phi_1: X_1 \rightarrow \tilde{X}$ at the base point P , we have a new base point $Q \in X_1$ such that $\mathfrak{m}_{X,o}\mathcal{O}_{X_1} = \mathfrak{m}_Q\mathcal{O}_{X_1}$. Let $\phi_2: X_2 \rightarrow X_1$ be the blowing up at the base point Q . Let $E_i \subset X_i$ be the exceptional set of ϕ_i , $Z_1 = \phi_1^*E + E_1$, and $Z_2 = \phi_1^*Z_1 + E_2$. Then the maximal ideal cycle on X_2 is Z_2 and $\mathcal{O}_{X_2}(-Z_2)$ has no base point. Hence $\text{mult}(X, o) = -Z_2^2 = 3$. Since $\text{embdim}(X, o) \leq \text{mult}(X, o) + 1 = 4$ (cf. [1]), and (X, o) is not Gorenstein, we have $\text{embdim}(X, o) = 4$, because any hypersurface is Gorenstein.

Let $h \in \mathfrak{m}_{X,o}$ be a general function. Then

$$\text{mult}(\{h = 0\}, o) = \text{mult}(X, o), \quad \text{embdim}(\{h = 0\}, o) = \text{embdim}(X, o) - 1.$$

By the formula of Morales [14, 2.1.4],

$$\delta(\{h = 0\}, o) = -(Z_K Z_2 + Z_2^2)/2 = 2 = \text{embdim}(\{h = 0\}, o) - 1.$$

Hence $(\{h = 0\}, o)$ is a partition curve $Y(3)$ in [2, §3].

This ends the proof of the theorem. \square

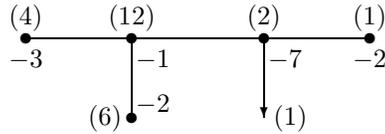
Example 4.3. We give defining equations of a Kodaira singularity with graph Γ . Let us recall [28, Example 6.3]. Let $(X', o) \subset (\mathbb{C}^4, o)$ be a singularity defined by

$$\text{rank} \begin{pmatrix} x & y & z \\ y - 3w^2 & z + w^3 & x^2 + 6wy - 2w^3 \end{pmatrix} < 2.$$

It is a numerically Gorenstein elliptic singularity. It shares the topological type the hypersurface singularity $(Y_2, o) := \{x^2 + y^3 + z^{13} = 0\} \subset (\mathbb{C}^3, o)$ with $p_g(Y_2, o) = 2$, however $p_g(X', o) = 1$. The exceptional set E' of the minimal resolution of (X', o) consists of two rational curve E'_1 and E'_2 with $E_1'^2 = -1$, $E_2'^2 = -2$, $E_1'E_2' = 1$ and E_1' has an ordinary cusp. The maximal ideal cycle is $2E_1' + E_2'$. The affine piece $V_1 \subset \mathbb{C}^5$ of the partial resolution (see [28, Example 6.3]) of (X', o) is defined by the equations

$$sx = y - 3w^2, \quad sy = z + w^3, \quad sz = x^2 + 6wy - 2w^3.$$

Consider the order of the coordinate functions on the exceptional set E' on V_1 . Then the order of s is zero, and the order of w is less than those of x, y, z . Hence $Z_{\max} = (w)_{E'}$. Note that $H := \text{div}(w) - (w)_{E'}$ intersects $E'_1 \setminus E'_2$ transversally. The graph of $\text{div}(w)$ on the minimal good resolution is as follows (the arrow corresponds to the strict transform of H):



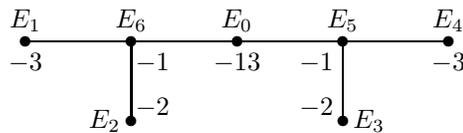
Let $\phi: (X, o) \rightarrow (X', o)$ be the double cover of X' branched along $w = 0$, namely, $\mathcal{O}_{X,o} = \mathcal{O}_{X',o}\{t\}/(t^2 - w)$. Then (X, o) is defined by

$$\text{rank} \begin{pmatrix} x & y & z \\ y - 3t^4 & z + t^6 & x^2 + 6t^2y - 2t^6 \end{pmatrix} < 2.$$

By the method of [16, III. Appendix 1], (X, o) has the resolution graph Γ , and $(t)_E = Z_{\max} = Z_{\min}$.

5. THE CASE $Z_{\max} = 2Z_{\min}$

Assume that \tilde{X} is the minimal good resolution and $Z_{\max} = 2Z_{\min}$ on \tilde{X} . We express the irreducible components of E as E_0, \dots, E_6 as below.



The cycle $E_i^* \in L$ is defined by $E_i^* E_i = -1$, $E_i^* E_j = 0$ for all $j \neq i$. (In general, E_i^* is an element of $L \otimes \mathbb{Q}$. In our case, $E_i^* \in L$ since the intersection matrix is unimodular.) E.g., $Z_{\min} = E_0^*$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-2Z_{\min}) \rightarrow \mathcal{O}_{\tilde{X}}(-Z_{\min}) \rightarrow \mathcal{O}_{Z_{\min}}(-Z_{\min}) \rightarrow 0$$

we have

$$(5.1) \quad h^1(\mathcal{O}_{\tilde{X}}(-2Z_{\min})) - h^1(\mathcal{O}_{\tilde{X}}(-Z_{\min})) = \chi(\mathcal{O}_{Z_{\min}}(-Z_{\min})) = 0.$$

Note that this equality holds whenever $Z_{\max} \geq 2Z_{\min}$.

I. The Gorenstein case.

Theorem 5.2. *Assume that $Z_{\max} = 2Z_{\min}$ on the minimal good resolution and (X, o) is Gorenstein. Then (X, o) is of splice type and the “leading form” of the splice diagram equations are given by*

$$z_1^2 z_2 + z_3^2 + z_4^3, \quad z_1^3 + z_2^2 + z_4^2 z_3,$$

where z_i corresponds to the end E_i . Furthermore, we have $\text{mult}(X, o) = 4$ and that $\mathcal{O}_{\tilde{X}}(-Z_{\max})$ has no base points.

The graph Γ satisfies the semigroup condition and we read the above defining equations from [23]. If X is of splice type, we have $\text{mult}(X, o) = 2 \cdot 2 = 4$, because the tangent cone is defined by the regular sequence z_3^2, z_2^2 . Furthermore, $\mathcal{O}_{\tilde{X}}(-Z_{\max})$ has no base points since $-Z_{\max}^2 = 4$ (or, by analysing the divisors E_1^* and E_4^* of z_1 and z_4). Therefore, it is sufficient to prove that the end curve condition is satisfied (see [24]).

Since (X, o) is Gorenstein, we have $p_g(X, o) = 3$ by Theorem 3.1. Therefore, from (2.10) and (5.1),

$$(5.3) \quad h^1(\mathcal{O}_{\tilde{X}}(-Z_{\min})) = h^1(\mathcal{O}_{\tilde{X}}(-2Z_{\min})) = 1.$$

Lemma 5.4. *Let $Z = E_4^*$. Then $\mathcal{O}_{\tilde{X}}(-Z)$ has no fixed component. In particular, there exists a function $f \in H^0(\mathcal{O}_{\tilde{X}}(-Z))$ such that $\text{div}(f) = Z + H$, where H is non-exceptional and $HE = HE_4 = 1$ (that is, H is a ‘cut’ of E_4), and hence the end curve condition at E_4 is satisfied.*

Proof. If $\mathcal{O}_{\tilde{X}}(-Z)$ has a fixed component, then every component of $E - E_4$ is also a fixed component because for any cycle $D > 0$ and the minimal anti-nef cycle D' such that $D' \geq D$, we have $H^0(\mathcal{O}_{\tilde{X}}(-D')) = H^0(\mathcal{O}_{\tilde{X}}(-D))$ (and if $D' > Z$ then $D' \geq Z + E$ too). We will show that E_6 cannot be a fixed component.

Since $Z > Z_{\min}$ (hence $h^1(\mathcal{O}_Z) \geq h^1(\mathcal{O}_{Z_{\min}}) = 2$), $Z_{\text{coh}} = Z_K$ and $C := Z_K - Z = E_0 + E_1 + E_2 + 2E_6 > 0$, we obtain that $h^1(\mathcal{O}_Z) = 2$. Thus

$$(5.5) \quad h^1(\mathcal{O}_{\tilde{X}}(-Z)) \geq p_g(X, o) - h^1(\mathcal{O}_Z) = 1.$$

Consider the exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\tilde{X}}(-Z - (C - E_6)) \rightarrow \mathcal{O}_{\tilde{X}}(-Z - E_6) \rightarrow \mathcal{O}_{C-2E_6}(-E_6) \rightarrow 0, \\ 0 \rightarrow \mathcal{O}_{\tilde{X}}(-Z - C) \rightarrow \mathcal{O}_{\tilde{X}}(-Z - (C - E_6)) \rightarrow \mathcal{O}_{E_6}(-(C - E_6)) \rightarrow 0. \end{aligned}$$

Since $h^1(\mathcal{O}_{C-2E_6}(-E_6)) = h^1(\mathcal{O}_{\tilde{X}}(-Z - C)) = 0$ and $h^1(\mathcal{O}_{E_6}(-(C - E_6))) = 1$, we obtain

$$(5.6) \quad 1 \geq h^1(\mathcal{O}_{\tilde{X}}(-Z - E_6)).$$

Therefore, (5.5) and (5.6) implies that $h^1(\mathcal{O}_{\tilde{X}}(-Z)) \geq h^1(\mathcal{O}_{\tilde{X}}(-Z - E_6))$.

This fact, and the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-Z - E_6) \rightarrow \mathcal{O}_{\tilde{X}}(-Z) \rightarrow \mathcal{O}_{E_6} \rightarrow 0$$

show that the restriction map $H^0(\mathcal{O}_{\tilde{X}}(-Z)) \rightarrow H^0(\mathcal{O}_{E_6})$ is non-trivial. Hence E_6 cannot be a fixed component. \square

Lemma 5.7. *Let $Z = E_3^*$. Then $\mathcal{O}_{\tilde{X}}(-Z)$ has no fixed component.*

Proof. Similarly as in the proof of the previous lemma, it is enough to verify that E_6 is not a fixed component.

There exists a computation sequence $\{Z_k\}_{k=0}^t$ from $Z_0 = Z + E_6$ to $Z_t = Z_K + Z_{\min} + E_5 + E_3$ such that $Z_{k+1} = Z_k + E_{i(k)}$, $Z_k E_{i(k)} > 0$, such that we add the base elements E_1, E_2, E_0 , and E_6 in this order. Then $Z_3 E_{i(3)} = 2$; at all the other steps $Z_k E_{i(k)} = 1$. From the exact sequences

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-Z_{i+1}) \rightarrow \mathcal{O}_{\tilde{X}}(-Z_i) \rightarrow \mathcal{O}_{E_{v(i)}}(-Z_i) \rightarrow 0,$$

we obtain that $h^1(\mathcal{O}_{\tilde{X}}(-Z_K - Z_{\min} - E_5 - E_3)) + 1 \geq h^1(\mathcal{O}_{\tilde{X}}(-Z - E_6))$. But, by a similar exact sequence, which connects $Z_K + Z_{\min}$ with Z_t (by adding E_5 and E_3 in this order) $h^1(\mathcal{O}_{\tilde{X}}(-Z_K - Z_{\min} - E_5 - E_3)) = h^1(\mathcal{O}_{\tilde{X}}(-Z_K - Z_{\min}))$, which is zero by Kodaira type vanishing. Hence

$$(5.8) \quad 1 \geq h^1(\mathcal{O}_{\tilde{X}}(-Z - E_6)).$$

Let $D = E_0 + E_1 + E_2 + 2E_6$. Then D is a minimally elliptic cycle on its support and thus $h^1(D) = 1$. Since $\mathcal{O}_{\tilde{X}}(-E_4^*)$ has no fixed component one has $H^0(\mathcal{O}_D(-E_4^*)) \neq 0$. This and $E_4^* D = 0$ imply that $\mathcal{O}_D(-E_4^*) \cong \mathcal{O}_D$. On the other hand, since $2Z - 3E_4^* = E_3 - E_4$, we obtain that

$$\mathcal{O}_D(-2Z) \cong \mathcal{O}_D(-3E_4^*) \cong \mathcal{O}_D.$$

Since $\text{Pic}(D)$ has no torsion, we obtain $\mathcal{O}_D(-Z) \cong \mathcal{O}_D$. Therefore,

$$(5.9) \quad h^1(\mathcal{O}_{\tilde{X}}(-Z)) \geq h^1(\mathcal{O}_D(-Z)) = 1.$$

Finally, from (5.8), (5.9) and the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-Z - E_6) \rightarrow \mathcal{O}_{\tilde{X}}(-Z) \rightarrow \mathcal{O}_{E_6} \rightarrow 0,$$

we obtain that E_6 cannot be a fixed component. \square

Therefore, the end curve condition is satisfied at all ends, and we finished the proof of Theorem 5.2.

II. The non-Gorenstein case.

Theorem 5.10. *Assume that $Z_{\max} = 2Z_{\min}$ on the minimal good resolution and (X, o) is not Gorenstein. Then $p_g(X, o) = 2$ and $Z_{\text{coh}} = E + E_5 + E_6$ on the minimal good resolution. Furthermore $\text{mult}(X, o) = 6$ and $\text{embdim}(X, o) = 7$.*

Assume that \tilde{X} is the minimal resolution. Then $Z_{\max} = 2E$. By Theorem 2.3, we have $h^1(\mathcal{O}_{2E}) = p_g(X, o)$. Clearly $h^1(\mathcal{O}_E) = h^1(\mathcal{O}_{2E})$ if and only if $p_g(X, o) = 2$; in this case, $Z_{\text{coh}} = E$ and the cohomological cycle on the minimal good resolution can be computed by [25, 2.6].

We assume that $h^1(\mathcal{O}_E) < h^1(\mathcal{O}_{2E})$, namely, $p_g(X, o) = 3$; we shall again deduce a contradiction.

From the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-2E) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{2E} \rightarrow 0,$$

and from $2E = Z_{\max}$, and $\chi(2E) = -1$, we have $h^1(\mathcal{O}_{\tilde{X}}(-2E)) = 1$. By (2.11), we have $h^1(\mathcal{O}_E(-2E)) = 1$ too. By duality, $h^0(\mathcal{O}_E(K + 3E)) = 1$ holds. Hence

$$(5.11) \quad \mathcal{O}_E(K + 3E) \cong \mathcal{O}_E.$$

Note that the groups of isomorphism classes of numerically trivial line bundles on \tilde{X} and $2E$ coincide, namely $H^1(\mathcal{O}_{\tilde{X}}) = H^1(\mathcal{O}_{2E})$. Hence the triviality of $\mathcal{O}_{2E}(K + 3E)$ would contradict to the fact that (X, o) is not Gorenstein.

We have the following exact sequence

$$(5.12) \quad 0 \rightarrow \mathcal{O}_E(K + 2E) \xrightarrow{\alpha} \mathcal{O}_{2E}(K + 3E) \xrightarrow{\beta} \mathcal{O}_E(K + 3E) \rightarrow 0$$

obtained by tensoring by $\mathcal{O}_{\tilde{X}}(K + 3E)$ the exact sequence

$$(5.13) \quad 0 \rightarrow \mathcal{O}_E(-E) \rightarrow \mathcal{O}_{2E} \rightarrow \mathcal{O}_E \rightarrow 0.$$

Note that from (5.13) we obtain $h^1(\mathcal{O}_E(-E)) = 1$ because $h^1(\mathcal{O}_{2E}) = 3$ by the assumption. Set $A := \mathcal{O}_E(K + 2E)$, $B := \mathcal{O}_E(K + 3E)$ and $N := \mathcal{O}_{2E}(K + 3E)$. Then, by (5.11), $A \cong \mathcal{O}_E(-E)$ and $B \cong \mathcal{O}_E$. Hence, both exact sequences (5.12) and (5.13) are extensions of B by A . It is sufficient to show the following.

Claim 1. For any nontrivial extension

$$0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0$$

of $\mathcal{O}_{\tilde{X}}$ -modules B by A , we necessarily have an isomorphism $M \cong \mathcal{O}_{2E}$.

Let Θ denote the bijection from the set of equivalence classes of extensions of B by A to $\text{Ext}^1(B, A)$. This map is given by $\Theta(0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0) = \delta(\text{Id}_B)$, where $\delta: \text{Hom}(B, B) \rightarrow \text{Ext}^1(B, A)$ is the connecting map of the long exact sequence obtained by the functor $\text{Hom}(B, \cdot)$. We denote the extension (5.12) by ξ . For any $a \in \mathbb{C}^*$, we define an extension $a \cdot \xi$ by

$$a \cdot \xi: \quad 0 \rightarrow A \xrightarrow{\alpha} N \xrightarrow{a^{-1}\beta} B \rightarrow 0.$$

Then $a \cdot \xi$ and $b \cdot \xi$ are equivalent if and only if $a = b$. We show that $a\Theta(\xi) = \Theta(a \cdot \xi)$. Here the first multiplication is in the \mathbb{C} -vector space $\text{Ext}^1(B, A)$.

Let us consider the injective resolution of ξ :

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_0 & \xrightarrow{\alpha_0} & I'_0 & \xrightarrow{\beta_0} & I''_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_1 & \xrightarrow{\alpha_1} & I'_1 & \xrightarrow{\beta_1} & I''_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Then the injective resolution of $a \cdot \xi$ is obtained by replacing β (resp. β_i) by $a^{-1}\beta$ (resp. $a^{-1}\beta_i$) in the diagram above. We denote by δ_β the connecting map associated with ξ . Applying the functor $\text{Hom}(B, \cdot)$ to the diagram corresponding to $a \cdot \xi$, we see that $\delta_{a^{-1}\beta}(\text{Id}_B) = a\delta_\beta(\text{Id}_B)$. Hence we obtain $\Theta(a \cdot \xi) = a\Theta(\xi)$. Since $\text{Ext}^1(B, A) \cong H^1(\mathcal{O}_E(-E)) \cong \mathbb{C}$, the above \mathbb{C}^* action on $\text{Ext}^1(B, A) \setminus \{0\}$ is transitive, namely $\mathbb{C}^* \rightarrow \mathbb{C}^*\Theta(\xi)$ is bijective onto $\text{Ext}^1(B, A) \setminus \{0\}$, or $\mathbb{C}^*\Theta(\xi) = \text{Ext}^1(B, A) \setminus \{0\}$. Hence the extensions (5.12) and (5.13) differ only by a non-zero constant multiplication (as above) and $\mathcal{O}_{2E}(K + 3E) \cong \mathcal{O}_{2E}$. This implies that the singularity is Gorenstein, a contradiction. In particular, we have proved Claim 1 and that $p_g(X, o) = 2$.

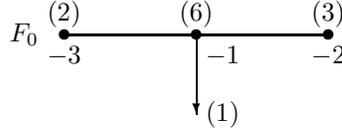
Next we compute the multiplicity and the embedding dimension. Since $p_g(X, o) = 2$, we have $h^1(\mathcal{O}_E) = h^1(\mathcal{O}_{2E}) = 2$. By (2.8) and (3.3), we have $h^0(\mathcal{O}_{2E}) = 1$ and $h^0(\mathcal{O}_E(-E)) = 0$. By (3.4), we have $h^1(\mathcal{O}_{\tilde{X}}(-2E)) = h^1(\mathcal{O}_{\tilde{X}}(-E)) = p_g(X, o) - 2 = 0$. By (2.8) and (2.11), we have $H^0(\mathcal{O}_{\tilde{X}}(-2E)) \rightarrow H^0(\mathcal{O}_E(-2E))$ is surjective and $h^0(\mathcal{O}_E(-2E)) = 1$. Therefore $\mathcal{O}_{\tilde{X}}(-2E)$ has base point. Let $g \in H^0(\mathcal{O}_{\tilde{X}}(-2E))$ be a general element and $\text{div}(g) = 2E + H$. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-E) \xrightarrow{\times g} \mathcal{O}_{\tilde{X}}(-3E) \rightarrow \mathcal{O}_H(-3E) \rightarrow 0.$$

Since $H^0(\mathcal{O}_{\tilde{X}}(-3E)) \rightarrow H^0(\mathcal{O}_H(-3E))$ is surjective, $\mathcal{O}_{\tilde{X}}(-3E)$ has no base point. Therefore there exists a function $h \in H^0(\mathcal{O}_{\tilde{X}}(-3E))$ such that $(h)_E = 3E$ and the image in $H^0(\mathcal{O}_E(-3E))$ is nonzero at the base points of $\mathcal{O}_{\tilde{X}}(-2E)$, namely,

at $E \cap H$. We resolve the base points and compute the multiplicity. We have the following three cases. Note that $HE = 2$.

- (1) Assume that $H \cap E$ has two distinct points p_1 and p_2 ; clearly these are smooth points of E . Let $\phi: Y \rightarrow \tilde{X}$ be the blowing up at $H \cap E$ and $F_i = \phi^{-1}(p_i)$. If Z denote the maximal ideal cycle on Y , then $Z = \phi^*(2E) + F_1 + F_2$ and $\mathcal{O}_Y(-Z)$ has no base points. Therefore $\text{mult}(X, o) = -Z^2 = 6$. Clearly the strict transform F_0 of E is the cohomological cycle and $\mathcal{O}_{F_0}(-Z) \cong \mathcal{O}_{F_0}$. Therefore Z is a p_g -cycle by [27, 3.10]. Hence $\text{embdim}(X, o) = -Z^2 + 1 = 7$ by [27, 6.2].
- (2) Assume that H intersects E at a smooth point $p \in E$. We have local coordinates x, y at p such that $E = \{x = 0\}$. Then, at p , we may assume that $h = x^3$ and $g = x^2(y^2 - xg_1)$ for some $g_1 \in \mathbb{C}\{x, y\}$ with $g_1(0, 0) \neq 0$; therefore, $\mathfrak{m}_{X, o}\mathcal{O}_{\tilde{X}} = (x^3, x^2y^2)\mathcal{O}_{\tilde{X}} = (x, y^2)\mathcal{O}_{\tilde{X}}(-2E)$. This base point can be resolved by two times of blowing ups; the graph of $\text{div}(g)$ is the following, where F_0 denote the strict transform of E .



By the same argument in (1), we obtain that $\text{mult}(X, o) = 6$ and $\text{embdim}(X, o) = 7$.

- (3) If H intersects E at a singular point of E , then H is nonsingular and the strict transform of H intersects transversally one of the (-3) -curves on the minimal good resolution. We may reset our situation as follows.

Let \tilde{X} be the minimal good resolution with exceptional set as in Section 5 and suppose that $Z_{\max} = (g)_E = E_4^*$ and $(h)_E = 3Z_{\min}$. By Lemma 6.2, $\mathcal{O}_{\tilde{X}}(-Z_{\min})$ has a base point, say P . Since $\text{coeff}_{E_4}(E_4^*) = 5$ and $\text{coeff}_{E_4}(3Z_{\min}) = 6$, we see that $\mathfrak{m}_{X, o}\mathcal{O}_{\tilde{X}} = \mathfrak{m}_P\mathcal{O}_{\tilde{X}}(-Z_{\max})$ and the base point is resolved by the blowing up at P . Then $\text{mult}(X, o) = -Z_{\max}^2 + 1 = 6$ and $\text{embdim}(X, o) = 7$ by the same argument as in (1).

6. THE CASE $Z_{\max} \neq Z_{\min}, 2Z_{\min}$

We assume that \tilde{X} is the minimal good resolution with exceptional set as in Section 5 and that $Z_{\max} \neq Z_{\min}, 2Z_{\min}$ on \tilde{X} . If the maximal ideal cycle on the minimal resolution is E , then the base point of $\mathcal{O}(-E)$ is a smooth point of E and thus $Z_{\max} = Z_{\min}$. Hence $\text{coeff}_{E_0}(Z_{\max}) = 2$ by Proposition 2.12. On the other hand, any anti-nef cycle on \tilde{X} with $\text{coeff}_{E_0} = 2$ is one of the following three cycles:

$$2Z_{\min} = 2E_0^*, \quad E_1^*, \quad E_4^*.$$

Hence we have to analyse the new cases when Z_{\max} equals either E_1^* or E_4^* . Since the two cases are symmetric, in the sequel we assume that $Z_{\max} = E_4^*$.

First we start with the following lemma.

Lemma 6.1. *For any $\ell \geq 1$ and for analytic structure supported by Γ*

(a) *the line bundle $\mathcal{O}_{\tilde{X}}(-(\ell + 2)Z_{\min})$ has no fixed component.*

(b) $h^1(\mathcal{O}_{\tilde{X}}(-(\ell + 2)Z_{\min})) = 0$.

Proof. (a) There exists a computation sequence starting from $E_4^* + \ell Z_{\min} + E_6$ and ending with $Z_K + \ell Z_{\min}$ by adding (in this order) E_1, E_2, E_6, E_0 , such that at the first three steps $Z_k E_{i(k)} = 1$ and at the last step $Z_k E_{i(k)} \leq 1$. Hence $h^1(\mathcal{O}_{\tilde{X}}(-E_4^* - \ell Z_{\min} - E_6)) \leq h^1(\mathcal{O}_{\tilde{X}}(-Z_K - \ell Z_{\min})) = 0$. In particular, from the

exact sequence $0 \rightarrow \mathcal{O}_{\tilde{X}}(-E_4^* - \ell Z_{\min} - E_6) \rightarrow \mathcal{O}_{\tilde{X}}(-E_4^* - \ell Z_{\min}) \rightarrow \mathcal{O}_{E_6} \rightarrow 0$,

$$\frac{H^0(\mathcal{O}_{\tilde{X}}(-E_4^* - \ell Z_{\min}))}{H^0(\mathcal{O}_{\tilde{X}}(-E_4^* - \ell Z_{\min} - E_6))} \cong \mathbb{C}.$$

Hence, there exists a function f with $\text{coeff}_{E_6}(f) = 12 + 6\ell$, $\text{coeff}_{E_0}(f) = 2 + \ell$ and $\text{coeff}_{E_5}(f) \geq 14 + 6\ell$. Symmetrically, there exists another function f' with $\text{coeff}_{E_6}(f') \geq 14 + 6\ell$, $\text{coeff}_{E_0}(f') = 2 + \ell$ and $\text{coeff}_{E_5}(f') = 12 + 6\ell$. Hence the divisor of $f + f'$ is $(\ell + 2)Z_{\min}$.

(b) There is a Laufer computation sequence starting from $Z_K + (\ell - 1)Z_{\min}$ and ending with $(\ell + 2)Z_{\min}$ such that at every step $Z_k E_{i(k)} = 1$. Hence $h^1(\mathcal{O}_{\tilde{X}}(-\ell + 2)Z_{\min}) = h^1(\mathcal{O}_{\tilde{X}}(-Z_K - (\ell - 1)Z_{\min})) = 0$. \square

Lemma 6.2. *If $Z_{\max} = E_4^*$ then $p_g(X, o) = 2$ (hence (X, o) is not Gorenstein), and $\mathcal{O}_{\tilde{X}}(-E_4^*)$ has a base point.*

Proof. Let $C = E_4^* - 2Z_{\min} = E_3 + E_4 + 2E_5$. In the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-E_4^*) \rightarrow \mathcal{O}_{\tilde{X}}(-2Z_{\min}) \rightarrow \mathcal{O}_C \rightarrow 0,$$

the assumption implies $H^0(\mathcal{O}_{\tilde{X}}(-E_4^*)) = H^0(\mathcal{O}_{\tilde{X}}(-2Z_{\min}))$, hence

$$(6.3) \quad h^1(\mathcal{O}_{\tilde{X}}(-E_4^*)) = 1 + h^1(\mathcal{O}_{\tilde{X}}(-2Z_{\min})).$$

Let $D = Z_K - E_4^* = E_0 + E_1 + E_2 + 2E_6$. Then we have $h^1(\mathcal{O}_D(-E_4^*)) = 1$ as in the proof of Lemma 5.7. From the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-Z_K) \rightarrow \mathcal{O}_{\tilde{X}}(-E_4^*) \rightarrow \mathcal{O}_D(-E_4^*) \rightarrow 0,$$

we obtain $h^1(\mathcal{O}_{\tilde{X}}(-E_4^*)) = 1$. By (6.3), we have $h^1(\mathcal{O}_{\tilde{X}}(-2Z_{\min})) = 0$. It follows from (5.1) and (2.10) that $p_g(X, o) = 2$.

Furthermore, (X, o) is not Gorenstein by Theorem 3.1.

There exists a computation sequence $\{Z_k\}$ starting from $E_4^* + E_4$ and ending with $3Z_{\min}$ such that $Z_k E_{i(k)} = 2$ at two steps and otherwise $= 1$. Since $h^1(\mathcal{O}_{\tilde{X}}(-3Z_{\min})) = 0$ (cf. Lemma 6.1(b)), we obtain $h^1(\mathcal{O}_{\tilde{X}}(-E_4^* - E_4)) = 2$. In particular, from the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-E_4^* - E_4) \rightarrow \mathcal{O}_{\tilde{X}}(-E_4^*) \rightarrow \mathcal{O}_{E_4}(-E_4^*) \rightarrow 0,$$

the image of the map $H^0(\mathcal{O}_{\tilde{X}}(-E_4^*)) \rightarrow H^0(\mathcal{O}_{E_4}(-E_4^*))$ is 1-dimensional. Hence $\mathcal{O}_{\tilde{X}}(-E_4^*)$ has a base point. \square

Let f be the generic element of $\mathfrak{m}_{X,o}$. Its divisor on \tilde{X} has the form $Z_{\max} + H$, where H is a cut of E_4 cutting it transversally in a unique point P . Then in local coordinates around P (with $\{x = 0\} = E$) f has the form $x^5 y$. By Lemma 6.1(a) there exists a function g with $(g)_E = 3Z_{\min}$, hence at P with local equation x^6 . Therefore, $\mathfrak{m}_{X,o} \mathcal{O}_{\tilde{X}} = \mathfrak{m}_P \mathcal{O}_{\tilde{X}}(-Z_{\max})$ and $\text{mult}(X, o) = -Z_{\max}^2 + 1 = 6$.

Next, $\text{embdim}(X, o) = 7$ by the same argument as in (1) of the previous section.

Remark 6.4. Assume that (X, o) is a singularity supported by Γ with $p_g = 2$. Then $h^1(\mathcal{O}_{\tilde{X}}(-2Z_{\min})) = h^1(\mathcal{O}_{\tilde{X}}(-3Z_{\min})) = 0$. Hence, from the exact sequence $0 \rightarrow \mathcal{O}_{\tilde{X}}(-3Z_{\min}) \rightarrow \mathcal{O}_{\tilde{X}}(-2Z_{\min}) \rightarrow \mathcal{O}_{Z_{\min}}(-2Z_{\min}) \rightarrow 0$ we obtain that

$$\frac{H^0(\mathcal{O}_{\tilde{X}}(-2Z_{\min}))}{H^0(\mathcal{O}_{\tilde{X}}(-3Z_{\min}))} \cong \mathbb{C}.$$

Since the divisors of the analytic functions are the anti-nef cycles, and the only anti-nef cycles C with $C \geq 2Z_{\min}$ and $C \not\geq 3Z_{\min}$ are $2Z_{\min}$, E_1^* , E_4^* , out of these three cycles *exactly one* appears as the divisor of an analytic function chosen by the analytic type. That divisor equals Z_{\max} .

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ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, REALTANODA
UTCA 13-15, H-1053, BUDAPEST, HUNGARY

ELTE - UNIVERSITY OF BUDAPEST, HUNGARY

BCAM - BASQUE CENTER FOR APPLIED MATHEMATICS, BILBAO, SPAIN

E-mail address: `nemethi.andras@renyi.mta.hu`

DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, YAMAGATA UNIVERSITY,
YAMAGATA, 990-8560, JAPAN.

E-mail address: `okuma@sci.kj.yamagata-u.ac.jp`